AP Calculus 2005 AB (Form B) FRQ Solutions

Louis A. Talman, Ph. D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

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1 Problem 1

1.1 Part a

We must first locate the right-hand corner of the region R—that is, the smallest positive value $x = a$ for which $1+\sin 2x = e^{x/2}$. Numerical solution gives $a = 1.13569$. The desired area is then

$$\int_0^a \left[(1 + \sin 2x) - e^{x/2}\right] \, dx \sim 0.42910, \tag{1}$$

where we have carried out the integration numerically. (The integral is elementary, but the upper limit is approximate—so there is little point in evaluating the integral symbolically. For the curious, doing so gives the value $(2 + a - 2e^{a/2} + \sin^2 a)$.)

1.2 Part b

Using the method of washers, we find that the volume generated by revolving the region R about the x-axis is

$$\pi \int_0^a \left[(1 + \sin 2x)^2 - e^x\right] \, dx \sim 4.26655, \tag{2}$$

having again calculated the integral numerically. Once again, we have found it convenient to evaluate the integral numerically. (Once again, the integral is elementary, but symbolic evaluation is tedious. If we invest the time and the energy, we find that the value of the integral is $\pi \left[1 + \frac{3}{2}a - e^a + 2\sin^2 a - \frac{1}{8}\sin 4a\right]$.)

1.3 Part c

The volume of this solid is

$$\frac{\pi}{2}\int_0^a \left[\frac{1}{2}(1+\sin 2x - e^{x/2})\right]^2 \, dx \sim 0.07766. \tag{3}$$

This integral, too, is elementary, but the calculation is pretty tedious, and we have done the integration numerically. Symbolic evaluation gives the value

$$\frac{\pi}{8}\left[\frac{52}{17}+\frac{3}{2}a+e^a-\cos 2a+\frac{4}{17}e^{a/2}\left(4\cos 2a-\sin 2a-17\right)-\frac{1}{8}\sin 4a\right] \tag{4}$$

for the integral.

2 Problem 2

2.1 Part a

At time $t = 15$, water is entering the tank at the rate of

$$W(15) = 95\sqrt{15}\sin^3\frac{15}{6} \sim 131.78231 \text{ gallons per hour,} \tag{5}$$

and is being removed from the tank at the rate of

$$R(15) = 275\sin^2\frac{15}{3} \sim 252.87234 \text{ gallons per hour.} \tag{6}$$

When $t = 15$, the removal rate is larger than the supply rate, so the amount of water in the tank is decreasing when $t = 15$.

2.2 Part b

The amount, $A(t)$, of water in the tank at time t s, by the Fundamental Theorem of Calculus,

$$A(t) = 1200 + \int_0^t \left[W(\tau) - R(\tau)\right], d\tau. \tag{7}$$

Thus, integrating numerically, we find that

$$A(18) \sim 1309.78818 \text{ gallons.} \tag{8}$$

So, to the nearest whole number, there are 1310 gallons of water in the tank at $t = 18$.

2.3 Part c

We seek the zeros of $A'(t) = W(t) - R(t)$ in the interval $(0, 18)$. Solving numerically, we find that these are $t \sim 6.49484$ and $t \sim 12.9748$. Because $A'(t)$ is defined for all $t \in (0, 18)$, we know that the absolute minimum value of $A(t)$ for $t \in [0, 18]$ must be one of the four values $A(0) = 1200$, $A(6.49484) \sim 525.24215$, $A(12.97482) \sim 1697.44124$, and $A(18) \sim 1309.78818$. (We have used (7) to calculate all but the first of these four values, carrying out the required integrations numerically.) Thus, the absolute minimum amount of water in the tank during the time interval $[0, 18]$ occurs when $t \sim 6.49484$; that minimum value is about 525.24215 gallons.

2.4 Part d

With A and R as defined above, we must solve for k in the equation

$$A(18) - \int_{18}^{k} R(\tau)\, d\tau = 0. \tag{9}$$

Note: Solution of (9) is not required. For the curious, numerical techniques give $k \sim 29.19242$ for the solution.

3 Problem 3

3.1 Part a

If $v(t) = \ln(t^2 - 3t + 3)$, then acceleration $a(t)$, being $v'(t)$ is given by

$$a(t) = v'(t) = \frac{2t - 3}{t^2 - 3t + 3}. \tag{10}$$

Thus,

$$a(4) = \frac{2 \cdot 4 - 3}{4^2 - 3 \cdot 4 + 3} = \frac{5}{7}. \tag{11}$$

3.2 Part b

The particle changes direction where v changes sign. This can occur only where $v(t) = 0$, or where $t^2 - 3t + 3 = 1$. Thus

$$t^2 - 3t + 3 = 1; \tag{12}$$
$$t^2 - 3t + 2 = 0; \tag{13}$$
$$(t-1)(t-2) = 0; \tag{14}$$

Thus, $v(t) = 0$ when $t = 1$ and when $t = 2$. The graph of $t^2 - 3t + 2$ is a parabola opening upward, so $v(t) > 0$ when $t < 1$ and when $t > 2$. On the other hand, $v(t) < 0$ when $1 < t < 2$. This means that the particle changes direction at $t = 1$ and at $t = 2$. The particle moves to the left when $v(t) < 0$, or when $1 < t < 2$.

3.3 Part c

The position $x(t)$ of the particle at time t is, by the Fundamental Theorem of Calculus, given by

$$x(t) = 8 + \int_0^t v(\tau)\,d\tau = 8 + \int_0^t \ln(\tau^2 - 3\tau + 3)\,d\tau \tag{15}$$

To find position at time $t = 2$, we integrate numerically:

$$x(2) = 8 + \int_0^2 \ln(\tau^2 - 3\tau + 3)\,d\tau \sim 8.36862 \tag{16}$$

Note: The integral is elementary, but tedious. We obtain $x(2) = 4 + \dfrac{\sqrt{3}}{2}\pi + \dfrac{1}{2}\ln 27$.

3.4 Part d

Average speed on the interval $[0, 2]$ is $\frac{1}{2}\int_0^2 |v(\tau)|\,d\tau$. Using what we have learned about the sign of v in Part b, above, we find that the required average speed is

$$\frac{1}{2}\left[\int_0^1 v(\tau)d\tau - \int_1^2 v(\tau)d\tau\right] \sim 0.37051. \tag{17}$$

We have integrated numerically, in order to avoid another tedious symbolic integration. The symbolic integration yields the value $\dfrac{1}{12}(9\ln 3 - \sqrt{3}\pi)$.

4 Problem 4

4.1 Part a

$g(-1) = \int_{-4}^{-1} f(t)\,dt$ is the negative of the area of the trapezoid defined by the x-axis, the vertical lines $x = -4$ and $x = -1$, and the line segment joining the points $(-4, -3)$ and $(-1, -2)$. Thus,

$$g(-1) = -\frac{1}{2} \cdot (3 + 2) \cdot 3 = -\frac{15}{2}. \tag{18}$$

By the Fundamental Theorem of Calculus, $g'(x) = f(x)$, so $g'(-1) = -2$. It also follows that $g''(x) = f'(x)$, if, and only if, the latter exists. Because of the corner in the graph of $f(x)$ at the point corresponding to $x = -1$, $f'(-1)$ does not exist. (In fact, $f'_{-}(-x) = 1/3$, while $f'_{+}(-1) = 2$.) Thus, $g''(-1)$ does not exist.

4.2 Part b

The inflection points of g occur where $g' = f$ has relative extrema. But f has just one relative extremum in the interval $(-4, 3)$, at $x = 1$—as is evident from the graph. Thus, the only inflection point for g is to be found at $x = 1$.

4.3 Part c

If

$$h(x) = \int_x^3 f(t)\,dt = -\int_3^x f(t)\,dt, \tag{19}$$

then the zeros of h are to be found at those values of x for which the graph of f over the interval whose endpoints are 3 and x has just as much area above the horizontal coordinate axis as below. These values are evidently $x = -1$ and $x = 1$. And, of course, we shouldn't forget the trivial solution $x = 3$.

4.4 Part d

With h as given in Part c, above, we have, by the Fundamental Theorem of Calculus, $h'(x) = -f(x)$. Therefore, h is decreasing on (the closures of) those intervals for which $-f(x) < 0$, or, equivalently, where $f(x) > 0$. From the graph, it is thus evident that h is decreasing on $[0, 2]$.

5 Problem 5

5.1 Part a

On the curve $y^2 = 2 + xy$, we treat y as though it is (at least, near each point on the curve) a function of x. Then, differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left[y^2\right] = \frac{d}{dx}\left[2 + xy\right] \tag{20}$$

$$2y\frac{dy}{dx} = y + x\frac{dy}{dx} \tag{21}$$

$$2y\frac{dy}{dx} - x\frac{dy}{dx} = y \tag{22}$$

$$\frac{dy}{dx} = \frac{y}{2y - x}, \tag{23}$$

at least as long as $2y - x \neq 0$. But if $2y - x = 0$, then $x = 2y$, and the equation $y^2 = 2 + xy$ becomes $y^2 = 2 + (2y)y$, or $y^2 = -2$, which is not possible for real values of y. We conclude that (23) gives $\frac{dy}{dx}$ at all points of the curve.

5.2 Part b

If

$$y' = \frac{1}{2} = \frac{y}{2y - x}, \text{ then} \tag{24}$$

$$2y - x = 2y, \tag{25}$$

and $x = 0$. Substituting this in the original equation, we find that $y^2 = 2$. The required points are therefore $(0, \sqrt{2})$ and $(0, -\sqrt{2})$.

5.3 Part c

If the tangent line to the curve $y^2 = 2 + xy$ is horizontal at a point (x_0, y_0), we must have

$$0 = y'(x_0) = \frac{y_0}{2y_0 - x_0}. \tag{26}$$

As we have seen in Part a, above, if (x_0, y_0) is a point on the curve, then $2y_0 + x_0 = 0$ is not possible, so (26) means that $y_0 = 0$. But then $0 = y_0^2 = 2 + x_0 y_0 = 2 + 0 = 2$, which means that $0 = 2$. The contradiction show that there can be no point on the curve $y^2 = 2 + xy$ where the tangent line is horizontal.

5.4 Part d

We differentiate the equation for the curve implicitly again, but this time we treat x and y both as functions of a third variable t, and we take the prime to mean differentiation with respect to t. We have

$$\frac{d}{dt}y^2 = \frac{d}{dt}\left[2+xy\right]; \tag{27}$$

$$2y\frac{dy}{dy} = y\frac{dx}{dt} + x\frac{dy}{dt}, \tag{28}$$

or $2yy' = yx' + xy'$. Putting $y = 3$, $y' = 6$, in both the original equation and the derived equation leads to the system of equations

$$9 = 2 + 3x \tag{29}$$

$$36 = 3x' + 6x. \tag{30}$$

From the first of these two, we see that $x = 7/3$, and substituting this for x in the second equation yields $36 = 3x' + 14$, whence $x' = 22/3$.

6 Problem 6

6.1 Part a

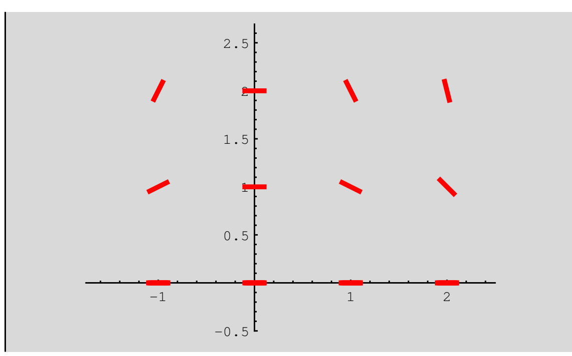

Figure 1: Problem 6, Part a

6.2 Part b

We are given a solution f of the differential equation $y' = -xy^2/2$ for which $f(-1) = 2$. Thus,

$$f'(-1) = -\frac{(-1)(2)^2}{2} = 2. \tag{31}$$

The equation of the line tangent to the graph of $y = f(x)$ at the point $(-1, 2)$ is therefore

$$y = 2 + 2(x+1), \text{ or} \tag{32}$$

$$y = 2x + 4. \tag{33}$$

6.3 Part c

If $y = f(x)$ is a solution of the differential equation $y' = -xy^2/2$ that satisfies $y(-1) = 2$, then

$$\frac{f'(x)}{[f(x)]^2} = -\frac{x}{2}. \tag{34}$$

Thus,

$$\int_{-1}^{x} \frac{f'(\xi)}{[f(\xi)]^2}\, d\xi = -\frac{1}{2}\int_{-1}^{x} \xi\, d\xi; \tag{35}$$

$$-\frac{1}{f(\xi)}\bigg|_{-1}^{x} = -\frac{\xi^2}{4}\bigg|_{-1}^{x}; \tag{36}$$

$$-\frac{1}{f(x)} - \left[-\frac{1}{f(-1)}\right] = -\frac{x^2}{4} - \left[-\frac{(-1)^2}{4}\right]; \tag{37}$$

$$f(x) = \frac{4}{1+x^2}. \tag{38}$$