AP Calculus 2005 AB FRQ Solutions

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1 Problem 1

1.1 Part a

The area of the region R is $\int_0^a \left[4^{-x} - \frac{1}{4} - \sin \pi x\right]\, dx$, where a is the smallest positive solution of the equation

$$1 + 4\sin \pi x = 4^{1-x}. \tag{1}$$

Numerical solution, and then a numerical integration, give

$$a \sim 0.17823, \text{ and} \tag{2}$$

$$\int_0^a \left[4^{-x} - \frac{1}{4} - \sin \pi x\right]\, dx \sim 0.06475. \tag{3}$$

1.2 Part b

The second smallest positive solution, b, of equation (1) is easily seen to be $b = 1$. The area of the region S is therefore given by

$$\int_a^1 \left[4^{-x} - \frac{1}{4} - \sin \pi x\right]\, dx \sim 0.41036, \tag{4}$$

where we have again carried out the integration numerically.

1.3 Part c

The volume of the solid generated when S is revolved about the horizontal ine $y = -1$ is (integrating numerically one more time)

$$\pi \int_a^1 \left[\left(\frac{1}{4} + \sin \pi x + 1 \right)^2 - \left(4^{-x} + 1\right)^2 \right] dx \sim 4.55876. \tag{5}$$

2 Problem 2

2.1 Part a

If the rate $R(t)$, in cubic yards per hour, at which sand is being removed by the tide at time t is given by

$$R(t) = 2 + 5 \sin \left(\frac{4\pi t}{25} \right), \tag{6}$$

then the amount, in cubic yards, of sand removed by the tide during the period $0 \le t \le 6$ is

$$\int_0^6 R(t)\, dt = 12 + \frac{125}{4\pi} - \frac{125}{4\pi} \cos \left(\frac{24}{25} \pi \right) \sim 31.81593 \tag{7}$$

2.2 Part b

$$Y(t) = 2500 - \int_0^t \left(2 + 5 \sin \frac{4\pi\tau}{25} \right) d\tau + \int_0^t \frac{15\tau}{1 + 3\tau}\, d\tau. \tag{8}$$

Note: Whether evaluation of the integrals is required is not clear. For the record,

$$\int_0^t \left(2 + 5 \sin \frac{4\pi\tau}{25} \right) d\tau = \left(2\tau - \frac{125}{4\pi} \cos \frac{4\pi\tau}{25} \right) \Big|_0^t \tag{9}$$

$$= \left(2t - \frac{125}{4\pi} \cos \frac{4\pi t}{25} \right) - \left(0 - \frac{125}{4\pi} \right), \tag{10}$$

while

$$\int_0^t \frac{15\tau}{1+3\tau}\,d\tau = \int_0^t \frac{5(1+3\tau)}{1+3\tau}\,d\tau - \int \frac{5}{1+3\tau}\,d\tau \tag{11}$$

$$= 5\int d\tau - \frac{5}{3}\int \frac{3\,d\tau}{1+3\tau} \tag{12}$$

$$= \left(5\tau - \frac{5}{3}\ln|1+3\tau|\right)\Big|_0^t \tag{13}$$

$$= 5t - \frac{5}{3}\ln|1+3t|. \tag{14}$$

Thus,

$$Y(t) = 2500 - \frac{125}{4\pi} + 3t + \frac{125}{4\pi}\cos\frac{4\pi t}{25} - \frac{5}{3}\ln|1+3t|. \tag{15}$$

2.3 Part c

When $t = 4$, the total amount of sand on the beach is changing at the rate

$$S(4) - R(4) = \frac{34}{13} - 5\sin\frac{16\pi}{25} \sim -1.90875 \text{ cubic yards per hour.} \tag{16}$$

2.4 Part d

A plot of $S(t) - R(t)$ over the interval $0 \leq t \leq 6$ shows that the rate of accumulation is negative when $t < t_0$ and positive when $t > t_0$, where t_0 is a certain value of t near $t = 5$. Solving numerically for t_0, we find that $t_0 \sim 5.11787$, and this must be the time when the amount of sand on the beach is minimal. This minimal amount is about

$$Y(t_0) \sim 2492.36948. \tag{17}$$

3 Problem 3

3.1 Part a

$$T'(7) \sim \frac{T(8) - T(6)}{8-6} = \frac{55-62}{8-6} = -\frac{7}{2}. \tag{18}$$

3.2 Part b

The average temperature of the wire is

$$\frac{1}{8}\int_0^8 T(x)\,dx \sim \frac{1}{8}\left[\frac{100+93}{2}+\frac{93+70}{2}(5-1)+\frac{70+62}{2}+\frac{62+55}{2}(8-6)\right] \tag{19}$$

$$\sim \frac{1211}{16}\text{ degrees Celsius.} \tag{20}$$

3.3 Part c

We are given that T is twice differentiable—though we are not told where. We take the statement to mean that T is twice differentiable, and, consequently that T' is continuous, on a domain that includes $[0,8]$, so that the problem is meaningful. By the Fundamental Theorem of Calculus,

$$\int_0^8 T'(t)\,dt = T(8)-T(0) = -45\text{ degrees Celsius.} \tag{21}$$

The integrand, $T'(x)$, is the (instantaneous) rate at which $T(x)$ changes per unit length at each point of the interval $[0,8]$, and the integral gives net temperature change over the same interval.

3.4 Part d

By hypothesis, T is continuous on $[1,5]$ and differentiable on $(1,5)$, so the Mean Value Theorem guarantees that there is a point $\xi \in (1,5)$ such that

$$T'(\xi) = \frac{T(5)-T(1)}{5-1} = -\frac{23}{4}. \tag{22}$$

By the same reasoning, there is a point $\eta \in (5,6)$ for which

$$T'(\eta) = \frac{T(6)-T(5)}{6-5} = -8. \tag{23}$$

We note that, necessarily, $0<\xi<\eta<8$. We apply the Mean Value Theorem still a third time, now on the interval $[\xi,\eta]$, and we obtain $\zeta \in (\xi,\eta)$ such that

$$T''(\zeta) = \frac{T'(\xi)-T'(\eta)}{\xi-\eta} = \frac{-8+(23/4)}{\xi-\eta} = \frac{-9}{4(\xi-\eta)} < 0. \tag{24}$$

Thus, the data in the table are not consistent with the assertion that $T''(x)>0$ throughout $0,8)$.

4 Problem 4

4.1 Part a

The derivative of f must be zero or undefined at any point of $(0, 4)$ where f has a relative extremum. Thus, $x = 1$ and $x = 2$ are the only values we need to consider. We find that $f'(x) > 0$ for all $x \neq 1$ in the interval $(0, 2)$. Consequently, f is increasing throughout that interval and can't have a relative extremum at $x = 1$. At $x = 2$, we find that $f(2)$ is meaningful, and that $f'(x)$ is positive on $(1, 2)$ but negative on $(2, 3)$. By the First Derivative Test, f has a relative maximum at $x = 2$.

4.2 Part b

See Figure 1

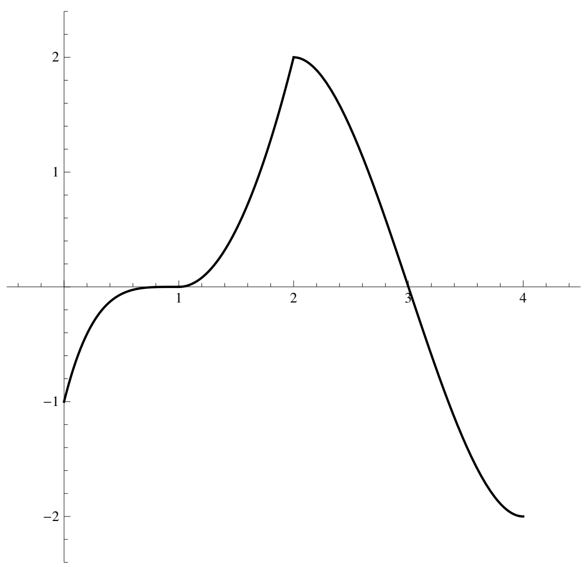

Figure 1: Problem 4, Part b

4.3 Part c

If

$$g(x) = \int_1^x f(t)\,dt, \tag{25}$$

then, by the Fundamental Theorem of Calculus, $g'(x) = f(x)$ Thus, g can have relative extrema only at points where $f(x) = 0$. At $x = 1$, $f(x)$ undergoes a sign change from negative to positive, and so at $x = 1$, $g(x)$ passes from a region where it is decreasing to a region where it is increasing. Consequently, g has a relative minimum at $x = 1$. Similar reasoning shows that g has a relative maximum at $x = 3$.

4.4 Part d

The function g has inflection points where g' as relative extrema. But g' is f and, according to Part a of this problem, f has a relative extremum only at $x = 2$. We conclude that g has just one inflection point, at $x = 2$.

5 Problem 5

5.1 Part a

$$\int_0^{24} v(t)\,dt = \frac{1}{2}(4-0)\cdot 20 + (16-4)\cdot 20 + \frac{1}{2}(34-16)\cdot 20 = 360 \text{ meters.} \tag{26}$$

The integral gives the distance, in meters, that the car travels during the time period $0 \leq t \leq 24$.

5.2 Part b

The definition of $v'(t_0)$ is

$$v'(t_0) = \lim_{h\to 9} \frac{v(t_0+h) - v(t_0)}{h}. \tag{27}$$

For the piecewise linear function given,

$$\lim_{h\to 0^-} \frac{v(4+h) - v(4)}{h} = 5, \text{ while} \tag{28}$$

$$\lim_{h\to 0^+} \frac{v(4+h) - v(4)}{h} = 0. \tag{29}$$

These one-sided limits are distinct, so the two-sided limit, which would be $v'(4)$, doesn't exist.

On the other hand, $v'(20) = -5/2$.

5.3 Part c

Acceleration, $a(t)$, is given by

$$a(t) = \begin{cases} t & \text{when } 0 < t < 4 \\ 0 & \text{when } 4 < t < 16 \\ -\dfrac{5}{2} & \text{when } 16 < t < 24. \end{cases} \tag{30}$$

5.4 Part d

The average rate of change of v over $8 \le t \le 20$ is

$$\frac{v(20) - v(8)}{20 - 8} = \frac{10 - 20}{20 - 8} = -\frac{5}{6}. \tag{31}$$

The hypotheses of the Mean Value Theorem require that a function be differentiable at every point of the interior of the interval on which we wish to apply the theorem, so we may not apply the Mean Value Theorem to the function v on the interval $[8, 20]$, because $v'(16)$ does not exist.

6 Problem 6

6.1 Part a

See Figure 2

6.2 Part b

At $(1, -1)$, we have $y' = -2(1)/(-1) = 2$, so the equation of the line tangent to the solution for which $y(1) = -1$ is

$$y = -1 + 2(x - 1) = 2x - 3. \tag{32}$$

This gives and approximate value for $y(1.1)$ of $y = 2 \cdot (1.1) - 3 = -0.8$.

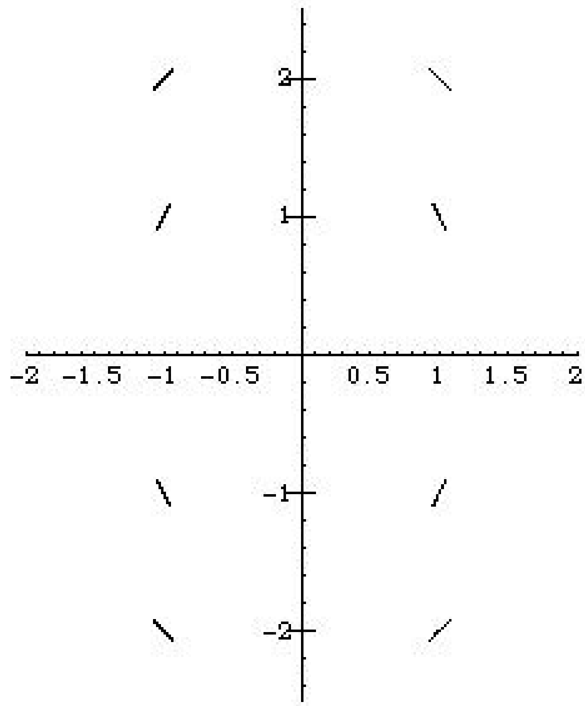

Figure 2: Problem 6, Part a

6.3 Part c

If f is the particular solution of $y' = -2x/y$ for which $f(1) = -1$, then

$$
\begin{aligned}
f(x)f'(x) &= -2x, \text{ so that} && (33)\\
\int_1^x f(\xi)f'(\xi)\,d\xi &= -2\int_1^x \xi\,d\xi; && (34)\\
\frac{1}{2}\left[f(\xi)\right]^2\Big|_1^x &= -\xi^2\Big|_1^x; && (35)\\
[f(x)]^2 - [f(1)]^2 &= -2x^2 + 2; && (36)\\
[f(x)]^2 &= 3 - 2x^2; \text{ and} && (37)\\
f(x) &= -\sqrt{3 - 2x^2}, && (38)
\end{aligned}
$$

where we have chosen the negative square root on the right side in order to satisfy the initial condition.