AP Calculus 2006 AB (Form B) FRQ Solutions

Louis A. Talman, Ph.D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

July 10, 2017

1 Problem 1

1.1 Part a

We must first find the intersection nearest the y-axis of the curve $y = f(x)$ with the negative x-axis. In order to do so, we solve the equation

$$0 = \frac{a^3}{4} - \frac{a^2}{3} - \frac{a}{2} + 3\cos a. \tag{1}$$

Solving numerically, we obtain $a \sim -1.37312$, and we find, also working numerically, that the area of the region R

$$\int_a^0 f(x)\,dx \sim 2.90309. \tag{2}$$

1.2 Part b

Using the method of washers, we find that the required volume is

$$\pi \int_a^0 \left([(f(x) - (-2)]^2 - [0 - (-2)]^2 \right) dx \sim 59.36140, \tag{3}$$

where we have again integrated numerically.

1.3 Part c

$$f'(x) = -\frac{1}{2} - \frac{2}{3}x + \frac{3}{4}x^2 - 3\sin x, \text{ so} \tag{4}$$

$$f'(0) = -\frac{1}{2}. \tag{5}$$

Thus, an equation for the line tangent to the curve $y = f(x)$ at the point $(0, 3)$ is $y = 3 - x/2$. Solving numerically, we find that this line also meets the the curve at $(b, f(b))$, where $f(b) = 3 - b/2$, or $b \sim 3.38987$. The required integral is therefore

$$\int_0^b \left[3 - \frac{1}{2}x - f(x)\right] dx \sim 6.98200 \tag{6}$$

Note: Evaluation is not required. Numerical integration gives the result shown.

2 Problem 2

2.1 Part a

The graph of f' is decreasing on the interval $(1.7, 1.9)$, so f is concave downward on that interval.

2.2 Part b

The equation $f'(x) = 0$ has a root at $c = \sqrt{\pi}$, or $c \sim 1.77245$. Moreover, for values of x sufficiently close to $x = c$, $f'(x) > 0$ when $x < c$ and $f'(x) < 0$ when $c < x$. By the First Derivative Test, f has a local maximum at $x = c$. There are two other points in the interval $[0, 3]$ where $f'(x) = 0$, but neither of these points can be a local maximum, again by the First Derivative Test.

2.3 Part c

The line tangent to the graph of f at $x = 2$ has equation $y = y_0 + m(x - 2)$, where $y_0 = f(2) \sim -0.45902$ and $m = f'(2) = e^{-1/2}\sin 4$ and (by the Fundamental Theorem of Calculus)

$$y_0 = 5 + \int_0^2 e^{-x/4} \sin x^2 \, dx. \tag{7}$$

Carrying out the integration numerically, we find that $y_0 \sim 5.62343$. The required equation is therefore, approximately,

$$y = 5.62343 - 0.45902(x - 2). \tag{8}$$

3 Problem 3

3.1 Part a

If $y = ax^2$, then $y' = 2ax$, so $x = 0$ gives both $y = 0$ and $y' = 0$, so that condition (i) is satisfied. However, if $x = 4$, then $1 = y' = 8a$, by condition (ii), so that $a = 1/8$. But then, using the other part of condition (ii), we find that $1 = y = (4)^2/8 = 2$, which is not possible. The curve $y = ax^2$ therefore can't satisfy condition (ii) for any choice of a.

3.2 Part b

Let $g(x) = cx^3 - \frac{1}{16}x^2$. Then condition (i) requires that

$$1 = g(4) = 64c - 1, \tag{9}$$

so that $c = 1/32$. Taking this value for c, we have

$$g(x) = \frac{1}{32}x^3 - \frac{1}{16}x^2; \tag{10}$$
$$g'(x) = \frac{3}{32}x^2 - \frac{1}{8}x; \tag{11}$$
$$g'(4) = \frac{3}{32} \cdot 4^2 - \frac{1}{8} \cdot 4 = \frac{3}{2} - \frac{1}{2} = 1. \tag{12}$$

Thus, $g(x) = \dfrac{1}{32}x^3 - \dfrac{1}{16}x$ satisfies condtion (ii).

3.3 Part c

If

$$g(x) = \frac{1}{32}x^3 - \frac{1}{16}x = \frac{1}{16}\left(\frac{1}{2}x^3 - x^2\right), \tag{13}$$

then

$$g'(x) = \frac{1}{16}\left(\frac{3}{2}x^2 - 2x\right) = \frac{x}{32}(3x-4). \tag{14}$$

On the interval $(0, 4/3)$, $x > 0$ while $(3x-4) < 0$. Consequently, $f'(x) < 0$ on $(0, 3/4)$, and it follows that f cannot be increasing on $(0, 4)$.

3.4 Part d

Let $h(x) = x^n/k$, where k is a nonzero constant and n is a positive integer. If h meets condition (ii), then $1 = h(4) = 4^n/k$, while $1 = h'(4) = n4^{n-1}/k$, too. Thus,

$$1 = \frac{k}{k} = \frac{4^n}{n4^{n-1}}, \tag{15}$$

so that $n = 4$. But if $n = 4$ and $1 = 4^4/k$, then $k = 4^4 = 256$. Thus, condition (ii), in conjunction with $h(x) = x^n/k$, forces $h(x) = x^4/256$. From $h(x) = x^4/256$, condition (i), $h(0) = 0 = h'(0)$ is immediate. Also, $h'(x) = x^3/64 > 0$ for all $x > 0$, and this means that h is increasing between $x = 0$ and $x = 4$. So condition (iii) is met.

4 Problem 4

4.1 Part a

The point $(22, f(22))$ lies at the midpoint of the segment whose endpoints are $(20, 15)$ and $(24, 3)$, and whose slope is $(15-3)/(20-24) = -3$. The segment is part of the tangent line to the curve $y = f(x)$ at the point $(22, f(22))$. Thus, $f'(22) = -3$ calories per minute per minute.

4.2 Part b

The function f is increasing only on the intervals $[0, 4]$ and $[12, 16]$. On the latter interval, its rate of increase is

$$\frac{15-9}{16-12} = \frac{3}{2} \text{ calories per minute per minute.} \tag{16}$$

When $0 \le t \le 4$, we have

$$f'(t) = -\frac{3}{4}t^2 + 3t, \text{ and} \tag{17}$$

$$f''(t) = -\frac{3}{2}t + 3. \tag{18}$$

Then $f''(t) = 0$ when $t = 2$; $f''(t) > 0$ when $0 < t < 2$, and $f''(t) < 0$ when $2 < t < 4$. Thus, f' is an increasing function on the interval $[0, 2]$ and a decreasing function on the interval $[2, 4]$. By the First Derivative Test, f' has a relative maximum at $x = 2$. The value of this maximum is $f'(2) = 3$, and this relative maximum must be an absolute maximum for the interval $[0, 4]$. This is larger than $f'(t)$ when $12 < t < 16$, so the maximal value of $f'(t)$, *i.e.* the maximal rate of increase for the rate at which calories are burned is 3 calories per minute per minute at time $t = 2$.

4.3 Part c

The total number of calories burned over the time interval $6 \le t \le 18$ is $\int_6^{18} f(t)\,dt$. We compute the areas of the relevant rectangles and the relevant trapezoid, and we find that

$$\int_6^{18} f(t)\,dt = 6 \cdot 9 + 4 \cdot \frac{15+9}{2} + 2 \cdot 15 = 132 \text{ calories}. \tag{19}$$

4.4 Part d

It is required that

$$\frac{1}{18-6}\int_6^{18} [f(t) + c]\ dt = 15. \tag{20}$$

But

$$\frac{1}{18-6}\int_6^{18} [f(t) + c]\ dt = \frac{1}{12}\int_6^{18} f(t)\,dt + \frac{1}{12}\int_6^{18} c\,dt \tag{21}$$

$$= \frac{1}{12} \cdot 132 + \frac{1}{12}ct\Big|_6^{18} \tag{22}$$

$$= 11 + c, \tag{23}$$

and this means that $15 = 11 + c$, so that $c = 4$.

5 Problem 5

5.1 Part a

See Figure 1.

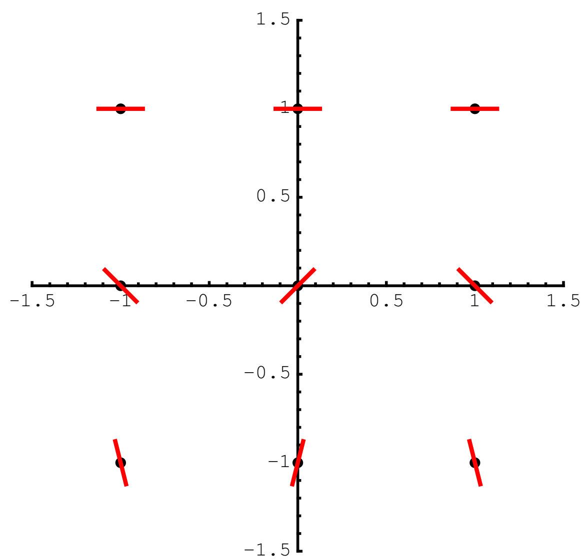

Figure 1: Problem 5, Part a

5.2 Part b

If $y \equiv c$ is a solution, then $y' \equiv 0$. But $y' = (y-1)^2 \cos \pi x$. It follows that $y \equiv 1$ is the required constant solution.

5.3 Part c

If $y = f(x)$, with $f(1) = 0$, is a solution of $y' = (y-1)^2 \cos \pi x$, then

$$f'(x) = [f(x) - 1]^2 \cos \pi x; \tag{24}$$

$$\frac{f'(x)}{[f(x)-1]^2} = \cos \pi x; \tag{25}$$

$$\int_1^x \frac{f'(\xi)}{[f(\xi)-1]^2}\, d\xi = \int_1^x \cos \pi \xi \, d\xi; \tag{26}$$

$$\left.\frac{1}{1-f(\xi)}\right|_1^x = \left.\frac{1}{\pi} \sin \pi \xi\right|_1^x; \tag{27}$$

$$\frac{1}{1-f(x)} - \frac{1}{1-f(1)} = \frac{1}{\pi} \sin \pi x; \tag{28}$$

$$\frac{1}{1-f(x)} = 1 + \frac{1}{\pi} \sin \pi x; \tag{29}$$

$$f(x) = \frac{\sin \pi x}{\pi + \sin \pi x}. \tag{30}$$

6 Problem 6

6.1 Part a

The integral $\int_{30}^{60} |v(t)|\, dt$ gives the total distance that the car travels during the interval $30 \le t \le 60$. By the Trapezoid Rule, we have

$$\int_{30}^{60} |v(t)|\, dt \sim \frac{14+10}{2} \cdot 5 + \frac{10+0}{2} \cdot 15 + \frac{0+10}{2} \cdot 10 = 185 \text{ feet/second.} \tag{31}$$

6.2 Part b

The integral $\int_0^{30} a(t)\, dt$ gives the change in the car's velocity during the time interval $0 \le t \le 30$.

$$\int_0^{30} a(t)\, dt = \int_0^{30} v'(t)\, dt = v(30) - v(0) = (-14) - (-20) = 6 \text{ feet/seccond.} \tag{32}$$

6.3 Part c

$v(0) = -20$, while $v(60) = 10$, so $v(0) < -5 < v(60)$. We are given that v is continous on $[0, 60]$, so the Intermediate Value Theorem guarantees the existence of a time $t_0 \in (0, 60)$ such that $v(t_0) = -5$.

Note: Continuity of the derivative is not needed here. Derivatives have the Intermediate Value Property—though this is a fact not ordinarily known to students in a first or second calculus course. Thus, I expect that a student who wants to make use of the IVP for derivatives must explicitly state the fact.

6.4 Part d

$v(0) = -20$ and $v(25) = -20$. The functions v and $v' = a$ are given continuous on $[0, 60]$, which contains $[0, 25]$. In particular, v is therefore continuous on $[0, 25]$ and differentiable on $(0, 25)$. By the Mean Value Theorem, there is a value $t_1 \in (0, 25)$ (and, *a fortiori,* in $(0, 60)$) such that

$$a(t_1) = v'(t_1) = \frac{v(25) - v(0)}{25 - 0} = 0. \tag{33}$$