AP Calculus 2007 AB (Form B) FRQ Solutions

Louis A. Talman, Ph.D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

July 2, 2017

1 Problem 1

1.1 Part a

The curve $y = e^{2x-x^2}$ intersects the line $y = 2$ where $2x - x^2 = \ln 2$, or, where $x = 1 \pm \sqrt{1 - \ln 2}$. Thus, the area of the region R is

$$\int_{1-\sqrt{1-\ln 2}}^{1+\sqrt{1-\ln 2}} \left(e^{2x-x^2} - 2\right)\,dx \sim 0.51414, \tag{1}$$

where we have carried out the integration numerically.

1.2 Part b

The curve $y = 3^{2x-x^2}$ intersects the line $y = 1$ where $2x - x^2 = \ln 1 = 0$, or at $x = 0$ and at $x = 2$. Thus, the sum of the areas of the regions R and S is the integral of $e^{2x-x^2} - 1$ from zero to two. From this, we subtract the integral of Part a, above, to obtain the area of the region S. Once again, we have no choice but to carry the integration out numerically.

$$\int_0^2 \left(e^{2x-x^2} - 1\right)\,dx - \int_{1-\sqrt{1-\ln 2}}^{1+\sqrt{1-\ln 2}} \left(e^{2x-x^2} - 2\right)\,dx \sim 1.54601. \tag{2}$$

1.3 Part c

Using the method of washers, the volume of the solid generated by revolving the region R about the line $y = 1$ is

$$\pi \int_{1-\sqrt{1-\ln 2}}^{1+\sqrt{1+\ln 2}} \left[\left(e^{2x-x^2} - 1 \right)^2 - 1 \right] dx \sim 4.14661. \tag{3}$$

Note: The integration was not required, but we've done it numerically to satisfy the reader's curiosity.

2 Problem 2

2.1 Part a

Acceleration is the derivative, taken with respect to time, of velocity. From $v(t) = \sin t^2$, we therefore see that acceleration is $v'(t) = 2t \cos t^2$. Thus, acceleration when $t = 3$ is $v'(3) = 6 \cos 9$. No units are given in the problem, so our answer contains none.

2.2 Part b

Total distance traveled is the integral of the magnitude of velocity over the time interval in question. We must carry out the integration numerically, here, and we obtain

$$\int_0^3 \left| \sin t^2 \right| dt \sim 1.70241 \tag{4}$$

2.3 Part c

Change in position is the integral of velocity over the time interval in question. To find the final position, we adjust by adding the initial position. Once again, integrating numerically, we obtain

$$5 + \int_0^3 \sin t^2 \, dt \sim 5.77356. \tag{5}$$

2.4 Part d

At the instant when the particle is farthest to the right, it must be at an end-point of the time interval or we must have $v(t) = 0$—and, in fact, velocity must change its sign from positive to negative. There are three points to consider in the given interval: $t = \sqrt{\pi}$, $t = \sqrt{3\pi}$, and $t = \sqrt{5\pi}$. (We reject $t = 0$ on the ground that the particle moves rightward when t is just larger than zero.) We could argue on the basis of the diagram, but it's easier, and less time-consuming, to just do the numerical integrations and compare the results. if x_k is position when $t = \sqrt{(2k-1)\pi}$, for $k = 1, 2, 3$, then

$$x_k = 5 + \int_0^{\sqrt{(2k-1)\pi}} \sin t^2 \, dt, \tag{6}$$

so that $x_1 \sim 5.89483$, $x_2 \sim 5.78826$, and $x_3 \sim 5.75244$. Thus, the particle is farthest right when $t = \sqrt{\pi}$.

3 Problem 3

3.1 Part a

We are given $W(v) = 55.6 - 22.1v^{0.16}$. So $W'(v) = -3.536v^{-0.84}$. Thus, $W'(20) = -0.28553$. This means that when the temperature is $32^\circ\ F$ and windspeed is 20 mph, the windchill is decreasing at the rate of $-0.28553^\circ F$ per mile per hour.

3.2 Part b

The average rate of change of W over the interval $[5, 60]$ is

$$\overline{W} \sim \frac{W(60) - W(5)}{60 - 5} \tag{7}$$

$$\sim \frac{13.05030281 - 27.00912318}{55} = -0.25380. \tag{8}$$

In order to obtain the value of v for which $W'(v) = \overline{W}$ we solve the equation $W'(v) = \overline{W}$ numerically, and we get

$$v \sim 23.011026 \tag{9}$$

Note: The precision of the numbers given in the statement of this problem makes this level of precision (or even the required level of three digits to the right of the decimal point) doubtful.

3.3 Part c

We put $v(t) = 20 + 5t$. Then

$$\frac{d}{dt}W[v(t)] = \frac{17.68}{(20+5t)^{0.84}}, \text{ so that} \tag{10}$$

$$\left.\frac{d}{dt}W[v(t)]\right|_{t=3} = -0.89220. \tag{11}$$

Thus, when $t = 3$, the rate of change of wind chill with respect to time is -0.89220 degrees Fahrenheit per hour.

4 Problem 4

4.1 Part a

Local maxima are to be found at points where the derivative undergoes a sign change from positive to negative. For this function, that happens twice: when $x = -3$ and when $x = 4$.

4.2 Part b

Any point where the derivative changes from increasing to decreasing, or *vice versa*, is an inflection point. For this function, we find such points at $x = -4$, $x = -1$, and $x = 2$.

4.3 Part c

If the derivative is both positive and increasing, then the function has positive slope and is concave upward. This function displays such behavior on the intervals $(-5, -5)$ and $(1, 2)$. (When we should include any endpoints depends on which choice, of several in common use, we have made for the definitions of upward and downward concavity.)

4.4 Part d

The absolute minimum value of $f(x)$ over the interval $[-5, 5]$ is $f(1) = 3$. We know that $x = 1$ gives a local minimum because the derivative changes sign from negative to positive at $x = 1$. The only other possibilities for minima are at the endpoints of the interval, because we have already (Part b, above) identified the other critical points in the interval as the locations of relative maxima. We have

$$f(-5) = 3 + \int_1^{-5} f'(x)\,dx = 3 - \frac{\pi}{2} + 2\pi = 3 + \frac{3}{2}\pi > 3, \tag{12}$$

the integral being the negative of the sum of the signed areas of two triangles. On the other hand, we have

$$f(5) = 3 + \int_1^{5} f'(x)\,dx = 3 + \frac{1}{\cancel{2}} \cdot 3 \cdot \cancel{2} - \frac{1}{2} \cdot 1 \cdot 1 = \frac{11}{2} > 3, \tag{13}$$

this integral being the sum of the areas of two signed triangles.

It follows that the absolute minimum value of $f(x)$ over the interval $[-5, 5]$ is to be found at $x = 1$. We have $f(1) = 3$.

5 Problem 5

5.1 Part a

See Figure 1.

5.2 Part b

Because $y' = \frac{1}{2}x + y - 1$, we have $y'' = \frac{1}{2} + y' = \frac{1}{2}x + y - \frac{1}{2}$. Solutions to the original differential equation must be concave upward in the region where $y'' > 0$, or, equivalently, where $\frac{1}{2}x + y - \frac{1}{2} > 0$. This is the region that lies above the line $y = \frac{1}{2}(1 - x)$, which is the line that passes through the point with coordinates $(0, 1/2)$ and through the point with coordinates $(1, 0)$.

5.3 Part c

If $y = f(x)$ is the solution to this differential equation for which $f(0) = 1$, then

$$f'(0) = \frac{1}{2} \cdot 0 + 1 - 1 = 0, \tag{14}$$

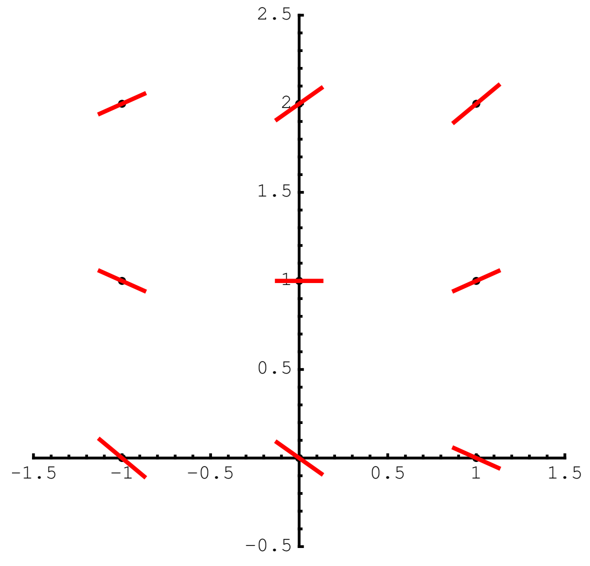

Figure 1: Slope Field

so f has a critical point at $x = 0$. From Part b, above, we have

$$f''(0) = \frac{1}{2} \cdot 0 + 1 - \frac{1}{2} = \frac{1}{2} > 0. \tag{15}$$

It follows from the Second Derivative Test that f has a relative minimum at $x = 0$.

5.4 Part d

If $y = mx + b$ is to be a solution of this differential equation, then $y' = m$ and it follows, from substituting m for y' and $mx + b$ for y, that the differential equation

$$y' = \frac{1}{2}x + y - 1 \text{ becomes} \tag{16}$$

$$m = \frac{1}{2}x + mx + b - 1, \text{ or} \tag{17}$$

$$0 = \left(m + \frac{1}{2}\right)x + (b - m - 1), \tag{18}$$

and equation (18) must hold for all values of x. But this is possible only if $m = -\frac{1}{2}$ and $b = m + 1 = \frac{1}{2}$.

Note: Part d of this problem can also be solved by finding the general solution for the differential equation. The calculation is as follows:

We rearrange the differential equation and multiply both sides by e^{-x}. Thus,

$$y' = \frac{1}{2}x + y - 1; \tag{19}$$

$$y' - y = \frac{1}{2}x - 1, \text{ or} \tag{20}$$

$$y'e^{-x} - ye^{-x} = \left(\frac{1}{2}x - 1\right)e^{-x}, \text{ which we rewrite} \tag{21}$$

$$\frac{d}{dx}\left(ye^{-x}\right) = \left(\frac{1}{2}x - 1\right)e^{-x}. \tag{22}$$

We now integrate (using integration by parts on the right) both sides to obtain

$$ye^{-x} = \frac{1}{2}(1 - x)e^{-x} + C, \text{ or} \tag{23}$$

$$y = \frac{1}{2}(1 - x) + Ce^{x}, \tag{24}$$

which is the general solution to the differential equation. Now it is easily seen that we must have $m = -\frac{1}{2}$ and $b = \frac{1}{2}$ if the solution is to have the form $y = mx + b$.

6 Problem 6

6.1 Part a

The function f is differentiable and so is continuous. (The problem statement doesn't say where, so we are free to assume *everywhere*.) By the Mean Value Theorem, there is a number ξ, $2 < \xi < 5$, such that

$$f'(\xi) = \frac{f(5) - f(2)}{5 - 2} = \frac{2 - 5}{5 - 2} = -1. \tag{25}$$

6.2 Part b

We have $g'(x) = f'[f(x)] \cdot f'(x)$. Thus

$$g'(2) = f'[f(2)] \cdot f'(2) = f'(5) \cdot f'(2), \text{ while} \tag{26}$$

$$g'(5) = f'[f(5)] \cdot f'(5) = f'(2) \cdot f'(5). \tag{27}$$

Thus, $g'(s) = g'(5)$. As the composition of continuous, differentiable functions, g is both continous and differentiable. Thus, by Rolle's Theorem, there must be a number $\eta \in (2, 5)$ such that $g'(\eta) = 0$.

6.3 Part c

From our calculation of $g'(x)$ in Part b, above, we see that the second derivative of g is given by

$$g''(x) = \frac{d}{dx}\big(f'[f(x)] \cdot f'(x)\big) = f''[f(x)] \cdot [f'(x)]^2 + f'[f(x)] \cdot f''(x). \tag{28}$$

The hypothesis that f'' vanishes identically therefore guarantees that g'' also vanishes identically. Because inflection points are to be found where the second derivative changes sign, g can have no such points.

6.4 Part d

If $h(x) = f(x) - x$, then $h(2) = f(2) - 2 = 5 - 2 = 3 > 0$, while $h(5) = f(5) - 5 = 2 - 5 = -3 < 0$. But f is given twice differentiable, so f must be continuous. Consequently, h which is the difference of two continuous functions, is also continuous. By the Intermediate Value Property of continuous functions, there must be a point $\lambda \in (2, 5)$ such that $h(\lambda) = 0$.