AP Calculus 2008 AB (Form B) FRQ Solutions

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1 Problem 1

1.1 Part a

The intersection points for the two curves have x-coordinates that are the solutions of the equation $\sqrt{x} = x/3$, which are $x = 0$ and $x = 9$. Thus, the region R lies over the interval $[0, 9]$, where $\sqrt{x} \geq x/3$. The required area is therefore

$$\int_0^9 \left(\sqrt{x} - \frac{x}{3}\right)\,dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{6}x^2\right]\Bigg|_0^9 \tag{1}$$

$$= \frac{2}{3}\cdot 27 - \frac{1}{6}\cdot 81 = \frac{9}{2}. \tag{2}$$

1.2 Part b

Let V denote the desired volume.

By the method of washers:

$$
\begin{aligned}
V &= \pi \int_0^3 \left[(3y+1)^2 - \left(y^2+1\right)^2 \right] \, dy && (3)\\
&= -\pi \int_0^3 \left(y^4 - 7y^2 - 6y\right) \, dy && (4)\\
&= -\pi \left[\frac{1}{5}y^5 - \frac{7}{3}y^3 - 3y^2 \right] \Bigg|_0^3 && (5)\\
&= -\pi \left[\frac{243}{5} - 63 - 27 \right] = \frac{207}{5}\pi. && (6)
\end{aligned}
$$

By the method of shells:

$$
\begin{aligned}
V &= 2\pi \int_0^9 \left(\sqrt{x} - \frac{x}{3}\right)(x+1) \, dx && (7)\\
&= 2\pi \int_0^9 \left(x^{1/2} - \frac{1}{3}x + x^{3/2} - \frac{1}{3}x^2 \right) \, dx && (8)\\
&= 2\pi \left[\frac{2}{3}x^{3/2} - \frac{1}{6}x^2 + \frac{2}{5}x^{5/2} - \frac{1}{9}x^3 \right] \Bigg|_0^9 && (9)\\
&= 2\pi \left(18 - \frac{27}{2} + \frac{486}{5} - 81 \right) && (10)\\
&= \frac{207}{5}\pi. && (11)
\end{aligned}
$$

1.3 Part c

Rewriting the equation of the curves in terms of y, we find that $y = \sqrt{x}$ becomes $x = y^2$, while $y = x/3$ becomes $x = 3y$. The areas of a cross section perpendicular to the y-axis is therefore $3y - y^2)^2$. then the required volume is

$$
\begin{aligned}
\int_0^3 \left(3y - y^2\right)^2 \, dy &= \int_0^3 \left(9y^2 - 6y^3 + y^4\right) \, dy && (12)\\
&= \left[3y^3 - \frac{3}{2}y^4 + \frac{1}{5}y^5 \right] \Bigg|_0^3 && (13)\\
&= 3 \cdot 27 - \frac{3}{2} \cdot 81 + \frac{1}{5} \cdot 243 = \frac{81}{10}. && (14)
\end{aligned}
$$

2 Problem 2

2.1 Part a

We integrate speed, the magnitude of velocity, to obtain distance traveled. (The problem gives speed, but the given speed is never zero, and this guarantees that travel is unidirectional. We take the direction of travel to be the positive direction.) The required distance is

$$x(2) = \int_0^2 120\left(1 - e^{-10t^2}\right)\, dt \sim 206.37005 \text{ kilometers.} \tag{15}$$

Note that the integral must be computed numerically, which we have done.

2.2 Part b

We have $g(x) = 0.05\, x\left(1 - e^{-x/2}\right)$. We must find the value of

$$\frac{d}{dt} g[x(t)] = g'[x(t)]x'(t) \tag{16}$$

when $t = 2$. But

$$g'(x) = \frac{d}{dx}\left[0.05x\left(1 - e^{-x/2}\right)\right] \tag{17}$$

$$= 0.05\left(1 - e^{-x/2}\right) + 0.025xe^{-x/2}, \text{ while it is given that} \tag{18}$$

$$x'(t) = r(t) = 120(1 - e^{-10t^2}). \tag{19}$$

We have $x(2)$ from equation (15), while $x'(2) = r(2) = 120(1 - e^{-40})$. Thus,

$$\left.\frac{d}{dt} g[x(t)]\right|_{t=2} = \left[6\left(1 - e^{-x(2)/2}\right) + 3\, x(2) e^{-x(2)/2}\right](1 - e^{-40}) \tag{20}$$

$$\sim 6.00000 \tag{21}$$

The rate of change, taken with respect to time, of the number of liters of gasoline used by the car when $t = 2$ hours is approximately 6.00000 liters/hour.

2.3 Part c

We begin by solving the equation $120(1 - e^{-10t^2}) = 80$ for t. It is easy to see that the following are equivalent:

$$120(1 - e^{-10t^2}) = 80; \tag{22}$$
$$1 - e^{-10t^2} = \frac{2}{3}; \tag{23}$$
$$e^{10t^2} = 3; \tag{24}$$
$$10t^2 = \ln 3; \tag{25}$$
$$t^2 = \frac{\ln 3}{10}. \tag{26}$$

The only positive solution is

$$t = \sqrt{\frac{\ln 3}{10}} \sim 0.33145. \tag{27}$$

Thus, speed reaches 80 km/hr when $t \sim 0.33145$ hours. At that time, position is given by

$$x\left[\sqrt{\ln(3)/10}\right] \sim 120 \int_0^{\ln(3)/\sqrt{10}} \left(1 - e^{-10\tau^2}\right) d\tau \sim 10.79410. \tag{28}$$

We carry out the required integration numerically again, and we find that the amount of fuel, in liters, consumed up to that time is

$$g\left(x\left[\sqrt{\ln(3)/10}\right]\right) \sim 0.53726. \tag{29}$$

3 Problem 3

3.1 Part a

The trapezoidal sum that approximates the area of the river cross section is

$$\frac{1}{2}\left[(0+7)\cdot 8 + (7+8)\cdot 6 + (8+2)\cdot 8 + (2+0)\cdot 2\right] = 115 \tag{30}$$

The area of the river cross section is about 115 square feet.

3.2 Part b

We integrate area times volumetric flow, with respect to time. Then we divide the result by the length of the time interval to obtain the average volumetric flow.

$$\frac{115}{120}\int_0^{120}\left(16+2\sin\sqrt{t+10}\right)\,dt \sim 1807.16972\ \mathrm{ft}^3/\mathrm{min}. \tag{31}$$

The integration is elementary, by way of the substitution $t+10=w^2$, followed by an integration by parts. Numerical integration is faster, and that is the technique we have used.

3.3 Part c

Once again, we integrate depth from 0 to 24:

$$\int_0^{24} 8\sin\frac{\pi x}{24}\,dx = -\frac{192}{\pi}\cos\frac{\pi x}{24}\Big|_0^{24} = -\frac{192}{\pi}\cos\pi + \frac{192}{\pi}\cos 0 = \frac{384}{\pi}. \tag{32}$$

Based on this model, the area of the cross section is $384/\pi \sim 122.23100$ square feet.

3.4 Part d

We must again integrate area times volumetric flow, this time using the area found in Part c, above, and with t varying from 40 to 60. We integrate numerically again, and we obtain

$$\frac{384}{\pi}\cdot\frac{1}{20}\int_{40}^{60}\left(16+2\sin\sqrt{t+10}\right)\,dt \sim 2181.91265. \tag{33}$$

The average volumetric flow during the interval $40 \le t \le 60$ is about 2181.91265 cubic feet per minute. This value exceeds the given safety limit of 2100 cubic feet per minute and indicates that water must be diverted.

4 Problem 4

4.1 Part a

By the Fundamental Theorem of Calculus and the Chain Rule, from

$$f(x) = \int_0^{3x}\sqrt{4+t^2}\,dt, \tag{34}$$

we see that

$$f'(x) = \frac{d}{dx}\left[\int_0^{3x} \sqrt{4+t^2}\,dt\right] \tag{35}$$

$$= \sqrt{4+9x^2}\cdot\frac{d}{dx}(3x) = 3\sqrt{4+9x^2}. \tag{36}$$

Then from $g(x) = f(\sin x)$ it follows from what we have seen above and, again, the Chain Rule that

$$g'(x) = f'(\sin x)\cos x \tag{37}$$

$$= 3\cos x\sqrt{4+9\sin^2 x}. \tag{38}$$

4.2 Part b

The slope of the tangent line to $y = g(x)$ at $x = \pi$ is

$$g'(\pi) = 3\cos\pi\sqrt{4+9\sin^2\pi} = -6 \tag{39}$$

An equation for the tangent line to the curve $y = g(x)$ at the point corresponding to $x = \pi$ is therefore

$$y = g(\pi) + g'(\pi)(x-\pi) = 0 - 6(x-\pi), \text{ or} \tag{40}$$

$$y = 6(\pi - x). \tag{41}$$

4.3 Part c

When $x > 0$, the value $f(x)$ is the integral of a positive quantity over an interval $[0, 3x]$, and from this it follows that f is an increasing function throughout the interval $[0, \infty)$. But the sine function carries the interval $[0, \pi]$ onto the interval $[0, 1]$, and the maximum value of $f(x)$ on this interval is $f(1) = f[\sin(\pi/2)]$. Therefore, the maximal value of $g(x)$ on $[0, \pi]$ is

$$g(\pi/2) = f[\sin(\pi/2)] = \int_0^3 \sqrt{4+t^2}\,dt. \tag{42}$$

Note: Evaluation of the integral is not required. However, a trig substitution followed by an integration by parts gives

$$\int_0^3 \sqrt{4+t^2}\,dt = \left[\frac{1}{2}t\sqrt{4+t^2} + 2\ln\left|\frac{1}{2}\left(t+\sqrt{4+t^2}\right)\right|\right]\Bigg|_0^3 \tag{43}$$

$$= \frac{3}{2}\sqrt{13} + 2\ln\frac{3+\sqrt{13}}{2}. \tag{44}$$

5 Problem 5

5.1 Part a

Inflections points for g occur at local extrema of g'. There are two such on the given graph: One at $x = 1$, and one at $x = 4$.

5.2 Part b

From the picture, we see that $g'(x) < 0$ throughout the intervals $[-3, -1)$ and $(2, 6)$. Consequently, g is decreasing on the intervals $[-3, -1]$ and $[2, 6]$, and g is increasing on the intervals $[-1, 2]$ and $[6, 7]$. [**Note:** A function that is continuous on an interval $[a, b]$ and increasing on (a, b) must necessarily be increasing on $[a, b]$.] It follows that the absolute maximum value of $g(x)$ must lie either at $x = 2$, which is the boundary between an interval where g increases to an interval where g decreases, or at one of the endpoints of the interval $[-3, 7]$.

We are given $g(2) = 5$. Making repeated use of the fact that the area of a triangle is one-half its altitude times its base, and that the Fundamental Theorem of Calculus guarantees us that $g(x) = g(2) + \int_2^x g'(\xi)\, d\xi$, we find that

$$g(-3) = 5 - \left(\frac{3}{2} - 4\right) = \frac{15}{2} \text{ and} \tag{45}$$

$$g(7) = 5 - 4 + \frac{1}{2} = \frac{3}{2}. \tag{46}$$

We now see that $g(-3) = 15/2$ gives the absolute maximum value for $g(x)$ when $-3 \leq x \leq 7$.

5.3 Part c

The average rate of change of $g(x)$ on the interval $[-3, 7]$ is

$$\frac{g(7) - g(-3)}{7 - (-3)} = \frac{3/2 - 15/2}{7 + 3} = -\frac{3}{5}, \tag{47}$$

where we have used the values of $g(-3)$ and $g(7)$ that we computed in Part b, above.

5.4 Part d

The average rate of change of $g'(x)$ on the interval $[-3, 7]$ is

$$\frac{g'(7) - g'(-3)}{7 - (-3)} = \frac{1 - (-4)}{7 - (-3)} = \frac{1}{2}, \tag{48}$$

were we have read $g'(-3) = -4$ and $g'(7) = 1$ from the given graph.

The Mean Value Theorem does not apply to the function g' on the interval $[-3, 7]$, because the hypotheses of that theorem require that $g''(x)$ exist for all values of x that lie in $(-3, 7)$. However, $g''(1)$ and $g''(4)$ do not exist for this function. (This can be seen by considering the left and right derivatives of g' at the points in question.)

6 Problem 6

6.1 Part a

Let

$$x^2 + 2x + y^4 + 4y = 5. \tag{49}$$

We suppose that this equation defines y implicitly as a function of x. Then, differentiating both sides of equation (49) with respect to x, we find that

$$2x + 2 + 4y^3 \frac{dy}{dx} + 4\frac{dy}{dx} = 0, \text{ whence} \tag{50}$$

$$\frac{dy}{dx} = -\frac{x + 1}{2(y^3 + 1)}. \tag{51}$$

6.2 Part b

At the point $(-2, 1)$, we substitute for x and y in equation (51) to find that

$$\frac{dy}{dx} = -\frac{-2 + 1}{2(1^3 + 1)} = \frac{1}{4}. \tag{52}$$

An equation for the line tangent, at $(-2, 1)$, to the curve with equation (49) is therefore

$$y = 1 + \frac{1}{4}(x + 2). \tag{53}$$

6.3 Part c

We begin anew with (49), which we now treat as giving x as an implicitly defined function of y. Differentiating both sides of the equation with respect to y then gives

$$2x\frac{dx}{dy} + 2\frac{dx}{dy} + 4y^3 + 4 = 0, \text{ or} \tag{54}$$

$$\frac{dx}{dy} = -\frac{2(y^3+1)}{x+1} \tag{55}$$

This derivative can vanish, so that the tangent line is vertical, only at points where $y = -1$. But we must be sure that $x + 1 \neq 0$ before we may draw conclusions about dx/dy. The corresponding values of x are then given by

$$x^2 + 2x + 1 - 4 = 5, \text{ or} \tag{56}$$

$$(x+1)^2 = 9, \tag{57}$$

whence $x = -4$ or $x = 2$. We conclude that the curve has vertical tangent lines at the points $(-4, -1)$ and $(2, -1)$.

6.4 Part d

At any point where this curve meets the x-axis, we must have $y = 0$, whence $x^2 + 2x + 0^4 + 4 \cdot 0 = 5$. Thus $x^2 + 2x + 1 = 6$, or $(x + 1)^2 = 6$, from which we see that $x = \sqrt{6} - 1$ or $x = -\sqrt{6} - 1$. But from equation (51), at $\left(\sqrt{6} - 1, 0\right)$ we have $y' = -\sqrt{6}/2$, while at $\left(-\sqrt{6} - 1, 0\right)$ we have $y' = \sqrt{6}/2$. From these calculations, it follows that this curve can't have a horizontal tangent at any of its x-intercepts.

Note: In my opinion, this problem (particularly Part c) perpetuates some of the difficulties in the way we treat implicit differentiation in our elementary calculus courses. The implicit differentiation technique depends for its justification on the Implicit Function Theorem, which very few students encounter before advanced calculus—or even intermediate analysis. We can't expect students of AP Calculus to know this theorem, but that's not a good reason why *we* should ignore its requirements. Among the hypotheses of that theorem is the requirement that, when we want to be sure that an equation $F(x, y) = 0$ defines a unique differentiable function $y(x)$ implicitly in some neighborhood of a point (x_0, y_0), we must know that the partial derivative F_y satisfies $F_y(x_0, y_0) \neq 0$. This means (among other things) that *it is not correct to conclude that a curve $F(x, y) = 0$ has a vertical tangent at a point (x_0, y_0) on the curve by using implicit differentiation to obtain y' and then noting that y', so obtained, is a fraction whose denominator vanishes at (x_0, y_0).* To see what kind of trouble this strategy can get us into, consider the curve $y^2 - x^2 = 0$ at the point $(0, 0)$.

For a deeper discussion of this and associated difficulties, see my short note titled “On Implicit Differentiation.” As of the date of this set of solutions, there is a link to this note at `http://sites.msudenver.edu/talmanl/ap-calculus-resources/`