AP Calculus 2008 AB FRQ Solutions

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1 Problem 1

1.1 Part a

The area of the region R is

$$\int_0^2 \left[\sin \pi x - \left(x^3 - 4x\right)\right]\,dx = \left[-\frac{1}{\pi}\cos \pi x - \left(\frac{1}{4}x^4 - 2x^2\right)\right]\Bigg|_0^2 \tag{1}$$

$$= \left[-\frac{1}{\pi}\cos 2\pi - (4-8)\right] - \left[-\frac{1}{\pi}\cos 0 - 0\right] = 4. \tag{2}$$

1.2 Part b

We must first find solutions, in the interval $[0, 2]$, of the equation $x^3 - 4x = -2$ to find the limits of integration. We do this numerically, and find that the solutions we need are $x_1 \sim 1.67513$ and $x_2 \sim 0.53919$.

The areas of that part of the region R which lies below the horizontal line $y = -2$ is given by the integral $\int_{x_2}^{x_1} \left[-2 - \left(x^3 - 4x\right)\right]\,dx$.

1.3 Part c

The area, $A(t)$, of a cross section of the solid perpendicular to the x-axis at $x = t$ is given by

$$A(t) = \left[\sin \pi t - \left(t^3 - 4t\right)\right]^2. \tag{3}$$

Thus, the volume of the solid is

$$\int_0^2 \left[\sin \pi t - \left(t^3 - 4t\right)\right]^2 \, dt \sim 9.97834, \tag{4}$$

where we have evaluated the integral numerically because, although the integral is elementary, the calculation is lengthy and requires integration by parts.

Note: The exact value of the integral is $\frac{1129}{105} - \frac{24}{\pi^3}$.

1.4 Part d

Under the conditions given, the pool is a region in three-dimensional space whose base is R and whose cross section perpendicular to the x-axis at $x = t$ has area $A(t)$ given by

$$A(t) = \left[\sin \pi t - (t^3 - 4t)\right] (3 - t). \tag{5}$$

The required volume is thus

$$\int_0^2 \left[\sin \pi t - (t^3 - 4t)\right] (3 - t) \, dt \sim 8.36995, \tag{6}$$

Where we have again integrated numerically to avoid a tedious calculation requiring integration by parts.

Note: The exact value of the integral is $\frac{116}{15} + \frac{2}{\pi}$.

2 Problem 2

2.1 Part a

At 5:30 pm, the rate at which the number of people standing in line was changing was approximately

$$\frac{L(7) - L(4)}{7 - 4} = \frac{150 - 126}{7 - 4} = 8 \text{ people per hour.} \tag{7}$$

2.2 Part b

The average number of people standing in line during the first four hours that tickets were on sale was

$$\frac{1}{4-0}\int_0^4 L(t)\,dt \sim \frac{1}{4}\left[\frac{120+156}{2}(1-0)+\frac{156+176}{2}(3-1)+\frac{176+126}{2}(4-3)\right] \tag{8}$$

$$\sim \frac{621}{4} = 155.25. \tag{9}$$

2.3 Part c

The function L is given twice differentiable on $[0,9]$. It is therefore continuous on $[a,b]$ and differentiable on (a,b) when $[a,b]$ is any subinterval of $[0,9]$, and we may apply the Mean Value Theorem to L on any such interval. There must be points, then, $\xi_1 \in (1,3)$ and $\xi_2 \in (3,4)$, such that

$$L'(\xi_1) = \frac{L(3)-L(1)}{3-1} = \frac{176-156}{3-1} > 0, \text{ and} \tag{10}$$

$$L'(\xi_2) = \frac{L(4)-L(3)}{4-3} = \frac{126-176}{1} < 0. \tag{11}$$

But L'' exists throughout $[0,9]$, so L' is a continuous function on $[\xi_1,\xi_2]$. By the Intermediate Value Theorem for continuous functions, there must be a number $\eta_1 \in (\xi_1,\xi_2)$ such that $L'(\eta_1)=0$. By similar reasoning there must $\xi_3 \in (4,7)$ for which $L'(\xi_3) > 0$, and so $\eta_2 \in (\xi_2,\xi_3)$ where $L'(\eta_2)=0$. Further, there must be $\xi_4 \in (7,8)$ for which $L'(\xi_4) < 0$, and this guarantees $\eta_3 \in (\xi_3,\xi_4)$ for which $L'(\eta_3)=0$.

We conclude that $L'(t)$ takes on the value 0 at least three times in the interval $(0,9)$.

Note: We can make this argument even if L is given merely differentiable instead of twice differentiable, although we can no longer depend on the continuity of L'. However, derivatives necessarily have the Intermediate Value Property in spite of the fact that they may fail to be continuous[1]. To see that this is so, suppose that f is differentiable on an interval (a,b) and let $a < \alpha < \beta < b$. Suppose that $f'(\alpha) < \lambda < f'(\beta)$. We let F be the function defined on $[\alpha,\beta]$ by

$$F(x) = f(x) - \lambda x, \text{ whence} \tag{12}$$

$$F'(x) = f'(x) - \lambda. \tag{13}$$

[1]This fact is not ordinarily a part of elementary calculus, and it is to be presumed that examinees who want to use it must state it explicitly.

Now F is continuous on $[\alpha, \beta]$, and so must have an absolute minimum on that interval—which must occur at either an endpoint or a critical point. But $F(\alpha)$ can't be a minimum because $F'_+(\alpha) = f'(\alpha) - \lambda < 0$. Similarly, we deduce that $F(\beta)$ can't be a minimum because $F'_-(\beta) > 0$. It follows that there must be a critical number $x_0 \in (\alpha, \beta)$—that is, a number x_0 for which $F'(x_0) = 0$. But $F'(x_0) = 0$ is equivalent to $f'(x_0) = \lambda$.•

2.4 Part d

If $T(t)$ denotes the number of tickets that have been sold by time t, we are given that $T(0) = 0$ and $T'(t) = 550te^{-t/2}$. By the Fundamental Theorem of Calculus,

$$T(t) = T(0) + \int_0^t T'(\tau)\, d\tau = 550 \int_0^t \tau e^{-\tau/2}\, d\tau \tag{14}$$

The integral is elementary, but requires integration by parts, so we integrate numerically to learn that $T(3) \sim 972.78412$. Thus, 973 tickets have been sold by 3:00 pm.

3 Problem 3

3.1 Part a

Let $V(t)$ be the volume of spilled oil at time t, $r(t)$ and $h(t)$ the radius and the height, respectively, of the spill. Then

$$V(t) = \pi\,[r(t)]^2\, h(t), \text{ whence} \tag{15}$$

$$V'(t) = 2\pi r(t)h(t)r'(t) + \pi\,[r(t)]^2\, h'(t). \tag{16}$$

We are given that $V'(t) = 2000$ cc/min for all t, and that at $r(t_0) = 100$ cm, $h(t_0) = 0.5$ cm, and $r'(t_0) = 2.5$ cm/min. Thus,

$$2000 = 2\pi \cdot 100 \cdot 0.5 \cdot 2.5 + \pi \cdot 100^2 \cdot h'(t_0), \text{ or} \tag{17}$$

$$h'(t_0) = \frac{8-\pi}{40\pi} \sim 0.03866 \text{ cm.min.} \tag{18}$$

3.2 Part b

Taking $t = 0$ to be the moment when the recovery device goes into action, we have

$$V'(t) = 2000 - 400\sqrt{t}. \tag{19}$$

Thus $V(t)$ has a critical point at $t = 25$, when $V'(t) = 0$. Because $V'(t) > 0$ for $t < 25$, but $V'(t) < 0$ when $t > 25$, it follows from the First Derivative Test that $V(t)$ is maximal when $t = 25$.

3.3 Part c

If there were $60,000$ cc of oil in the slick at the moment $t = 0$, when the recovery device began to operate, then, by the Fundamental Theorem of Calculus, we must have

$$V(t) = V(0) + \int_0^t V'(\tau)\, d\tau. \tag{20}$$

From what we saw in Part b, above, we must therefore have

$$V(t) = 60000 + \int_0^t \left[2000 - 400\sqrt{\tau}\right]\, d\tau. \tag{21}$$

Note: Evaluation of the integral is not required. For the curious,

$$60000 + \int_0^t \left[2000 - 400\sqrt{\tau}\right]\, d\tau = 60000 + \frac{400}{3}\left(15 - 2\sqrt{t}\right) t. \tag{22}$$

4 Problem 4

4.1 Part a

Applying the Fundamental Theorem of Calculus to what we are given we find that

$$x(t) = -2 + \int_0^t v(\tau)\, d\tau. \tag{23}$$

This means that $x(3) = -10$, $x(5) = -7$, and $x(6) = -9$. From the figure and the other information given, we have $x'(t) = v(t) < 0$ for $0 < t < 3$ and for $5 < t < 6$, while $x'(t) > 0$ for $3 < t < 5$. Thus, x is decreasing when $0 \le t \le 3$ and when $5 \le t \le 6$, while x is increasing when $3 \le t \le 5$. thus, the particle is farthest to the left when $t = 3$, and its position at that instant is $x = -10$.

Note: If function continuous on $[a, b]$ is increasing (respectively, decreasing) on (a, b), it is necessarily increasing (respectively, decreasing) on $[a, b]$. We should thus include the endpoints. In the past, the readers haven't taken this subtlety into account.

4.2 Part b

Because $x(0) = -2$ and $x(3) = -10$, (see Part a, above), the particle moves through $x = -8$ at least once (leftward bound) when $0 < t < 3$. Because $x(3) = -10$ and $x(5) = -7$ (see Part a again) it moves through $x = -8$ again (rightward bound) at some time in the interval $(3, 5)$. Because $x(5) = -7$ and $x(6) = -9$ (see Part a again) it moves through -8 still again (now leftward bound) at some time in the interval $(5, 6)$. The existence of these times is guaranteed, in each case, because the differentiable function x must be continuous on $[0, 6]$, and continuous functions have the intermediate value property[2]. That these three instances are the only instances is guaranteed by the fact the x must be monotonic on each of the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ because velocity, the derivative of x, doesn't change sign at a point interior to any of these intervals.

4.3 Part c

Let $\sigma(t)$ denote the particle's speed at time t. Then

$$\sigma(t) = |v(t)|, \text{ so that} \tag{24}$$

$$[\sigma(t)]^2 = [v(t)]^2, \text{ and} \tag{25}$$

$$2\sigma(t)\sigma'(t) = 2v(t)v'(t), \text{ or, provided } \sigma(t) \neq 0, \tag{26}$$

$$\sigma'(t) = \frac{v(t)}{\sigma(t)}v'(t) = \frac{v(t)v'(t)}{|v(t)|}. \tag{27}$$

The denominator of this last fraction is positive ($\sigma(t)$ being non-zero), so the sign of $\sigma'(t)$ is the same as the sign of the product $v(t)v'(t)$. On the interval $(2, 3)$, we see from the graph that $v(t) < 0$, but that $v(t)$ is increasing, so that $v'(t) > 0$. It follows that $v(t)v'(t) < 0$ on $(2, 3)$, and, therefore, that speed is decreasing on $(2, 3)$.

4.4 Part d

Acceleration if $v'(t)$. Thus, acceleration is negative on intervals where $v(t)$ is decreasing. From the graph and what we have been given about it, acceleration is negative on $[0, 1)$ and on $(4, 6]$, and only on those intervals.

[2] See the remarks in the Note to Problem 2, Part c.

5 Problem 5

5.1 Part a

See Figure 1.

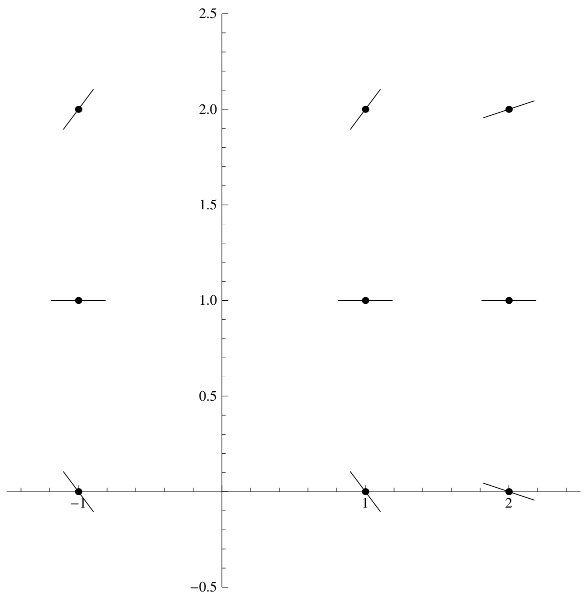

Figure 1: Problem 5, Part a

5.2 Part b

We have

$$y'(x) = \frac{y(x) - 1}{x^2}, \text{ together with} \tag{28}$$

$$y(2) = 0. \tag{29}$$

We note that $y(x) \equiv 1$ gives a solution to (28), but that this solution doesn't satisfy (29). We may therefore discard this solution and assume that $y(x) \neq 1$. Under this assumption, we may write (28) as

$$\frac{y'(x)}{y(x) - 1} = \frac{1}{x^2}, \tag{30}$$

from which we deduce that

$$\int_2^x \frac{y'(\xi)}{y(\xi)-1}\,d\xi = \int_2^x \frac{d\xi}{\xi^2}, \text{ or} \tag{31}$$

$$\ln|y(\xi)-1|\Big|_2^x = -\frac{1}{\xi}\Big|_2^x, \text{ which is equivalent to} \tag{32}$$

$$\ln|y(x)-1| - \ln|y(2)-1| = -\frac{1}{x} + \frac{1}{2}. \tag{33}$$

Applying (29) and noting that $\ln|-1| = 0$, we now see that

$$|y(x)-1| = e^{1/2}e^{-1/x}. \tag{34}$$

But, again from (29), we have $y(2) - 1 = -1 < 0$, so that $y(x) < 1$ for all x in its domain. Thus, we can rewrite (34) as

$$1 - y(x) = e^{1/2}e^{-1/x}, \text{ and we see that} \tag{35}$$

$$y(x) = 1 - \frac{\sqrt{e}}{e^{1/x}}. \tag{36}$$

5.3 Part c

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty}\left[1 - \frac{\sqrt{e}}{e^{1/x}}\right] \tag{37}$$

$$= 1 - \frac{\sqrt{e}}{\lim_{x\to\infty} e^{1/x}} = 1 - \sqrt{e}. \tag{38}$$

6 Problem 6

6.1 Part a

$$f'(e^2) = \frac{1 - \ln e^2}{(e^2)^2} = \frac{1-2}{e^4} = -e^{-4}, \tag{39}$$

so an equation of the tangent line to the curve $y = f(x)$ at the point where $x = e^2$ is

$$y = 2e^{-2} - e^{-4}(x - e^2). \tag{40}$$

6.2 Part b

$f'(x) = 0$ when $(1 - \ln x)/x^2 = 0$, so the x-coordinate of the critical point of f is $x = e$. We note that $f'(x) > 0$ when $x < e$, but that $f'(x) < 0$ when $x > e$, and it follows from the First Derivative Test that f has a local maximum at $x = e$.

6.3 Part c

from

$$f'(x) = \frac{1 - \ln x}{x^2}, \text{ we have} \tag{41}$$

$$f''(x) = \frac{(-1/x) \cdot x^2 - (2x)(1 - \ln x)}{(x^2)^2} = \frac{2\ln x - 3}{x^3}. \tag{42}$$

We note that $f''(x) < 0$ when $x < e^{3/2}$ but that $f''(x) > 0$ when $x > e^{3/2}$. The function f therefore has an inflection point at $x = e^{3/2}$.

6.4 d

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\ln x}{x} = \lim_{x \to 0^+} \left[\frac{1}{x} \cdot \ln x\right] = -\infty. \tag{43}$$

because $1/x \to \infty$ and $\ln x \to -\infty$ as $x \to 0^+$.