AP Calculus 2009 AB (Form B) FRQ Solutions

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1 Problem 1

1.1 Part a

At time t, the radius in centimeters of the tree, $R(t)$, is given by

$$R(t) = 6 + \frac{1}{16}\int_0^t \left(3 + \sin\tau^2\right)\, d\tau. \tag{1}$$

Carrying out the integration numerically with $t = 3$, we find that $R(3) \sim 6.61085$.

1.2 Part b

We have $A(t) = \pi\,[R(t)]^2$, whence, by implicit differentiation, $A'(t) = 2\pi R(t)R'(t)$. Thus,

$$A'(3) = 2\pi R(3)R'(3) \tag{2}$$

$$= 2\cdot\pi\cdot 6.61085\cdot\frac{1}{16}\cdot(2+\sin 9) \sim 8.85811. \tag{3}$$

Thus, the area is then increasing at a rate of about 8.85811 square centimeters per year.

Note: This is something of a misstatement. Very few texts give a definition for the phrase "increasing at a point".

1.3 Part c

The integral

$$\int_0^3 A'(t)\,dt = A(3) - A(0) \sim \pi\left[(6.61085)^2 - 36\right] \sim 24.20075 \tag{4}$$

represents the change, in square centimeters, in the area of the tree's cross-section at the given height over the time period $0 \le t \le 3$. The area of the cross-section is 24.2065 square centimeters larger when $t = 3$ than it was when $t = 0$.

2 Problem 2

2.1 Part a

Let $D(t)$ denote the distance, in meters, from the road to the edge of the water at time t hours after the beginning of the storm. We are given $D(0) = 35$, $D'(t) = \sqrt{t} + \cos t - 3$. By the Fundamental Theorem of Calculus,

$$\begin{aligned}
D(t) &= 35 + \int_0^t D'(\tau)\,d\tau && (5)\\
&= 35 + \int_0^t \left(\sqrt{\tau} + \cos\tau - 3\right)\,d\tau && (6)\\
&= 35 + \left[\frac{2}{3}\tau^{3/2} + \sin\tau - 3\tau\right]\Bigg|_0^t && (7)\\
&= 35 - 3t + \frac{2}{3}t^{3/2} + \sin t. && (8)
\end{aligned}$$

Substituting 5 for t, we obtain $D(5) \sim 26.49464$. Thus, at the end of the five-hour storm, the distance from road to water is about 26.49464 meters.

2.2 Part b

If $f'(4) = 1.007$, then $D''(4) = 1.007$, so after four hours of the storm, the rate at which distance from road to water is changing is increasing at 1.007 meters per hour per hour.

2.3 Part c

We are to find the absolute minimum of $f(t)$ on the interval $[0, 5]$. Such a minimum lies at either a critical point or an endpoint. The critical points for f are the zeros of

$$f'(t) = \frac{1}{2\sqrt{t}} - \sin t. \tag{9}$$

We solve numerically and find these critical points are $t \sim 0.66186$ and $t \sim 2.84038$. We find

$$\begin{aligned} f(0) &= -2, && (10) \\ f(0.66186) &\sim -1.39760, && (11) \\ f(2.84038) &\sim -2.26963, && (12) \\ f(5) &\sim -0.48027. && (13) \end{aligned}$$

The smallest of these is $f(2.84038)$, so the distance from water to road was decreasing most rapidly about 2.84038 hours after the storm began.

2.4 Part d

If sand is restored to the beach in such a way that the rate of change of the distance from water to road is $g(p)$ meters per day, where p is the number of days since pumping began, then, by the Fundamental Theorem of Calculus the number of days, P, of pumping required to restore the original disance between road and water, satisfies (approximately) the equation

$$35 = D(5) + \int_0^P g(p)\, dp. \tag{14}$$

3 Problem 3

We don't appear to have been given quite enough information to solve this problem. We must assume that the line segment and the curved portion of the curve meet at the point $(0, 2)$. In what follows, we make this assumption.

3.1 Part a

The line segment that gives the portion of the curve that lies to the left of the y-axis has slope $2/3$, so

$$\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \frac{2}{3}. \tag{15}$$

On the other hand, it is apparent that

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} < 0. \tag{16}$$

The left- and right-hand limits of the difference quotient for f at zero being different, $f'(0)$ cannot exist, and f is not differentiable at $x = 0$.

3.2 Part b

The average rate of change of f over the interval $[a, 6]$ is $\dfrac{f(6)-f(a)}{6-a}$. This can be zero only if $f(a) = f(6) = 1$ while $a \neq 6$. The horizontal line through $(5, f(1)) = (6, 1)$ intersects the curve in just two other points, so there are just two values of a for which the average rate of change of f over $[a, 6]$ is zero.

3.3 Part c

We note that f is continuous on $[3, 6]$ and differentiable on $(3, 6)$, so the hypotheses of the Mean Value Theorem are satisfied. Hence there is a number $c \in [3, 6]$ such that

$$f'(c) = \frac{f(6)-f(3)}{6-3} = \frac{1}{3}. \tag{17}$$

Thus, we may take $a = 3$.

3.4 Part d

If

$$g(x) = \int_0^x f(t)\,dt, \tag{18}$$

then $g'(x) = f(x)$ and $g''(x) = f'(x)$. Thus, g is concave upward on intervals where $f'(x) > 0$ or, more generally, on intervals where f' is increasing. We conclude that g is concave upward on $(-4, 0)$ and on $(3, 6)$. Whether we may conclude that g is concave upward on the closures of these intervals depends upon which of several definitions of concavity we choose. In the past, the readers haven't paid any attention to this subtlety.

4 Problem 4

4.1 Part a

The area of R is

$$\int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{4}x^2\right]\Big|_0^4 = \frac{2}{3}\cdot 8 - \frac{1}{4}\cdot 16 = \frac{4}{3}. \tag{19}$$

4.2 Part b

The volume described is

$$\int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)^2 dx = \int_0^4 \left(x - x^{3/2} + \frac{x^2}{4}\right) dx \tag{20}$$

$$= \left[\frac{1}{2}x^2 - \frac{2}{5}x^{5/2} + \frac{1}{12}x^3\right]\Big|_0^4 \tag{21}$$

$$= \frac{16}{2} - \frac{64}{5} + \frac{64}{12} = \frac{8}{15}. \tag{22}$$

4.3 Part c

The volume generated by revolving R about the line $y = 2$ is

$$\pi \int_0^4 \left[\left(2 - \frac{x}{2}\right)^2 - \left(2 - \sqrt{x}\right)^2\right] dx = \pi \int_0^4 \left(4\sqrt{x} - 3x + \frac{1}{4}x^2\right) dx \tag{23}$$

$$= \pi \left[\frac{8}{3}x^{3/2} - \frac{3}{2}x^2 + \frac{1}{12}x^3\right]\Big|_0^4. \tag{24}$$

Alternately, we may write the volume as

$$2\pi \int_0^2 (2y - y^2)(2 - y)\, dy = 2\pi \int_0^1 (4y - 4y^2 + y^3)\, dy. \tag{25}$$

Note: Those who were unable to comply with the instruction not to evaluate their integral should have found that the volume in question is $8\pi/3$.

5 Problem 5

5.1 Part a

If

$$g(x) = e^{f(x)}, \tag{26}$$

then

$$g'(x) = f'(x)e^{f(x)}, \tag{27}$$

so

$$g'(1) = f'(1)e^{f(1)} = -4e^2. \tag{28}$$

Hence, an equation for the line tangent to the curve $y = g(x)$ at the point corresponding to $x = 1$ is

$$y = g(1) + g'(1)(x-1), \tag{29}$$

or

$$y = e^2 - 4e^2(x-1). \tag{30}$$

5.2 Part b

By the First Derivative Test, g has a local maximum at any point where $g'(x)$ changes sign from positive to negative. But $g'(x) = f'(x)e^{f(x)}$, and $e^{f(x)}$ is always positive. Therefore the local maxima of g are to be found at points where $f'(x)$ changes sign from positive to negative. From the graph given, we see that there is just one such point in the interval $(-1.2, 3.2)$: $x = -1$. The function g therefore has a local maximum only at $x = 1.1$ in the interval $(-1.2, 3.2)$.

5.3 Part c

Because

$$g''(x) = e^{f(x)} \left(\left[f'(x) \right]^2 + f''(x) \right) \tag{31}$$

and $e^{f(x)} > 0$, the sign of $g''(x)$ is the same as the sign of $\left([f'(x)]^2 + f''(x) \right)$. Now, as is given, $(f'(-1))^2 = 0$, and we see from the graph that, f' being a decreasing function in a neighborhood of $x = -1$, it must be the case that $f''(-1) < 0$. So $g''(-1) < 0$.

5.4 Part d

The average rate of change of g' over the interval $[1, 3]$ is $[g'(3) - g'(1)]/[3 - 1]$. But, as we saw in Part a of this problem, above, $g'(1) = -4e^2$. We also have $g'(3) = f'(3)e^{f(3)} = 0$, $f'(30) = 0$ being given. The desired average rate of change is

$$\frac{0 - (-4e^2)}{2} = 2e^2. \tag{32}$$

6 Problem 6

6.1 Part a

Acceleration at time $t = 36$ is approximately

$$\frac{v(36 + 4) - v(36 - 4)}{(36 + 4) - 36 - 4)} = \frac{7 - (-4)}{8} = \frac{11}{8}. \tag{33}$$

6.2 Part b

By the Fundamental Theorem of Calculus, the difference, $x(40) - x(20)$, between the particle's position when $t = 40$ and its position when $t = 20$ satisfies

$$x(40) - x(20) = \int_{20}^{40} v(t)\, dt. \tag{34}$$

We approximate the integral with a trapezoidal sum using the three intervals given in the table:

$$\int_{20}^{40} v(t)\,dt \sim \frac{1}{2}\left([v(25)+v(20)](32-25)+[v(32)+v(25](32-25)+[v(40)+v(32)](40-32)\right) \tag{35}$$

$$\sim \frac{1}{2}\left[(-18)\cdot 5+(-12)\cdot 7+(3)\cdot 8\right] = -75 \text{ meters.} \tag{36}$$

6.3 Part c

The particle must change direction somewhere in the interval $[8, 20]$, because the velocity function $v(t)$ takes on a positive value when $t = 8$ and a negative value when $t = 20$. We are given that this velocity function is differentiable, and this implies its continuity—so the velocity function has the Intermediate Value Property. The velocity function must also change sign somewhere in the interval $[32, 40]$, because $v(32) = -4 < 0$ and $v(40) = 7 > 0$.

6.4 Part d

By the Fundamental Theorem of Calculus,

$$x(t) = x(0) + \int_0^t v(\tau)\,d\tau = 7 + \int_0^t v(\tau)\,d\tau. \tag{37}$$

But if acceleration, which is v', is positive on $(0, 8)$ and $v(0) = 3$, then $v(\tau) \geq 3$ for all $\tau \in [0, 8]$, because v must be an increasing function on that interval. Thus,

$$x(8) = 7 + \int_0^8 v(\tau)\,d\tau \geq 7 + \int_0^8 3\,d\tau = 31 > 30. \tag{38}$$

Alternate Solution: Velocity is the derivative of the distance function, and the existence of velocity on $[0, 8]$ therefore guarantees that the distance function is continuous on that interval and differentiable on $(0, 8)$. If $x(8) \leq 30$, there is, by the Mean Value Theorem, a $t_0 \in (0, 8)$ such that

$$v(t_0) = \frac{x(8)-x(0)}{8-0} \leq \frac{30-7}{8} = \frac{23}{8} < 3. \tag{39}$$

The function v is also continuous and differentiable, so there is $t_1 \in (0, t_0)$ such that

$$v'(t_1) = \frac{v(t_0)-v(0)}{t_0-0} = \frac{v(t_0)-3}{t_0} < 0. \tag{40}$$

But $v'(t_1)$ is acceleration at $t = t_1$, and $t_1 \in (0, t_0) \subseteq (0, 8)$, an interval where we know that $v'(t) > 0$. The contradiction shows that $x(8) \leq 30$ is not possible, so it must be the case that $x(8) > 30$.