AP Calculus 2009 AB FRQ Solutions

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1 Problem 1

1.1 Part a

At time $t = 7.5$, the acceleration of Caren's bicycle is

$$\frac{0.3 - 0.2}{7 - 8} = -\frac{1}{10} \text{ miles per minute per minute.} \tag{1}$$

1.2 Part b

The integral $\displaystyle\int_0^{12} |v(t)|\,dt$ gives, in miles, the total distance that Caren traveled during the period $0 \le t \le 12$. The value of this integral is $\dfrac{9}{5}$.

1.3 Part c

Her turn-around time corresponds to the point on the graph where the sign of her velocity changes from positive to negative. That's $t = 2$ minutes.

1.4 Part d

Caren lives $\displaystyle\int_5^{12} v(t)\,dt = \frac{7}{5}$ miles from school because she left home at $t = 5$, arrived at school at $t = 12$, traveled in one direction only, and the distance she traveled during that time is given by the integral. Larry's distance is the integral of his velocity over the interval $[0, 12]$, or

$$\frac{\pi}{15} \int_0^{12} \sin\left(\frac{\pi t}{12}\right)\, dt = -\frac{\pi}{15} \cdot \frac{12}{\pi} \cos\left(\frac{\pi t}{12}\right)\Big|_0^{12} = \frac{8}{5}. \tag{2}$$

Larry lives farther from school than Caren, who lives only $\frac{7}{5}$ miles away.

2 Problem 2

2.1 Part a

At time $t = 2$, the auditorium contains

$$\int_0^2 \left(1380t^2 - 675t^3\right)\, dt = \left[460t^3 - \frac{675}{4}t^4\right]\Big|_0^2 = 460 \cdot 8 - 675 \cdot 4 = 980 \text{ people.} \tag{3}$$

2.2 Part b

We are given

$$R(t) = 1380t^2 - 675t^3, \text{ so that} \tag{4}$$

$$R'(t) == 2760t - 2025t^2 = 15(184 - 135t). \tag{5}$$

Thus, R is increasing on the interval $\left[0, \frac{184}{135}\right]$ and decreasing on the interval $\left[\frac{184}{135}, 2\right]$, because $R'(t) > 0$ on the interior of the first of those intervals and $R'(t) < 0$ on the interior of the second. It follows that the maximal rate at which people enter the auditorium occurs at $t = \frac{184}{135}$ hours.

2.3 Part c

We have $R(t) = 1380t^2 - 675t^3$, and $w'(t) = (2 - t)R(t)$. By the Fundamental Theorem of Calculus, the difference $w(2) - w(1)$ is given by

$$w(2) - w(1) = \int_1^2 w'(t)\, dt = \int_1^2 (2 - t)(1380t^2 - 675t^3)\, dt \tag{6}$$

$$= \int_1^2 (2760t^2 - 2730t^3 + 675t^4)\, dt = \left(920t^3 - \frac{1365}{2}t^4 + 135t^5\right)\Big|_1^2 \tag{7}$$

$$= 760 - \frac{745}{2} = \frac{775}{2} \text{ hours.} \tag{8}$$

2.4 Part d

From Part a of this problem, above, we know that there are 980 people in the auditorium at time $t = 2$. We also know that the total wait time for these 980 people is

$$\int_1^2 w'(t)\,dt = \left(920t^3 - \frac{1365}{2}t^4 + 135t^5\right)\Big|_0^2 = 760 \text{ hours.} \tag{9}$$

Consequently, average waiting time is

$$\frac{760}{980} = \frac{38}{49} \text{ hours.} \tag{10}$$

3 Problem 3

3.1 Part a

Mighty's profit on a cable of length k meters is

$$P(k) = 120k - 6\int_0^k \sqrt{x}\,dx \tag{11}$$

$$= 120k - 6\left(\frac{2}{3}x^{2/3}\right)\Big|_0^k = 4k\left(30 - \sqrt{k}\right). \tag{12}$$

Thus, profit on a 25-meter cable is $P(25) = 2500$ dollars.

3.2 Part b

The integral $\int_{25}^{30} 6\sqrt{x}\,dx$ gives the cost, in dollars, to Mighty for building the last five meters of a 30-meter cable.

3.3 Part c

We saw in Part a of this problem, above, that Mighty's profit, in dollars, on the sale of a cable that is k meters long is

$$P(k) = 120k - 6\int_0^k \sqrt{x}\,dx = 4k(30 - \sqrt{k}). \tag{13}$$

3.4 Part d

We first observe that profit for a cable k meters long, computed above, is non-negative only when $0 \leq k \leq 900$ and is zero at the endpoints of this interval. Thus, the absolute maximum profit does not occur at any endpoint of the interval (or outside the interval). However, the profit function is continuous, so there must be an absolute maximum profit that occurs for some $k_0 \in (0, 100)$. Thus, k_0 must be a critical point for the profit function $P(k) = 4k(30 - \sqrt{k})$. Noting that $P'(k) = 120 - 6\sqrt{k}$ vanishes only when $k = 400$, we conclude that maximum profit occurs when $k = 400$. This maximum profit is therefore $P(400) = 16,000$ dollars.

We also notice that if cables longer than 900 ever become popular, Mighty should rethink its pricing structure.

4 Problem 4

4.1 Part a

The desired area is

$$\int_0^2 (2x - x^2)\, dx = \left(x^2 - \frac{1}{3}x^3\right)\bigg|_0^2 = 4 - \frac{8}{3} = \frac{4}{3}. \tag{14}$$

4.2 Part b

The desired volume is

$$\int_0^2 \sin\frac{\pi x}{2}\, dx = -\frac{2}{\pi}\cos\frac{\pi x}{2}\bigg|_0^2 = -\frac{2}{\pi}(-1 - 1) = \frac{4}{\pi}. \tag{15}$$

4.3 Part c

Solving the first equation for y in terms of x gives $x = y/2$, while doing the same with the second gives $x = \sqrt{y}$. The length of the base of the square corresponding to $y = y_0$ is therefore $\sqrt{y_0} - y_0/2$, and the desired volume is therefore $\int_0^4 \left(\sqrt{y} - \frac{1}{2}y\right)^2\, dy$.

Evaluation of the integral was not required. However,

$$\int_0^4 \left(\sqrt{y} - \frac{1}{2}y\right)^2 \, dy = \int_0^4 \left(y - y^{3/2} + \frac{1}{4}y^2\right) \, dy = \left(\frac{1}{2}y^2 - \frac{2}{5}y^{5/2} + \frac{1}{12}y^3\right)\Big|_0^4 \tag{16}$$

$$= 8 - \frac{64}{5} + \frac{16}{3} = \frac{8}{15}. \tag{17}$$

5 Problem 5

5.1 Part a

$$f'(4) \sim \frac{f(5) - f(3)}{5 - 3} = \frac{-2 - 4}{5 - 3} = -3. \tag{18}$$

5.2 Part b

$$\int_2^{13} \left[3 - 5f'(x)\right] dx = \left[3x - 5f(x)\right]\Big|_2^{13} \tag{19}$$

$$= \left[3 \cdot 13 - 5 \cdot f(13)\right] - \left[3 \cdot 2 - 5 \cdot f(2)\right] \tag{20}$$

$$= (39 - 30) - (6 - 5) = 8. \tag{21}$$

5.3 Part c

The desired left Riemann sum is

$$f(2) \cdot (3 - 1) + f(3) \cdot (5 - 3) + f(3) \cdot (8 - 5) + f(8) \cdot (13 - 8) = 1 + 8 - 6 + 15 \tag{22}$$

$$= 18. \tag{23}$$

5.4 Part d

An equation for the line tangent to the curve $y = f(x)$ at the point on the curve that corresponds to $x = 5$ is

$$y = f(5) + f'(5)(x - 5), \text{ or} \tag{24}$$

$$y = -2 + 3(x - 5). \tag{25}$$

Now $f''(x) < 0$ for all x in the interval $[5, 8]$, so the curve is concave downward throughout that interval; thus, the tangent line at $x = 5$ lies above the curve on $[5, 8]$. That is, when $5 \le x \le 8$, we have $f(x) \le -2 + 3(x - 5)$. Consequently, $f(7) \le -2 + 3(7 - 5) = 4$.

On the other hand, $f''(x) < 0$ on $[5, 8]$, and this implies that the curve $y = f(x)$, being concave downward there, lies above the secant line determined by the point $(5, f(5)) = (5, -2)$ and the point $(8, f(8)) = (8, 3)$. An equation for this secant line is

$$y = f(5) + \frac{f(8) - f(5)}{8 - 5}(x - 5), \text{ or} \tag{26}$$

$$y = -2 + \frac{5}{3}(x - 5). \tag{27}$$

Consequently, when $5 \le x \le 8$, we have $-2 + \frac{5}{3}(x - 5) \le f(x)$. Thus,

$$\frac{4}{3} = -2 + \frac{5}{3}(7 - 5) \le f(7). \tag{28}$$

6 Problem 6

6.1 Part a

The graph of f has an inflection point at those values of x for which f' has a local extreme, because at such points f'' must undergo a change of sign. From what we are given, we see that f' has local extremes at $x = -2$ and at $x = 0$. Therefore, f has an inflection point at $x = 2$ and another at $x = 0$.

6.2 Part b

For any x in $[-4, 4]$, the Fundamental Theorem of Calculus gives

$$f(x) = f(0) + \int_0^x f'(\xi)\, d\xi. \tag{29}$$

From what we are given about the geometry of the f' curve, we see that

$$f(-4) = f(0) + \int_0^{-4} f'(\xi)\, d\xi = 5 - \left(8 - \frac{1}{2} \cdot \pi \cdot 2^2\right) = 2\pi - 3. \tag{30}$$

On the other hand,

$$f(4) = f(0) + \int_0^4 f'(\xi)\, d\xi = 5 + \int_0^4 (5e^{-\xi/3} - 3)\, d\xi \tag{31}$$

$$= 5 + \left(-15e^{-\xi/3} - 3\xi\right)\Big|_0^4 = 8 - 15e^{-4/3}. \tag{32}$$

6.3 Part c

Arguing again from the geometry of the f' curve, we know that $f'(x) \geq 0$ for all x in the interval $[-4, 3\ln(5/3)]$, and is actually positive at all but two of the points in that interval. Hence, f is an increasing function on $[-4, 3\ln(5/3)]$. We also know that $f'(x) \leq 0$ when $x \in [3\ln(5/3), 4]$ and has just one zero in that interval. Consequently, f is decreasing on $[-4, 3\ln(5/3)]$. From this it follows that f has an absolute maximum for $[-4, 4]$ at $x = 3\ln(5/3)$.