AP Calculus 2010 AB (Form B) FRQ Solutions

Louis A. Talman, Ph.D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

June 5, 2017

1 Problem 1

1.1 Part a

The area of the region R is

$$\int_0^2 [6 - 4\ln(3-x)]\ dx \sim 6.81665, \tag{1}$$

where we have carried out the integration numerically.

Note: The integral is elementary, but is most easily accomplished using integration by parts, which isn't on the AB syllabus. We have

$$\int_0^2 [6 - 4\ln(3-x)]\ dx = 6x\Big|_0^2 - 4(x-3)\ln(3-x)\Big|_0^2 + 4\int_0^2 dx = 20 - 12\ln 3 \tag{2}$$

$$= 4(5 - 3\ln 3).. \tag{3}$$

1.2 Part b

The volume obtained by revolving R about the line $y = 8$ is given by

$$\pi \int_0^2 \left([8 - 4\ln(3-x)]^2 - 4\right)\ dx \sim 168.17954, \tag{4}$$

where we have once again integrated numerically.

Note: The integral is elementary, and symbolic integration is possible. However it is lengthy, and numerical integration saves a fair amount of time. Symbolic integration gives

$$\pi \int_0^2 \left([8 - 4\ln(3-x)]^2 - 4 \right) \, dx = 24\pi \left[13 - 2(6 - \ln 3) \ln 3 \right]. \tag{5}$$

1.3 Part c

The volume of this solid is

$$\int_0^2 [6 - 4\ln(3-x)]^2 \, dx \sim 26.26660. \tag{6}$$

Note: Once more, we have carried out the integration numerically, though a symbolic integration is possible. Once again, the numerical integration saves time. For the curious,

$$\int_0^2 [6 - 4\ln(3-x)]^2 \, dx = 16 \left[20 - 3(6 - \ln 3) \ln 3 \right]. \tag{7}$$

2 Problem 2

2.1 Part a

We need the values of x in the interval $[0.12, 1]$ for which $g'(x) = 0$. A plot suggests that the zeros of $g'(x)$ are $x \sim 0.17$ and $x \sim 0.36$. Numerical approximation then gives us $x \sim 0.16340$ and $x \sim 0.35943$. So the points where this curve has horizontal tangent lines are to be found at $x \sim 0.16340$ and at $x \sim 0.35943$.

2.2 Part b

The graph of g is concave downward on intervals where $g''(x) < 0$. Numerical solution of the equation $g''(x) = 0$ gives $x \sim 0.12946$, $x \sim 0.22273$ and $x = 1$.

The function g'' is continuous on $[0.12, 1]$, so it can change sign only at points where it vanishes. Numerical solution gives the relevant zeros of $g''(x)$ as $x \sim 0.12946$, $x \sim 0.22273$, and $x = 1$. We have $g''(0.12) \sim 38.61 > 0$, $g''(0.19) \sim -18.0192 < 0$, and $g''(0.8) \sim 0.25935 > 0$. Thus, g is concave downward on $(0.12946, 0.22273)$. Whether to include the endpoints in this interval depends on which of several definitions of *concave downward* one uses—and the readers know this.

2.3 Part c

The Fundamental Theorem of Calculus tells us that

$$g(x) = g(1) + \int_1^x g'(\xi)\,d\xi = 2 + \int_1^x g'(\xi)\,d\xi, \tag{8}$$

it being given that $g(1) = 2$. A numeric integration gives $g(0.3) \sim 1.54601$.

Direct evaluattion gives $g'(0.3) \sim -0.47216$.

We can therefore write the equation of the tangent line at $(2, g(2))$, with three decimal places of accuracy as $y = 1.546 - 0.472(x - 0.3)$.

2.4 Part d

From what we have seen in Part b, above, the curve $y = g(x)$ must be concave upward on $(0.22273, 1.000)$, so the tangent line at $x = 0.3$ must lie below the curve throughout that interval—and, *a fortiori,* on the interval $(0.3, 1)$.

3 Problem 3

3.1 Part a

The three-equal-interval midpoint Riemann sum that approximates the amount of water pumped into the pool during the time interval $0 \le t \le 12$ is

$$f(2)\cdot 4 + f(6)\cdot 4 + f(10)\cdot 4 = 660. \tag{9}$$

Consequently, about 660 cubic feet of water was pumped into the pool in the given twelve-hour interval.

3.2 Part b

If water leaked at the rate $R(t) = 25e^{-0.05t}$ cubic feet per hour, then, during the interval $0 \le t \le 12$, the amount of water lost was, in cubic feet,

$$25\int_0^{12} e^{-t/20}\,dt = -500e^{-t/20}\Big|_0^{12} = 500(1 - e^{-3/5}). \tag{10}$$

3.3 Part c

The total amount of water, in cubic feet, in the pool at the end of the twelve-hour period is thus about

$$1000 + 660 - 500(1 - e^{-3/5}) \sim 1434.40582. \tag{11}$$

To the nearest cubic foot, at time $t = 12$ the pool contains about 1434 cubic feet of water.

3.4 Part d

The rate at which the volume of water in the pool is increasing is $P(t) - R(t)$ cubic feet per hour. When $t = 8$ this rate is $60 - 25e^{-2/5} \sim 43.24200$ cubic feet per hour.

The relationship between the height, h, of water in the tank and the volume, V, of water in the tank is $V = \pi r^2 h = 144\pi h$. Thus,

$$h = \frac{V}{144\pi}, \text{ and} \tag{12}$$

$$\frac{dh}{dt} = \frac{1}{144\pi}\frac{dV}{dt}. \tag{13}$$

Setting $t = 8$ in this latter equation gives (from what we have seen above)

$$\left.\frac{dh}{dt}\right|_{t=8} = \frac{1}{144\pi}(60 - 25e^{-2/5}) \sim 0.09558 \text{ feet per hour.} \tag{14}$$

4 Problem 4

4.1 Part a

The squirrel's direction changes when its velocity changes sign. That happens only at those values of t where the graph of velocity crosses the t-axis. There are two such places: $t = 9$ and $t = 15$.

4.2 Part b

The squirrel's distance from Building A at time T is the integral of its velocity from 0 to T. This integral is the algebraic sum of the signed areas associated with the appropriate regions between the velocity curve and the t-axis, taking the regions above the axis to have positive area and assigning negative areas to the regions below the t-axis. The area under the velocity curve on the interval $[0, 9]$ is clearly larger than the area below the axis on the interval $[9, 15]$, and this latter area is clearly smaller than the area above the axis on the interval $[15, 18]$. Thus, the squirrel is farthest from Building A when $t = 9$, and this distance is the area enclosed by the trapezoid whose vertices are $(0, 0)$, $(2, 20)$, $(7, 20)$, and $(9, 0)$. The area of this trapezoid is 140, so at time $t = 9$, the squirrel is 140 feet from Building A and is closer at all other times.

4.3 Part c

Summing the magnitudes of the signed areas, we find that the squirrel has traveled a total distance of $140 + 50 + 25 = 215$ feet.

4.4 Part d

On the interval $7 < t < 10$, the squirrel's velocity is given by a line, of slope

$$\frac{20 - (-10)}{7 - 10} = -10, \tag{15}$$

which passes through the point $(7, 20)$, and therefore has equation

$$v(t) = 20 - 10(t - 7) = 90 - 10t \text{ feet per second.} \tag{16}$$

For the squirrel's acceleration during this interval, we have

$$a(t) = v'(t) = -10 \text{ feet per second per second.} \tag{17}$$

For distance, $x(t)$, from Building A, we have, when $7 \leq t \leq 10$,

$$\begin{aligned} x(t) &= x(7) + \int_7^t v(\tau)\, d\tau && (18) \\ &= 120 + \int_7^t (90 - 10\tau)\, d\tau && (19) \\ &= -5t^2 + 90t - 265 \text{ feet.} && (20) \end{aligned}$$

5 Problem 5

5.1 Part a

See Figure 1.

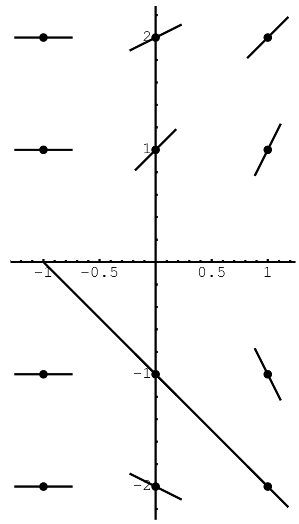

Figure 1: Problem 5, Part a

5.2 Part b

The points in the plane for which $\frac{dy}{dx} = -1$ are those points in the domain of the fraction $(x+1)/y$ for which the value of the fraction is -1. The set of all such points is the set of all (x, y) for which $x + 1 = -y$ and $y \neq 0$. This is the set of all but one of the points on the straight line of slope -1 that passes through the point $(0, -1)$. The exceptional point is the one with coordinates $(-1, 0)$.

5.3 Part c

Solving the initial value problem $y' = (x+1)/y$; $y(0) = -2$, we suppose that the solution is $y = f(x)$. Then

$$f'(x) = \frac{x+1}{f(x)}, \text{ or} \tag{21}$$

$$f(x) \cdot f'(x) = x + 1. \tag{22}$$

Now we integrate both sides of (22) from 0 to x, so that

$$\int_0^x f(t)\, f'(t)\, dt = \int_0^x (t+1)\, dt, \tag{23}$$

$$\frac{1}{2}[f(t)]^2\bigg|_0^x = \left(\frac{t^2}{2} + t\right)\bigg|_0^x, \tag{24}$$

$$[f(x)]^2 - 4 = x^2 + 2x. \tag{25}$$

Solving for $f(x)$ and remembering that $f(0) = -2 < 0$, we obtain

$$f(x) = -\sqrt{x^2 + 2x + 4}. \tag{26}$$

6 Problem 6

6.1 Part a

The particle R moves to the right when $r'(t) > 0$. But

$$r'(t) = 3t^2 - 12t + 9 \tag{27}$$

$$= 3(t-1)(t-3), \tag{28}$$

and this is positive only when the factors $(t-1)$ and $(t-3)$ have the same sign—or when either $t < 1$ or $t > 3$. Thus, for $0 \le t \le 6$, the particle R moves to the right when $0 \le t < 1$ and when $3 < t \le 6$.

6.2 Part b

The two particles move in th opposite directions when the derivatives $p'(t)$ and $r'(t)$ have opposite signs, or when the product $p'(t)r'(t) < 0$. But

$$p'(t)r'(t) = -\frac{\pi}{2} \cdot 3(t-1)(t-3)\sin\left(\frac{\pi t}{4}\right), \tag{29}$$

and, $0 \le t \le 6$ being given, this product is negative on $[0, 1)$ and on $(3, 4)$. We conclude that the two particles are moving in opposite directions both when $0 \le t < 1$ and when $3 < t < 4$.

6.3 Part c

We have

$$p''(t) = -\frac{\pi^2}{8}\cos\left(\frac{\pi t}{4}\right), \tag{30}$$

Thus, acceleration when $t = 3$ is

$$p''(3) = \left(-\frac{\pi^2}{8}\right)\cdot\left(-\frac{\sqrt{2}}{2}\right) > 0. \tag{31}$$

The acceleration is positive, so velocity is increasing when $t = 3$.