AP Calculus 2010 AB FRQ Solutions

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June 3, 2017

1 Problem 1

1.1 Part a

At 6 AM, $\int_0^6 7t\, e^{\cos t}\, dt$ cubic feet of snow have accumulated. Integrating numerically, we find that 142.17469 cubic feet have accumulated by 6 AM.

1.2 Part b

At 8 AM, snow is falling at the rate of $56e^{\cos 8}$ cubic feet per hour, but Janet is removing it at the rate of 108 cubic feet per hour. So at 8 AM, the rate of change of the volume of snow on the driveway is $56e^{\cos 8} - 108 \sim -59.58297$ cubic feet per hour.

1.3 Part c

We have

$$h(t) = \begin{cases} 0 & \text{when } 0 \le t \le 6. \\ 125(t-6) & \text{when } 6 \le t < 7. \\ 125 + 108(t-7) & \text{when } 7 \le t \le 9. \end{cases} \tag{1}$$

1.4 Part d

$7\int_0^9 te^{\cos t}\, dt - 341$ is the total amount, in cubic feet, of snow on the driveway at $t = 9$. Numeric integration gives this as 26.33461 cubic feet of snow at 9 AM.

2 Problem 2

2.1 Part a

At $t = 6$, the approximate rate at which entries were being made is given by the fraction

$$\frac{E(7) - E(5)}{7 - 5} = \frac{21 - 13}{2} = 4 \text{ hundred entries per hour.} \tag{2}$$

2.2 Part b

The trapezoidal approximation is

$$\frac{1}{8}\int_0^8 E(t)\,dt \sim \frac{1}{16}[(4+0)\cdot 2 + (13+4)\cdot 3 + (21+13)\cdot 2 + (23+21)\cdot 1 = \frac{171}{16}. \tag{3}$$

This means that the average rate of deposits at any time during the 8-hour period was about $171/16$ hundreds of entries per hour.

2.3 Part c

The number $U(t)$, of entries not processed at a given time t, $8 \leq t \leq 12$, is given by

$$U(t) = 2300 - 100\int_8^t P(\tau)\,d\tau, \tag{4}$$

where $P(t) = t^3 - 30t^2 + 298t - 976$. Thus,

$$\begin{aligned} U(t) &= 2300 - 100\left(\frac{1}{4}\tau^4 - 10\tau^3 + 149\tau^2 - 976\tau\right)\Big|_8^t & (5)\\ &= -2tt^2 + 1000t^3 - 14900t^2 + 97600t - 234500., & (6) \end{aligned}$$

and

$$U(12) = 700. \tag{7}$$

According to this model, 700 entries remain unprocessed at midnight.

2.4 Part d

We want the maximal rate at which entries were being processed in the time interval $[8, 12]$. Such a maximum must lie at a critical point or at an end-point. The critical points are the solutions of the equation $P'(t) = 3t^2 - 60t + 298 = 0$ or

$$t = \frac{30 \pm \sqrt{6}}{3}. \tag{8}$$

Thus, the critical points are at t_1 and t_2, where $t_1 \sim 9.18350$ and $t_2 \sim 10.81650$. We find that

$$\begin{align} P(8) &= 0; \tag{9}\\ P(t_1) &\sim 5.08866; \tag{10}\\ P(t_2) &\sim 2.91134; \text{ and} \tag{11}\\ P(12) &= 8. \tag{12} \end{align}$$

The largest of these numbers must be the maximum, so the entries were being processed most quickly at midnight.

3 Problem 3

3.1 Part a

From $t = 0$ to $t = 2$, the rate at which people arrive appears to increase linearly from 1000 per hour to 1200 per hour. The total number that arrive during this period is $\int_0^2 r(\tau)\,d\tau$, which is about $\frac{1}{\cancel{2}}(1000 + 1200) \cdot \cancel{2} = 2200$ people. From $t = 2$ to $t = 3$, the rate at which people arrive appears to decrease linearly from 1200 per hour to 800 per hour. The total number that arrive during this period is $\int_2^3 r(\tau)\,d\tau$, which is about $\frac{1}{2}(1200 + 800)(3 - 2) = 1000$. Consequently, the total number of arrivals for $0 \le t \le 3$ is about

$$2200 + 1000 = 3200. \tag{13}$$

3.2 Part b

During the period from $t = 2$ to $t = 3$, the arrival rate exceeds 800 people per hour, while people move onto the ride at the rate of 800 people per hour. Consequently, the number of people waiting in line is increasing during this period.

3.3 Part c

Prior to the time $t = 3$, $r(t) > 800$, and the arrival rate exceeds the processing rate of 800 people per hour—meaning that the length of the line increases during this period. When $t > 3$, however, $r(t) < 800$, and the length of the line decreases. Consequently, the line is longest when $t = 3$.

3.4 Part d

In order to find the earliest time t at which there is no longer a line for the ride, we must find the smallest positive value of t that satisfies the equation

$$700 + \int_0^t f(\tau)\, d\tau = 800t. \tag{14}$$

4 Problem 4

4.1 Part a

The area of R is

$$\int_0^9 \left[6 - 2\sqrt{x}\right]\, dx = \left[6x - \frac{4}{3}x^{3/2}\right]\Bigg|_0^9 = 18. \tag{15}$$

4.2 Part b

The volume of the solid generated by revolving R about the line $y = 7$ is given by

$$\pi \int_0^9 \left[\left(7 - \sqrt{x}\right)^2 - 1\right]\, dx = \pi \int_0^9 \left(48 - 14\sqrt{x} + x\right)\, dx. \tag{16}$$

Note: Evaluation of the integral is not required. However,

$$\pi \int_0^9 \left(48 - 14\sqrt{x} + x\right)\, dx = \pi \left(48x - 14 \cdot \frac{2}{3}x^{3/2} + \frac{x^2}{2}\right)\Bigg|_0^9 \tag{17}$$

$$= \pi \left(48 \cdot 9 - 14 \cdot \frac{2}{3} \cdot 27 + \frac{81}{2}\right) = \frac{441}{2}\pi. \tag{18}$$

4.3 Part c

The volume of this solid is $\frac{3}{16}\int_0^6 y^4\,dy$.

Note: Evaluation of this integral isn't required either. However

$$\frac{3}{16}\int_0^6 y^4\,dy = \frac{3}{16}\cdot\frac{y^5}{5}\bigg|_0^6 = \frac{1458}{5} \tag{19}$$

5 Problem 5

5.1 Part a

By the Fundamental Theorem of Calculus,

$$g(3) = g(0) + \int_0^3 g'(x)\,dx = 5 + \pi + \frac{3}{2} = \frac{13}{2} + \pi. \tag{20}$$

Similarly,

$$g(-2) = 5 + \int_0^{-2} g'(x)\,dx = 5 - \pi. \tag{21}$$

5.2 Part b

The inflection points of a function such as this one must lie at the relative extrema of its derivative. These lie at $x = 2$ (where the derivative changes from a decreasing function to an increasing function and concavity changes from downward to upward as we move from left to right) and at $x = 3$ (where the derivative changes from an increasing function to a decreasing function and concavity changes from upward to downward as we move from left to right).

5.3 Part c

The critical points of h are the zeros of $h'(x) = g'(x) - x$. Thus, h has a critical point at each x-value where the line $y = x$ touches the graph given for $g'(x)$. There are two such points: $x = \sqrt{2}$ and $x = 3$. We note that $x < g'(x)$ when $x < \sqrt{2}$, and that $x \geq g'(x)$ when $x > \sqrt{2}$. This means that $h'(x) > 0$ for $x < \sqrt{2}$, while $h'(x) \leq 0$ when $x > \sqrt{2}$. It follows from the First Derivative Test that h has a local maximum at $x = \sqrt{2}$.

On any small interval centered at $x = 3$, we see that $g'(x) \le x$, or $h'(x) \le 0$. Hence, $x = 3$ gives neither a local maximum nor a local minimum for h—also by the First Derivative Test.

6 Problem 6

6.1 Part a

The solution, $y = f(x)$ of $y' = xy^3$ that passes through the point $(1, 2)$ has there a tangent line whose slope is $y'\big|_{(1,2)} = 1 \cdot (2)^3 = 9$. An equation for that line is therefore

$$y = 2 + 8(x - 1). \tag{22}$$

6.2 Part b

Using the tangent line at $(1, 2)$ to approximate $y = f(x)$ through that point yields

$$f(1.1) \sim 2 + 8 \cdot (1.1 - 1) = 2.8 \tag{23}$$

as an approximate value for $f(1.1)$ on the curve when $x = 1.1$.

Because f is a solution of $y' = yx^3$,

$$\begin{aligned} y'' &= \frac{d}{dx}(y') && (24)\\ &= \frac{d}{dx}\left(yx^3\right) && (25)\\ &= y' \cdot x^3 + y \cdot 3x^2 && (26)\\ &= (yx^3) \cdot x^3 + 3yx^2 = yx^2(x^4 + 3). && (27) \end{aligned}$$

But $y = f(x) > 0$ when $1 < x < 1.1$, and both x^2 and $x^4 + 3$ are positive when $1 < x < 1.1$, so that $y'' = y \cdot x^2 \cdot (x^4 + 3) > 0$ when $1 < x < 1.1$. This means that the curve $y = f(x)$ is concave upward, so that it lies above its tangent line in that region. Hence, the approximation we have just given is an underestimate for $f(1.1)$.

6.3 Part c

We have

$$\begin{aligned} f'(x) &= x[f(x)]^3, \text{ so that } f(x) \equiv 0 \text{ or} \qquad (28)\\ \frac{f'(x)}{[f(x)]^3} &= x. \qquad (29) \end{aligned}$$

The zero solution doesn't satisfy $f(1) = 2$ so we will integrate both sides of (29) from 1 to x, making use of the fact that $f(1) = 2 > 0$ and using the continuity (which follows from its differentiability) of f to know that we can keep x close enough to 1 that we can be sure that $f(t) > 0$ for all t between 1 and x. This gives

$$\begin{aligned} \int_1^x \frac{f'(t)}{[f(t)]^3}\,dt &= \int_1^x t\,dt; \qquad (30)\\ -\frac{1}{\cancel{2}[f(t)]^2}\bigg|_1^x &= \frac{t^2}{\cancel{2}}\bigg|_1^x; \qquad (31)\\ \frac{1}{4} - \frac{1}{[f(x)]^2} &= x^2 - 1; \qquad (32)\\ [f(x)]^2 &= \frac{4}{5-4x^2}. \qquad (33) \end{aligned}$$

The values of $f(x)$ must be positive, at least when x is near 1, so we take the positive square root, and we arrive at the solution

$$f(x) = \frac{2}{\sqrt{5-4x^2}}. \qquad (34)$$

Note: We could have solved Part b of this problem by making a forward reference to this solution and actually computing $f(1.1) = 5$. And, yes: the approximation of Part b is pretty miserable.