# AP Calculus 2011 AB, Form B, FRQ Solutions

Louis A. Talman, Ph.D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

May 31, 2017

## 1 Problem 1

### 1.1 Part a

According to the model, the height of the water in the can at the end of the $60$-day period is

$$\int_0^{60} [2\sin(0.03t) + 1.5]\, dt = \left[-\frac{2}{0.03}\cos(0.03t) + 1.5t\right]\Bigg|_0^{60} \tag{1}$$

$$= \left(-\frac{200}{3}\cos(9/5) + 90\right) + \frac{200}{3} = \left[\frac{470}{3} - \frac{200}{3}\cos\left(\frac{9}{5}\right)\right] \text{ mm.} \tag{2}$$

### 1.2 Part b

The average rate of change in the height of water in the can over the $60$-day period is

$$\frac{1}{60}\int_0^{60} S'(t)\, dt = \frac{1}{60}\left[\frac{470}{3} - \frac{200}{3}\cos\left(\frac{9}{5}\right)\right] = \left[\frac{47}{18} - \frac{10}{9}\cos\left(\frac{9}{5}\right)\right] \text{ mm/day}, \tag{3}$$

where we have inserted the value of the integral from equation (2).

### 1.3 Part c

The volume $V(t)$ of water in the can at time $t$ is given by

$$V(t) = 100\pi S(t), \text{ so} \tag{4}$$
$$V'(t) = 100\pi S'(t). \tag{5}$$

Consequently,

$$V'(7) = 100\pi S'(7) = 150\pi + 200\pi \sin\left(\frac{21}{100}\right) \text{ cubic mm/sec.} \tag{6}$$

### 1.4 Part d

We have $M'(t) = \frac{1}{400}(9t^2 - 60t + 330)$. Using $S'(t)$ as given, we find that

$$M'(0) - S'(0) = \frac{33}{40} - \frac{3}{2} = -\frac{27}{40} < 0, \text{ while} \tag{7}$$
$$D(60) = M'(60) - S'(60) = \frac{2853}{40} - 2\sin\left(\frac{9}{5}\right) > \frac{2853}{40} - 2 > 69 > 0. \tag{8}$$

Because $D$ is a continous function on $[0, 60]$, it follows from the Intermediate Value Theorem that there is a time $t_0 \in (0, 60)$ such that $D(t_0) = 0$, which is to say that $M'(t_0) = S'(t_0)$, or the two rates are the same.

## 2 Problem 2

### 2.1 Part a

We have

$$\lim_{t\to 5^-} r(t) = \lim_{t\to 5^-} \frac{600t}{t+3} = \frac{3000}{8} = 357, \text{ while} \tag{9}$$
$$\lim_{t\to 5^+} \left[1000e^{-0.2t}\right] \sim 367.9. \tag{10}$$

The two one-sided limits are different, so the function $r$ has no limit at $t = 5$. The function is therefore not continuous at $t = 5$.

### 2.2 Part b

The average rate at which the tank drains over the interval $[0,8]$ is given by the integral

$$\frac{1}{8}\int_0^8 r(t)\,dt = \frac{1}{8}\left[\int_0^5 \frac{600t}{t+3}\,dt + \int_5^8 1000e^{-0.2t}\,dt\right] \sim 258.05274, \tag{11}$$

which we have evaluated by numerical integration. The average rate of drainage is thus $258.05274$ liters per hour.

### 2.3 Part c

We have

$$r'(t) = \frac{1800}{(t+3)^2}, \text{ so that} \tag{12}$$

$$r'(3) = \frac{1800}{6^2} = 50 \text{ liters per hour per hour.} \tag{13}$$

This is the rate at which the rate of drainage is changing when $t = 3$.

### 2.4 Part d

The time $A$ at which the amount of water in the tank is $9000$ liters must satisfy the equation

$$9000 + \int_0^A r(t)\,dt = 12000. \tag{14}$$

## 3 Problem 3

### 3.1 Part a

The area of the pictured region $R$ is

$$\int_0^4 \sqrt{x}\,dx + \int_4^6 (6-x)\,dx = \frac{2}{3}x^{3/2}\Big|_0^4 + \left(6x - \frac{x^2}{2}\right)\Big|_4^6 = \frac{16}{3} + 2 = \frac{22}{3}. \tag{15}$$

### 3.2 Part b

A cross section of this solid perpendicular to the $y$-axis at $y = t$ is a rectangle whose height is $2t$ and whose base extends from the curve $x = y^2$ to the curve $x = 6 - y$. The area of such a cross section is therefore $2t\left[(6-t) - t^2\right]$, so the required integral is $2\int_0^2 \left[6t - t^2 - t^3\right]\, dt$.

**Note:** Evaluation of this integral is not required. For the curious,

$$2\int_0^2 \left[6t - t^2 - t^3\right]\, dt = 2\left[3t^2 - \frac{1}{3}t^3 - \frac{1}{4}t^4\right]\Bigg|_0^2 \tag{16}$$

$$= 2\left[12 - \frac{8}{3} - 4\right] = \frac{32}{3}. \tag{17}$$

### 3.3 Part c

The slope of the line $y = 6 - x$ is $-1$, so we seek a point on the curve $y = \sqrt{x}$ where $y' = 1$. But $y' = \frac{1}{2}x^{-1/2}$ = 1 when $x^{-1/2} = 2$, or, equivalently, when $x = \frac{1}{4}$. The point $P$ therefore has coordinates $\left(\frac{1}{4}, \frac{1}{2}\right)$.

## 4 Problem 4

### 4.1 Part a

The function $f$ has a single critical point in $(0, \infty)$, where $f'(x) = (4-x)x^{-3} = 0$. This critical point is at $x = 4$. Now $f'(x) > 0$ for $x \in (0, 4)$, while $f'(x) < 0$ when $x \in (4, \infty)$. (A continuous function that is increasing (respectively, decreasing) on an open interval is necessarily increasing (respectively, decreasing) on the closure of that interval. Consequently, $f$ is increasing on $(0, 4]$ and decreasing on $[4, \infty)$. It follows that $f$ has a relative maximum at $x = 4$.

### 4.2 Part b

If $f'(x) = (4-x)x^{-3}$, then

$$f''(x) = -x^{-3} - 3(4-x)x^{-4} = 2(x-6)x^{-4}. \tag{18}$$

Consequently, $f''(x) < 0$ when $x \in (0, 6)$ and $f''(x) > 0$ when $x \in (6, \infty)$. Therefore, $f$ is concave upward on $(6, \infty)$ and $f$ is concave downward on $(0, 6)$. (Note: whether 6 belongs in these intervals of concavity depends on the definition of ″upward [downward] concavity″ we adopt. Texts vary in this respect.)

### 4.3 Part c

By the Fundamental Theorem of Calculus,

$$f(x) = f(1) + \int_1^x f'(t)\,dt = 2 + \int_1^x \left[4t^{-3} - t^{-2}\right]\,dt \tag{19}$$

$$= 2 + \left(-2t^{-2} + t^{-1}\right)\Big|_1^x \tag{20}$$

$$= 2 + \left(-2x^{-2} + x^{-1}\right) - (-1) \tag{21}$$

$$= 3 - 2x^{-2} + x^{-1}. \tag{22}$$

## 5 Problem 5

### 5.1 Part a

Ben's acceleration at time $t = 5$ is approximately

$$\frac{v(10) - v(0)}{10 - 0} = \frac{2.3 - 2.0}{10} = 0.03 \text{ meters per second per second.} \tag{23}$$

### 5.2 Part b

The integral $\int_0^{60} |v(t)|\,dt$ is the integral of Ben's speed. It measures the total distance Ben has traveled over the interval $0 \leq t \leq 60$. We have

$$\int_0^{60} |v(t)|\,dt \sim 2.0 \cdot (10 - 0) + 2.3 \cdot (40 - 10) + 2.5 \cdot (60 - 4) = 139, \tag{24}$$

so the total distance Ben traveled during this minute is about 139 meters.

### 5.3 Part c

We have

$$\frac{B(60)-B(40)}{60-40} = \frac{49-9}{60-40} - \frac{40}{20} = 2. \tag{25}$$

We may apply the Mean Value Theorem here, because we are given that $B$ is a twice differentiable function, and this latter fact guarantees that $B$ is continuous on $[40,60]$ and differentiable on $(40,60)$—which are the hypotheses of the Mean Value Theorem. Thus, there must be a time $t_0 \in (40,60)$ when $v(t_0) = B'(t_0) = 2$.

**Note:** We are cheating a bit, but this has to be what the examiners expected. We haven't been told just *where* $B$ is twice-differentiable or what the domain of $B$ is, and it's not really clear what it would mean for $B''(60)$ to exist if the domain of $B$ is $[0,60]$. We adopt the convention that the problem takes differentiability at an end-point to be the appropriate one-sided differentiability there; if we don't do so, our conclusion that $B$ is continuous at $t = 60$ is unsupportable.

### 5.4 Part d

From $L^2 = 144 + B^2$, we find that $2LL' = 2BB' = 2Bv$. Thus, when $t = 40$ we have

$$2LL' = 2Bv = \cancel{2} \cdot 9 \cdot \frac{5}{\cancel{2}} = 45. \tag{26}$$

However, when $t = 40$, we also have $L^2 = 144 + 81 = 225$, so that $L = 15$. Thus, at $t = 40$, $45 = 2LL' = 2 \cdot 15 \cdot L'$, and $L' = \frac{45}{30} = \frac{3}{2}$ meters per second.

## 6 Problem 6

### 6.1 Part a

We note first that $\int_{-2\pi}^{4\pi} g(x)\,dx$ is the area of the pictured triangle, or $\frac{1}{\cancel{2}} \cdot 6\pi \cdot \cancel{2}\pi = 6\pi^2$. On the other hand,

$$\int_{-2\pi}^{4\pi} \cos\frac{x}{2}\,dx = 2\sin\frac{x}{2}\bigg|_{-2\pi}^{4\pi} = 2\sin(2\pi) - 2\sin(-\pi) = 0. \tag{27}$$

Consequently, $\int_{-2\pi}^{4\pi} f(x)\,dx = 6\pi^2$.

**Note:** We can also use the symmetries of the cosine function to compute the integral that appear in (27). Doing the calculation above is probably faster than explaining how the symmetries yield a zero integral.

### 6.2 Part b

We have $f'(x) = 1 + \frac{1}{2}\sin\frac{x}{2}$ when $-2\pi < x < 0$; $f'(x) = -\frac{1}{2} + \frac{1}{2}\sin\frac{x}{2}$ when $0 < x < 4\pi$. Thus $f'(\pi) = 0$ and $f'(x)$ is undefined when $x = 0$ because $g$ is not differentiable at $x = 0$. (This is because $g'_{-}(0) = -1$ while $g'_{+}(0) = -\frac{1}{2}$, both of which are easily seen from the definition of $g$.) These give the only two critical points of $f$.

### 6.3 Part c

If $h(x) = \int_0^{3x} g(t)\,dt$, then, by the Fundamental Theorem of Calculus and the Chain Rule, $h'(x) = 3g(3x)$. Therefore

$$h'\left(-\frac{\pi}{3}\right) = 3g(-\pi) = 3\pi. \tag{28}$$