AP Calculus 2011 AB FRQ Solutions

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1 Problem 1

1.1 Part a

Speed, $s(t) = |v(t)|$, satisfies $[s(t)]^2 = [v(t)]^2$, whence $2s(t)s'(t) = 2v(t)v'(t)$. But $s(t) \geq 0$, so $s'(t) > 0$ only when $v(t)v'(t) > 0$. Here,

$$v(t)v'(t) = v(t)a(t) \tag{1}$$

$$= \left(2\sin e^{t/4} + 1\right)\left(\frac{1}{2}e^{t/4}\cos e^{t/4}\right). \tag{2}$$

Thus,

$$v(5.5) \cdot v'(5.5) \sim 0.61591 > 0, \tag{3}$$

and it follows that speed is increasing when $t = 5.5$.

1.2 Part b

Average velocity over the interval $0 \leq t \leq 6$ is

$$\frac{1}{6}[x(6) - x(0)] = \frac{1}{6}\int_0^6 v(t)\,dt \tag{4}$$

$$= \frac{1}{6}\int_0^6 \left(2\sin e^{t/4} + 1\right)\,dt. \tag{5}$$

Integrating numerically, we find that the average velocity over $[0, 6]$ is approximately 1.94938.

1.3 Part c

Total distance traveled over the interval $0 \le t \le 6$ is

$$\int_0^6 |v(t)|\,dt = \int_0^6 \sqrt{[v(t)]^2}\,dt. \tag{6}$$

Another numerical integration gives this total distance as approximately $t_0 = 12.57326$.

1.4 Part d

We seek the unique t_1 in $0 \le t_1 \le 6$ for which velocity changes sign. This can happen only where $v(t_1) = 0$, from which we see that $e^{t_1/4} = 7\pi/6$ or

$$t_1 = 4\ln\left(\frac{7}{6}\pi\right) \sim 5.19552 \tag{7}$$

We are given that $x(0) = 2$, and, by the Fundamental Theorem of Calculus, the position we want is

$$x(t_1) = x(0) + \int_0^{t_1} v(\tau)\,d\tau \tag{8}$$

$$= 2 + \int_0^{t_1} \left(2\sin e^{\tau/4} + 1\right)\,d\tau. \tag{9}$$

Another numerical integration gives $x(t_1) \sim 14.13477$ as the approximate position of the particle at the instant when it changes its direction of motion.

2 Problem 2

2.1 Part a

The rate at which the temperature of the tea is changing at time $t = 3.5$ is given, approximately, by the difference quotient

$$\frac{H(3.5+1.5) - H(3.5-1.5)}{(3.5+1.5) - (3.5-15)} = \frac{52-60}{3} = -\frac{8}{3} \text{ degrees per minute.} \tag{10}$$

2.2 Part b

The average value $\bar{T}$ of the temperature of the tea, in degrees Celsius, is

$$\bar{T} = \frac{1}{10}\int_0^{10} H(t)\,dt. \tag{11}$$

The trapezoidal approximation for this integral is

$$\begin{aligned}
&\frac{1}{10}\cdot\frac{1}{2}\sum_{k=1}^{4}\left[H(t_{k-1})+H(t_k)\right](t_k - t_{k-1}) &(12)\\
&= \frac{1}{20}\left[(66+60)(2-0)+(60+52)(5-2)+(52+44)(9-5)+(44+43)(10-9)\right] &(13)\\
&= \frac{1059}{20}. &(14)
\end{aligned}$$

2.3 Part c

By the Fundamental Theorem of Calculus, $\int_0^{10} H'(t)\,dt = H(10) - H(0) = -23$. Thus,the amount by which the temperature changed over the interval $0 \le t \le 10$ is $-23°$ C.

2.4 Part d

$B(t)$ is given, again by the Fundamental Theorem of Calculus, by

$$B(t) = 100 - 13.84\int_0^{t} e^{-0.173\tau}\,d\tau. \tag{15}$$

Therefore

$$\begin{aligned}
B(10) &= 100 - 13.84\int_0^{10} e^{-0.173\tau}\,d\tau &(16)\\
&= 100 - 13.84\left(-\frac{1}{0.173}e^{-0.173\tau}\right)\Bigg|_0^{10} \sim 34.18275. &(17)
\end{aligned}$$

We seek $H(10) - B(10) = 43 - 34.18275 = 8.81725$. So the biscuits are about $8.81725°$ C. cooler than the tea at time $t = 10$.

3 Problem 3

3.1 Part a

If $f(x) = 8x^3$, then $f(1/2) = 1$, $f'(x) = 24x^2$, and $f'(1/2) = 6$. An equation for the line tangent to the curve $y = f(x)$ at the point where $x = 1/2$ is therefore

$$y = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right), \text{ or} \tag{18}$$

$$y = 1 + 6\left(x - \frac{1}{2}\right). \tag{19}$$

3.2 Part b

The area of the region R is

$$\int_0^{1/2} \left(\sin \pi x - 8x^3\right)\, dx = \left(-\frac{1}{\pi}\cos \pi x - 2x^4\right)\Big|_0^{1/2} \tag{20}$$

$$= \left(-0 - \frac{1}{8}\right) - \left(-\frac{1}{\pi} - 0\right) = \frac{8-\pi}{8\pi}. \tag{21}$$

3.3 Part c

The volume, V_R, of the solid generated by rotating the region R about the horizontal line $y = 1$ is

$$V_R = \pi \int_0^{1/2} \left[(1 - 8x^3)^2 - (1 - \sin \pi x)^2\right]\, dx. \tag{22}$$

Note: Evaluation of this integral is not required, but, for the curious,

$$\begin{aligned}
\pi \int_0^{1/2} & \left[(1-8x^3)^2 \; - (1-\sin\pi x)^2\right]\, dx && (23)\\
&= \pi \int_0^{1/2} \left[64x^6 - 16x^3 + 2\sin\pi x - \sin^2\pi x\right]\, dx && (24)\\
&= \pi \int_0^{1/2} \left[64x^6 - 16x^3 + 2\sin\pi x - \frac{1}{2} + \frac{1}{2}\cos 2\pi x\right]\, dx && (25)\\
&= \pi \left[\frac{64}{7}x^7 - 4x^4 - \frac{x}{2} - \frac{2}{\pi}\cos\pi x + \frac{1}{4\pi}\sin 2\pi x\right]\Bigg|_0^{1/2} && (26)\\
&= \pi\left(\frac{1}{14} - \frac{1}{2}\right) - \pi\left(-\frac{2}{\pi}\right) = 2 - \frac{3}{7}\pi. && (27)
\end{aligned}$$

4 Problem 4

4.1 Part a

$$g(-3) = -6 + \int_0^{-3} f(t)\, dt = -6 - \frac{1}{4}\pi \cdot 3^2 = -6 - \frac{9}{4}\pi; \qquad (28)$$

$$g'(x) = \frac{d}{dx}\left[2x + \int_0^x f(t)\, dt\right] = 2 + f(x). \qquad (29)$$

$$G'(3) = 2 + f(-3) = 2. \qquad (30)$$

4.2 Part b

The absolute maximum of g must occur at an endpoint of the interval $[-4, 3]$ or at a critical point interior to that interval. But $g'(x) = 2 + f(x)$, and this is simply the curve $y = f(x)$ shifted 2 units upward. Note that all of the shifted curve that lies to the left of the y-axis lies above the x-axis, so that $g'(x) > 0$ when x lies to the left of the y-axis—and for a substantial interval just to the right of the y-axis. For $0 \leq x \leq 3$, we then have $g'(x) = 5 - 2x$, so that $g'(x) = 0$ when $x = \frac{5}{2}$. Thus, $g'(x) > 0$ for $-4 \leq x < \frac{5}{2}$, negative for $\frac{5}{2} < x \leq 3$, and zero when $x = \frac{5}{2}$. The latter value is the only critical value for g. It is clear, on geometric ground, that the area under g' on the interval $[-4, \frac{5}{2}]$ is positive and exceeds, in magnitude, the area between the g' curve and the x-axis on the interval $[\frac{5}{2}, 2]$. Consequently, $0 = f(-4) < g(\frac{5}{2})$ and $g(3) < g(\frac{5}{2})$. The absolute maximum therefore occurs at $x = \frac{5}{2}$.

4.3 Part c

The function g' [see Part b, above, for an explicit description of g'] is increasing on $[-4, 0]$ and decreasing on $[0, 3]$. Inflection points are to be found where the monotonicity of the derivative changes, so $x = 0$ is the location of the only inflectiion point for this curve.

4.4 Part d

We have $f(-4) = -1$ and $f(3) = -3$. The average rate of change of f on the interval $[-4, 3]$ is therefore

$$\frac{f(3) - f(-4)}{4 - (-3)} = \frac{(-3) - (-1)}{7} = -\frac{2}{7}. \tag{31}$$

That $f'(c) = -\frac{2}{7}$ fails for all c in $(-4, 3)$ doesn't contradict the Mean Value Theorem because $f'(0)$ doesn't exist. The hypotheses of the Mean Value Theorem require, among other things, that a function f be differentiable on $(-4, 3)$ before we may apply the theorem to that function on the interval $[-4, 3]$. This is not so for this f, so there is no contradiction.

5 Problem 5

5.1 Part a

We are given

$$W'(t) = \frac{1}{25}[W(t) - 300], \tag{32}$$

so $W'(0) = \frac{1400-300}{25} = 44$, and the equation for the line tangent to the solution curve for the initial value problem, in (t, w) coordinates, at $t = 0$ is $w = W(0) + W'(0)(t - 9) = 1400 + 44t$. When $t = \frac{1}{4}$, this gives $w = 1400 + 11 = 1411$, so the approximate amount of solid waste at the end of the first three months of 2010 is 1411 tons.

5.2 Part b

Differentiating both sides of (32, we see that

$$\frac{d^2W}{dt^2} = \frac{1}{25} \cdot \frac{d}{dt}\left[W(t) - 300\right] \tag{33}$$

$$= \frac{1}{25} W'(t), \text{ which, again by (32), is} \tag{34}$$

$$\frac{d^2W}{dt^2} = \frac{1}{625}\left[W(t) - 300\right]. \tag{35}$$

Thus, $W''(0) = \frac{44}{25} > 0$, and, $W''(t)$ being continuous, the solution curve must be concave upward near $t = 0$. This means that the tangent line to the curve at $t = 0$ lies below the curve, so the estimate given in Part a is an underestimate.

5.3 Part c

Now $W(0) = 1400$, so $W(0) - 300 > 0$ and W, as the solution to a differential equation, is continuous near $\tau = 0$. Thus, $W(\tau) - 300 > 0$ in some open interval, I, centered at the origin.

For choices of t lying in I and τ lying between 0 and t, we may rewrite (32) as

$$\frac{W'(t)}{W(t) - 300} = \frac{1}{25}, \tag{36}$$

which means that

$$\int_0^t \frac{W'(\tau)}{W(\tau) - 300}\, d\tau = \int_0^t \frac{1}{25}\, d\tau. \tag{37}$$

Thus

$$\ln\left|W(\tau) - 300\right| \Big|_0^t = \frac{\tau}{25}\Big|_0^t \tag{38}$$

Thus, for our choice of t, we may rewrite (38) as

$$\ln\left[W(t) - 300\right] - \ln(1400 - 300) = \frac{t}{25}, \text{ or} \tag{39}$$

$$\ln\left[\frac{W(t) - 300}{1100}\right] = \frac{t}{25}. \tag{40}$$

This leads to

$$W(t) = 300 + 1100e^{t/25}. \tag{41}$$

6 Problem 6

6.1 Part a

We are given

$$f(x) = \begin{cases} 1 - 2\sin x & \text{when } x \le 0; \\ \\ e^{-4x} & \text{when } x > 0. \end{cases} \tag{42}$$

Thus

1. $f(0) = 1 - 2\sin 0 = 1$, so that 0 lies in the domain of f;
2. $\lim_{x\to 0^-}(1 - 2\sin x) = 1$;
3. $\lim_{x\to 0^+} e^{-4x} = 1$.

Thus, $\lim_{x\to 0} f(x) = 1 = f(0)$, and it follows that f is continuous at $x = 0$.

6.2 Part b

When $x < 0$, $f'(x) = -2\cos x$. When $x > 0$, $f'(x) = -4e^{-4x}$. We note that $f'(x) = -3$ is not possible when $x < 0$ because $-2\cos x \ge -2$. Consequently, we look for a positive number x for which $-4e^{-4x} = -3$ or $e^{-4x} = \frac{3}{4}$. From this latter equation, we see that we must have $-4x = \ln\frac{3}{4}$ or $x = -\frac{1}{4}\ln\frac{3}{4}$.

6.3 Part c

The average value of f on the interval $[-1, 1]$ is

$$\frac{1}{1-(-1)} \int_{-1}^{1} f(t)\,dt = \frac{1}{2} \int_{-1}^{0} f(t)\,dt + \frac{1}{2} \int_{0}^{1} f(t)\,dt \tag{43}$$

$$= \frac{1}{2} \int_{-1}^{0} (1 - 2\sin t)\,dt + \frac{1}{2} \int_{0}^{1} e^{-4t}\,dt \tag{44}$$

$$= \frac{1}{2}(t + 2\cos t)\Big|_{-1}^{0} - \frac{1}{8} e^{-4t}\Big|_{0}^{1} \tag{45}$$

$$= \frac{1}{2}\left[(0+2) - (-1 + 2\cos 1)\right] - \frac{1}{8}\left[e^{-4} - 1\right] \tag{46}$$

$$= \frac{(13 - 8\cos 1)e^{4} - 1}{8e^{4}}. \tag{47}$$