AP Calculus 2012 AB FRQ Solutions

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1 Problem 1

1.1 Part a

According to the data in the table, $W(15) = 67.9°$ F, while $W(9) = 61.8°$ F. Therefore,

$$W'(12) \sim \frac{W(15) - W(9)}{15 - 9} = \frac{67.9 - 61.8}{6} = \frac{61}{60}. \tag{1}$$

This means that, 12 minutes after the heating began, the temperature of the water in the tub is increasing at roughly $61/60$ degrees Fahrenheit per minute.

1.2 Part b

By the Fundamental Theorem of Calculus,

$$\int_0^{20} W'(t)\,dt = W(20) - W(0) = 71.0 - 55.0 = 16.0. \tag{2}$$

Thus, the water temperature has increased by about $16.0°$ F. during the first twenty minutes of heating.

1.3 Part c

Using a left Riemann sum and the data from the table, we can approximate

$$\frac{1}{20}\int_0^{20} W(t)\,dt \sim \frac{1}{20}\left[55.0\cdot(4-0)+57.1\cdot(9-4)+61.8\cdot(15-9)+67.9\cdot(20-15)\right]. \tag{3}$$

The value of this sum is $60.79°$ F. We were given that W is an increasing function on the interval in question, so the value of $W(t)$ at the left-hand end-point of each of the subintervals we have used is the minimum of $W(t)$ in that subinterval. Consequently, the left Riemann sum underestimates the integral for the average value of W.

1.4 Part d

By the Fundamental Theorem of Calculus,

$$W(25) = W(20) + \int_{20}^{25} W'(t)\,dt \tag{4}$$

$$= W(20) + 0.04\int_{20}^{25}\left[\sqrt{t}\cos(0.06t)\right]\,dt. \tag{5}$$

Integrating numerically, we find that $W(25) \sim 73.04315°$ F.

2 Problem 2

2.1 Part a

We begin by solving numerically for x_0, the x-coordinate of the point where the curves $y = \ln x$ and $y = 5 - x$ intersect, and we find that $x_0 \sim 3.69344$. Then the corresponding y-coordinate, y_0 is given by $y_0 = 5 - x_0 \sim 1.30656$.

We rewrite the equations of the curves as $x = 5 - y$ and $x = e^y$. Then the area we seek is therefore

$$\int_0^{y_0}\left[(5-y)-e^y\right]dy \sim 2.98580, \tag{6}$$

where we have carried out the integration numerically. (Symbolic integration is possible, but there is little point in doing so because we know y_0 only approximately.)

2.2 Part b

The volume of the solid we have been given is then

$$\int_0^{x_0} (\ln x)^2\, dx + \int_{x_0}^{5} (5-x)^2\, dx \sim 4.78402. \tag{7}$$

(We have done the integrations numerically, although it was not required to carry them out at all. (Symbolic integration is possible, but there is little point in doing so because we know x_0 only approximately.))

2.3 Part c

If the horizontal line $y = k$ divides the region R into two pieces of equal area, we must have

$$\int_0^{k} \left[(5-y) - e^y\right]\, dy = \int_k^{y_0} \left[(5-y) - e^y\right]\, dy. \tag{8}$$

Solution of this equation for k is not required,. We know one of the limits of integration only approximately, and numerical methods give $k \sim 0.42100$.

3 Problem 3

3.1 Part a

The value $g(2)$ is the negative of the area bounded by the lines $y = 0$, $y = (x-1)/2$, and $x = 2$. The region is a triangle of base 1, altitude $1/2$, so $g(2) = -1/4$.

The value $g(-2)$ is the sum of, on the one hand, the area of the triangular region bounded by the lines $y = 0$, $y = -3(x+1)$, and $x = -2$, and, on the other hand, the area of a semi-circular region of radius 1. Thus $g(-2) = (3+\pi)/2$.

3.2 Part b

We have $g(x) = \int_1^x f(t)\, dt$, so it follows from the Fundamental Theorem of Calculus that $g'(x) = f(x)$. Hence $g'(-3) = f(-3)$, and we read the latter from the graph: Thus, $g'(-3) = f(-3) = 2$.

From our conclusion above that $g'(x) = f(x)$, it follows that $g''(x) = f'(x)$ wherever the latter exists. But the graph of $y = f(x)$ is a straight line of slope 1 in the vicinity of the point $(-3, 2)$, so $g''(-3) = f'(-3) = 2$.

3.3 Part c

The line tangent to $y = g(x)$ is horizontal only where $g'(x) = f(x)$ [as found above in Part b] is 0. From the graph, we see that $f(x) = 0$ in just two places: where $x = -1$ and where $x = 1$. Thus, $x = -1$ and $x = 1$ give the only horizontal tangent lines to the curve $y = g(x)$.

As x increases through $x = 1$, $g'(x) = f(x)$ doesn't change sign. By the First Derivative Test, g has neither a relative minimum nor a relative maximum at $x = 1$.

3.4 Part d

The curve $y = g(x)$ has inflection points where the second derivative, $g''(x)$, undergoes a change of sign. We saw in Part b, above, that $g''(x) = f'(x)$, and we can read the sign of the latter from the graph. Hence, g has inflection points at $x = -2$, $x = 0$, and $x = 1$.

4 Problem 4

4.1 Part a

If $f(x) = \sqrt{25 - x^2}$ on $[-5, 5]$, then

$$f'(x) = \frac{-x}{\sqrt{25 - x^2}} \tag{9}$$

on $(-5, 5)$.

4.2 Part b

From Part a, we have

$$f'(-3) = -\frac{-3}{\sqrt{25 - 9}} = \frac{3}{4}. \tag{10}$$

So the line tangent to the curve $y = f(x)$ at the point where $x = -3$ (and $y = 4$) has equation

$$y = 4 + \frac{3}{4}(x+3). \tag{11}$$

4.3 Part c

We have $\lim_{x\to-3^-}\sqrt{25-x^2} = 4$. Also, $\lim_{x\to-3^+}(x+7) = 4$. Consequently, the limit of $g(x)$ as $x \to -3$ exists and is $4 = f(-3) = g(-3)$. We conclude that $\lim_{x\to-3} g(x) = g(-3)$, and g is continuous at $x = -3$.

4.4 Part d

We let $u = 25 - x^2$. Then $du = -2x\,dx$, or $x\,dx = -\dfrac{1}{2}\,du$. Moreover, when $x = 0$, $u = 25$, and when $x = 5$, $u = 0$. Therefore

$$\int_0^5 x\sqrt{25-x^2}\,dx = -\frac{1}{2}\int_{25}^0 \sqrt{u}\,du = -\frac{1}{3}u^{3/2}\Big|_{25}^0 = \frac{125}{3}. \tag{12}$$

5 Problem 5

5.1 Part a

We suppose that $B(t_1) = 40$, while $B(t_2) = 70$. Because

$$B'(t) = \frac{1}{5}[100 - B(t)], \tag{13}$$

we have

$$B'(t_1) = \frac{1}{5}[100 - B(t_1)] = 12 > 5 = \frac{1}{5}[100 - B(t_2)] = B'(t_2). \tag{14}$$

It follows that the bird is growing faster when it weighs 40 grams than when it weighs 70 grams.

5.2 Part b

From $B'(t) = \frac{1}{5}[100 - B(t)]$, we obtain

$$B''(t) = -\frac{1}{5}B'(t) = -\frac{1}{25}[100 - B(t)] \tag{15}$$

But this quantity is negative when $B(t) < 100$, and, because $B(0) = 20$, this means that the graph of B must be concave downward on some interval immediately to the right of $t = 0$. The given graph doesn't have these properties, and so can't be the graph of B.

5.3 Part c

If $B(0) = 20$, then, B being the solution of a differential equation, is a continuous function and $100 - B(t) > 0$ on some open interval, I, centered at $t = 0$.

From $B'(t) = \frac{1}{5}[100 - B(t)]$, we have for all τ in I,

$$\frac{B'(\tau)}{100 - B(\tau)} = \frac{1}{5}, \text{ whence, for any } t \text{ in } I, \tag{16}$$

$$\int_0^t \frac{B'(\tau)}{100 - B(\tau)}\, d\tau = \frac{1}{5}\int_0^t d\tau. \tag{17}$$

Integrating, and making use of the fact that $B(0) = 20 < 100$, we see that

$$-\ln[100 - B(\tau)]\Big|_0^t = \frac{1}{5}\tau\Big|_0^t, \text{ or} \tag{18}$$

$$\ln 80 - \ln[100 - B(t)] = \frac{t}{5}, \text{ which we rewrite as} \tag{19}$$

$$\ln[100 - B(t)] = \ln 80 - \frac{1}{5}t. \tag{20}$$

From this it follows that

$$100 - B(t) = 80e^{-t/5}, \text{ or} \tag{21}$$

$$B(t) = 100 - 80e^{-t/5}. \tag{22}$$

6 Problem 6

6.1 Part a

We are given that $v(t) = \cos(\pi t/6)$ for $0 \leq t \leq 12$ for a certain particle moving along the x-axis. Observing the conventions (not mentioned in the statement of the problem) that the x-axis is horizontal with positive direction pointing toward the right, we see that the particle moves leftward when $v(t) < 0$, or when $0 > \cos(\pi t/6)$. Thus, we see that we need that portion of the interval $[0, 12]$ where $\cos(\pi t/6) < 0$, or, using standard properties of the cosine function, when $3 < t < 9$.

6.2 Part b

The total distance traveled is the integral of speed, $|v(t)|$. Thus, the distance traveled when $0 \leq t \leq 6$ is $\displaystyle\int_0^6 |\cos t|\, dt$.

6.3 Part c

Acceleration is $v'(t) = -\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{6}t\right)$.

Let $S(t)$ denote speed at time t. Then $S = |v| \geq 0$, and $S^2 = v^2$. Thus, $2SS' = 2vv'$, and $S' = vv'/S$. We have $v(4) = \cos(2\pi/3) < 0$ and $v'(4) = -(\pi/6)\sin(2\pi/3) > 0$, so it follows that $S' = vv'/S < 0$. Hence, speed is decreasing when $t = 4$.

Note: In fact, the notion *decreasing* is usually defined only for functions on an interval, and not at a point. Thus, it would be more appropriate to say that, because S' is continuous at $t = 4$ and $S'(4) < 0$. So $S'(t)$ must be negative on some open interval centered at $t = 4$— which guarantees that S is decreasing on some open interval centered at $t = 4$.

6.4 Part d

The position $x(t)$ of the particle at time t is given by $x(t) = x(0) + \int_0^t v(\tau)\, d\tau$. Therefore,

$$x(4) = -2 + \int_0^4 \cos\left(\frac{\pi}{6}\tau\right)\, d\tau \tag{23}$$

$$= -2 + \frac{6}{\pi} \sin\left(\frac{\pi}{6}\tau\right)\Big|_0^4 \tag{24}$$

$$= \frac{3\sqrt{3}}{\pi} - 2. \tag{25}$$