AP Calculus 2014 AB FRQ Solutions

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1 Problem 1

1.1 Part a

We have $A(t) = 6.687(0.931)^t$, where t is measured in days and $A(t)$ is measured in pounds. So the average rate of change of $A(t)$ over the interval $0 \le t \le 30$ is

$$\frac{A(30) - A(0)}{30 - 0} \sim \frac{0.7829279 - 6.687}{30} \sim -0.19680 \text{ pounds per day.} \tag{1}$$

1.2 Part b

We have $A'(t) = 6.687 \cdot (0.931)^t \cdot \ln(0.931) = 0.47809376 \cdot (0.931)^t$, so $A'(15) \sim -0.16359$. Thus, after 15 days have passed, the amount of grass clippings remaining in the bin is changing at about the rate of -0.164 pounds per day.

1.3 Part c

The average amount of grass clippings in the bin over the interval $0 \le t \le 30$ is

$$\frac{1}{30}\int_0^{30} A(\tau)\,d\tau,$$

so we must solve for t in the equation

$$30A(t) = \int_0^{30} A(\tau)\,d\tau. \tag{2}$$

We solve numerically and obtain $t \sim 12.41477$. Thus, we need 12.41477 days.

1.4 Part d

The linear approximation $L(t)$ to A at $t = 30$ is

$$L(t) = A(30) + A'(30)(t - 30), \text{ or} \tag{3}$$
$$L(t) \sim 0.78293 - 0.05598(t - 30). \tag{4}$$

To find the approximate time at which there will be 0.5 pounds of grass clippings remaining in the bin, we must solve the equation $L(t) = 0.5$ for t. Doing so, we find that, according to this model, there will be 0.5 pounds of grass clippings in the bin when $t = 35.05443$ days.

2 Problem 2

Let R be the region enclosed by the horizontal line $y = 4$ and the graph of the equation $y = x^4 - 2.3x^3 + 4$.

2.1 Part a

The volume of the solid generated when R is rotated about the horizontal line $y = -2$ is, using the method of washers, $\pi \int_a^b \left([4+2]^2 - [f(x)+2]^2\right) dx$, where $a < b$ are the x-coordinates of the intersections of the two curves. We find these limits by solving the equation $f(x) = 4$, or $x^4 - 2.3x^3 + 4 = 4$, for x. From this solution we see that $a = 0$ and $b = 2.3$. The desired volume is therefore (integrating numerically)

$$\pi \int_0^{2.3} \left(36 - [f(x)+2]^2\right) dx = \pi \int_0^{2.3} \left(27.6x^2 - 17.29x^4 + 4.6x^6 - x^8\right) dx \tag{5}$$
$$\sim 98.86789, \tag{6}$$

2.2 Part b

If R is the base of a solid, each of whose cross-sections perpendicular to the x-axis is an isosceles right triangle with a leg in R, then we may take the altitude and the base of each cross-section to be $[4 - f(x)]$. Thus, the volume of this solid is

$$\frac{1}{2} \int_0^{2.3} [4 - f(x)]^2 \, dx \sim 3.57372 \tag{7}$$

2.3 Part c

The equation we must solve is

$$\int_0^k [4 - f(x)] \; dx = \int_k^{2.3} [4 - f(x)] \; dx. \tag{8}$$

Solution of (8) is not required, but a numerical solution is easily obtained. Integration, followed by algegraic simplfication, allows us to rewrite (8) as

$$0.4k^5 - 1.15k^4 + 3.2181715 = 0, \tag{9}$$

and then numerical solution gives $k \sim 1.57824$.

3 Problem 3

We are given a function graphically; it can be written as

$$f(x) = \begin{cases} -x - 3, & -5 \le x \le -3; \\ \\ \dfrac{4}{3}x + 4, & -3 < x \le 0; \\ \\ 4 - 2x, & 0 < x \le 4. \end{cases} \tag{10}$$

We put $g(x) = \displaystyle\int_{-3}^{x} f(t)\, dt$.

3.1 Part a

The integral that gives $g(3)$ is the sum of the (signed) areas of the triangle whose vertices are $(-5, 2)$, $(-3, 0)$, and $(-5, 0)$; the triangle whose vertices are $(-3, 0)$, $(2, 0)$, and $(0, 4)$; and the triangle whose vertices are $(2, 0)$, $(3, -2)$, and $(3, 0)$. Thus, $g(3) = 2 + 10 + (-1) = 11$.

3.2 Part b

The function f is, by the Fundamental Theorem of Calculus, the function g'. So f', which gives the slope of f, is g''. Thus, g is both increasing and concave down where f is positive and f' is negative. The intervals in question are therefore $(-5, -3)$ and $(0, 2)$.

3.3 Part c

With $h(x) = \dfrac{g(x)}{5x}$, we have

$$h'(x) = \frac{5xg'(x) - 5g(x)}{25x^2}, \text{ so} \tag{11}$$
$$h'(3) = \frac{\cancel{5} \cdot 3 \cdot g'(3) - \cancel{5} \cdot g(3)}{\cancel{5} \cdot 3} \tag{12}$$
$$= \frac{3 \cdot f(3) - g(3)}{3} \tag{13}$$
$$= \frac{3 \cdot (-2) - 11}{3} = -\frac{17}{3}. \tag{14}$$

3.4 Part d

If $p(x) = f\left(x^2 - x\right)$, then, by the Chain Rule,

$$p'(x) = (2x - 1)f'\left(x^2 - x\right). \tag{15}$$

Thus,

$$p'(1) = [2 \cdot (-1) - 1] \cdot f'\left[(-1)^2 - (-1)\right] \tag{16}$$
$$= (-3) \cdot f'(2) = (-3) \cdot (-2) = 6. \tag{17}$$

4 Problem 4

4.1 Part a

The average acceleration of the train over the interval $2 \leq t \leq 8$ is

$$\frac{v_A(8) - v_A(2)}{8 - 2} = \frac{(-120) - 100}{6} = -\frac{220}{6} = -\frac{110}{3} \text{ meters/min/min.} \tag{18}$$

4.2 Part b

We are given that the velocity function, v_A, is differentiable in its domain, so it is also continuous there. Now $v_A(5) = 40$ and $v_A(8) = -120$ are given in the table, so the Intermediate Value Property of continuous functions guarantees that there is a number ξ in the interval $(5, 8)$ for which $v_A(\xi) = -100$.

Note: Continuity of the derivative is not needed here; derivatives have the Intermediate Value Property—even though they need not be continuous functions. This fact is not ordinarily known to students at the level of AP Calculus, so a student who wants to use it should state it explicitly.

4.3 Part c

Under the conditions given, if $s_A(t)$ denotes the distance of train A from Origin Station at time t, then s_A is given by

$$s_A(t) = 300 + \int_2^t v_A(\tau)\, d\tau. \tag{19}$$

The train's distance from Origin Station at time $t = 12$ is thus given by

$$s_A(12) = 300 + \int_2^{12} v_A(\tau)\, d\tau. \tag{20}$$

The trapezoidal approximation, using the three subintervals given in the table, is

$$s_A(12) = 300 + \frac{1}{2}[(100 + 40) \cdot 3 + (40 - 120) \cdot 3 + (-120 - 150) \cdot 4] = -150 \text{ meters.} \tag{21}$$

4.4 Part d

Let $s_B(t)$ denote the distance of train B from Origin Station at time t. The distance $S(t)$ between the two trains then satisfies the equation

$$S^2 = s_A^2 + s_B^2. \tag{22}$$

Implicit differentiation with respect to t gives

$$2S(t)S'(t) = 2s_A(t)s_A'(t) + 2s_B(t)s_B'(t) \tag{23}$$
$$= 2s_A(t)v_A(t) + 2s_B(t)v_B(t). \tag{24}$$

Substituting $t = 2$ and using what has been given in the problem, we find that $S'(2) = 160$ meters per minute.

5 Problem 5

5.1 Part a

By the First Derivative Test, a differentiable function defined for all real numbers has a relative minimum only at a point where its derivative is zero, negative on some interval to the left, and positive on some interval to the right. From the table given, we see that the function f has a relative minimum in $[-2, 3]$ only at $x = 1$.

5.2 Part b

Because we know that f is twice differentiable on the real line, we know that f'' exists everywhere, making f' continuous. Moreover, we are given that $f'(-1) = 0 = f'(1)$. Thus, f' satisfies the hypotheses of Rolle's theorem on $[-1, 1]$, and it follows that there must be a number c in $(-1, 1)$ such that $f''(c) = 0$.

5.3 Part c

$$h'(x) = \frac{d}{dx}\left(\ln[f(x)]\right) = \frac{f'(x)}{f(x)}, \text{ so} \tag{25}$$

$$h'(3) = \frac{f'(3)}{f(3)} = \frac{1/2}{7} = \frac{1}{14}. \tag{26}$$

5.4 Part d

$$f'[g(x)]g'(x) = \frac{d}{dx}f[g(x)], \tag{27}$$

so

$$\int_{-2}^{3} f'[g(x)]g'(x)\,dx = f[g(x)]\Big|_{-2}^{3} \tag{28}$$

$$= f[g(3)] - f[g(-2)] \tag{29}$$

$$= f(1) - f(-1) \tag{30}$$

$$= 2 - 8 = -6. \tag{31}$$

6 Problem 6

6.1 Part a

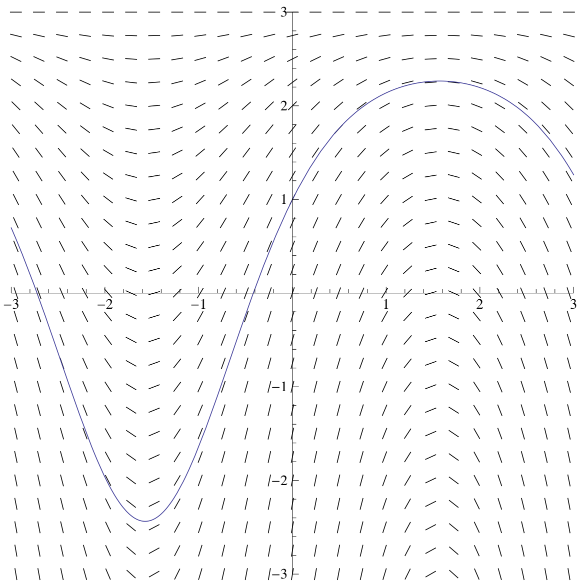

Figure 1: Problem 6, Part a

6.2 Part b

If f is the solution to the differential equation $y' = (3-y)\cos x$ through $(0,1)$, then $f'(0) = (3-1)\cos 0 = 2$ so the line tangent to the solution curve at $(0,1)$ has equation

$$y = 1 + 2(x - 0). \tag{32}$$

The ordinate of the point on this line that corresponds to $x = 0.2$ gives the approximate value of $f(0.2)$ that we desire. Thus, $f'(0.2) \sim 1 + 2(0.2) = 1.4$.

6.3 Part c

If f is the solution to the initial value problem $y' = (3-y)\cos x$ with $y(0) = 1$, then $f'(x) = [3 - f(x)]\cos x$ on some open interval centered at $x = 0$ where f must be continuous.

Because $f(0) = 1 \neq 3$ we may assume that $3 - f(x) > 0$ on that interval. For all x in such an interval, we must have

$$\int_0^x \frac{f'(\xi)}{3 - f(\xi)}\, d\xi = \int_0^x \cos\xi\, d\xi, \text{ or} \tag{33}$$

$$-\ln|3 - f(\xi)|\Big|_0^x = \sin\xi\Big|_0^x \tag{34}$$

Equivalently,

$$\ln 2 - \ln|3 - f(x)| = \sin x, \text{ or} \tag{35}$$

$$\frac{2}{|3 - f(x)|} = e^{\sin x}, \text{ and} \tag{36}$$

$$|3 - f(x)| = 2e^{-\sin x}. \tag{37}$$

We must write $|3 - f(x)| = 3 - f(x)$ because $3 - f(x) > 0$, and it follows that

$$f(x) = 3 - 2e^{-\sin x} \tag{38}$$

is the solution we seek.