AP Calculus 1998 BC FRQ Solutions

Louis A. Talman, Ph.D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

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1 Problem 1

1.1 Part a

The area of the region R is

$$\int_0^2 \left(8 - x^{2/3}\right)\,dx = \left(8x - \frac{2}{5}x^{5/2}\right)\Big|_0^2 = 16 - \frac{8}{5}\sqrt{2} \sim 13.73726. \tag{1}$$

1.2 Part b

The volume of the solid generated by revolving R about the x-axis is

$$\pi\int_0^2 \left(8 - x^{3/2}\right)^2\,dx = \pi\int_0^2 \left(64 - 16x^{3/2} + x^3\right)\,dx \tag{2}$$

$$= \pi\left(64x - \frac{32}{5}x^{5/2} + \frac{1}{4}x^4\right)\Big|_0^2 \tag{3}$$

$$= \left(132 - \frac{128}{5}\sqrt{2}\right)\pi \sim 300.95243 \tag{4}$$

1.3 Part c

We have

$$\pi\int_0^k \left(8 - x^{3/2}\right)^2\,dx = \pi\left(64k - \frac{32}{5}k^{5/2} + \frac{1}{4}k^4\right). \tag{5}$$

Equating this quantity to half of the volume given in (4) and solving numerically, we find that $k \sim 0.80489$.

2 Problem 2

2.1 Part a

We observe first that

$$\lim_{x \to -\infty} 2xe^{2x} = \lim_{x \to -\infty} \frac{2x}{e^{-2x}}. \tag{6}$$

Numerator and denominator of this last fraction both become infinite as $x \to -\infty$, so we may attempt l'Hôpital's Rule. This gives

$$\lim_{x \to -\infty} \frac{2x}{e^{-2x}} = \lim_{x \to -\infty} \frac{2}{-2e^{-2x}} = 0, \tag{7}$$

and we conclude that

$$\lim_{x \to -\infty} 2xe^{2x} = 0. \tag{8}$$

2.2 Part b

If $f(x) = 2xe^{2x}$, then $f'(x) = (2 + 4x)e^{2x}$, which is defined for all real x. Thus, $f'(x) = 0$ only when $x = -1/2$, so f has just one critical point—which lies at $x = -1/2$. But $e^{2x} > 0$ for all x, while $2 + 4x < 0$ on $(-\infty, -1/2)$ but $2x + 4 > 0$ on $(-1/2, \infty)$. So $f'(x) < 0$ on $(-\infty, -1/2)$, and $f'(x) > 0$ on $(-1/2, \infty)$. Because f is everywhere continuous[1], it follows that f is a strictly decreasing function on $(-\infty, -1/2]$, but that f is a strictly increasing function on $[-1/2, \infty)$. That is, if $x < -1/2$ then $f(x) > f(-1/2)$ while if $x > -1/2$ then $f(x) > f(-1/2)$. Consequently, $f(-1/2) = -e^{-1}$ is an absolute minimum for $f(x)$.

2.3 Part c

By our conclusion in Part b, above, the observation that $\lim_{x \to \infty} 2xe^{2x} = \infty$, and the continuity of f, we see that the range of f is $[-e^{-1}, \infty)$.

[1] Continuity allows us to extend our conclusions of monotonicity to the finite endpoints of both intervals

2.4 Part d

We put $f_b(x) = bxe^{bx}$, and we find that ${f_b}'(x) = (b + b^2x)e^{bx}$. If $b > 0$, we argue as in Parts a and b, above, and we find that f_b has an absolute minimum at $x = -1/b$. This minimum is $f_b(-1/b) = -e^{-1}$, which does not depend on b. If $b < 0$, we obtain the same result after the change of variables $u = -x$, which amounts to a reflection about the y-axis.

3 Problem 3

3.1 Part a

The third degree Taylor polynomial, $T_3(x)$ for f about $x = 0$ is

$$\begin{aligned} T_3(x) &= f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(0)x^3 && (9) \\ &= 5 - 3x + \frac{1}{2}x^2 + \frac{2}{3}x^3. && (10) \end{aligned}$$

Thus,

$$f(1.2) \sim T_3(1.2) = 4.42533. \tag{11}$$

3.2 Part b

We can obtain the fourth degree Taylor polynomial, $P_3(x)$, about $x = 0$ for $g(x) = f(x^2)$ by substituting x^2 for x in $T_3(x)$, found above, and then truncating. This gives

$$P_3(x) = 5 - 3x^2 + \frac{1}{2}x^4 \tag{12}$$

3.3 Part c

We can obtain the third degree Taylor polynomial $Q_3(x)$ for

$$h(x) = \int_0^x f(t)\,dt \tag{13}$$

by integrating $T_3(x)$ term by term and truncating. We obtain

$$Q_3(x) = 5x - \frac{3}{2}x^2 + \frac{1}{6}x^3. \tag{14}$$

3.4 Part d

It is not possible to determine $h(1) = \int_0^1 f(t)\,dt$ from what it given.

If, on the one hand,

$$f(x) = 5 - 3x + \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{6}x^4, \tag{15}$$

then f meets all of the given conditions, and

$$\begin{aligned} h(1) &= \int_0^1 \left(5 - 3t + \frac{1}{2}t^2 + \frac{2}{3}t^3 - \frac{1}{6}t^4\right)\,dt & (16)\\ &= \left(5t - \frac{3}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{6}t^4 - \frac{1}{30}t^5\right)\Bigg|_0^1 & (17)\\ &= \frac{19}{5}. & (18) \end{aligned}$$

If, on the other hand,

$$f(x) = 5 - 3x + \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{6}x^5, \tag{19}$$

then, again, f meets all of the given conditions, but

$$\begin{aligned} h(1) &= \int_0^1 \left(5 - 3t + \frac{1}{2}t^2 + \frac{2}{3}t^3 - \frac{1}{6}t^5\right)\,dt & (20)\\ &= \left(5t - \frac{3}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{6}t^4 - \frac{1}{36}t^6\right)\Bigg|_0^1 & (21)\\ &= \frac{137}{6}. & (22) \end{aligned}$$

4 Problem 4

4.1 Part a

See Figure 1

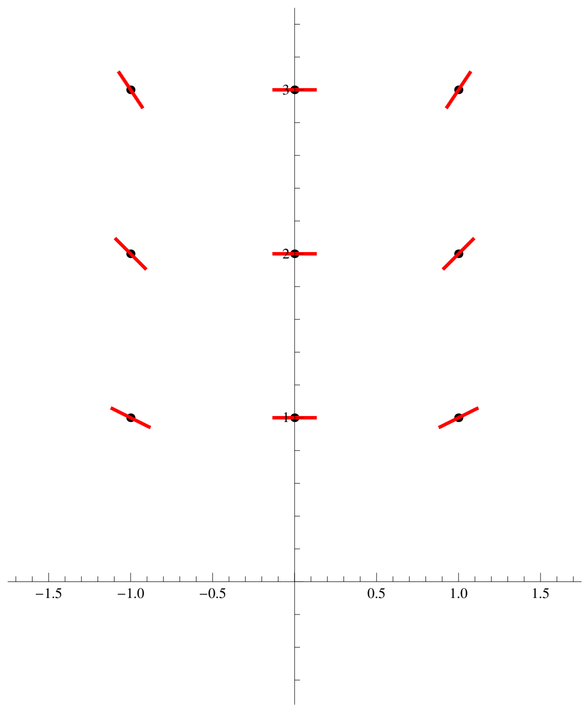

Figure 1: Problem 4, Part a

4.2 Part b

The Euler's Method recursion for this problem is given by

$$h = 0; \tag{23}$$
$$x_0 = 0; \tag{24}$$
$$y_0 = 3; \tag{25}$$
$$x_k = x_{k-1} + h, \text{ when } k > 0; \tag{26}$$
$$y_k = y_{k-1} + \frac{x_{k-1}y_{k-1}}{2}h, \text{ when } k > 0. \tag{27}$$

Thus,

$$x_1 = 0 + 0.1 = 0.1; \tag{28}$$
$$y_1 = 3 + \frac{0 \cdot 3}{2} \cdot (0.1) = 3; \tag{29}$$
$$x_2 = 0.1 + 0.1 = 0.2; \tag{30}$$
$$y_2 = 3 + \frac{(0.1) \cdot 3}{2} \cdot (0.1) = 3.015. \tag{31}$$

So $f(0.2) \sim 3.015$.

4.3 Part c

Let f be a particular solution to the differential equation $2y' = xy$, that satisfies $f(0) = 3$. Then $2f'(x) = xf(x)$ in some open interval I centered at $x = 0$. Because $f(0) = 3$ and f is the solution to a differential equation, we may assume that f is continuous on I, and, indeed, that $f(x) > 0$ on I. Thus, throughout I,

$$\frac{f'(x)}{f(x)} = \frac{x}{2}, \tag{32}$$

and if x is any point of I,

$$\int_0^x \frac{f'(\xi)}{f(\xi)}\, d\xi = \frac{1}{2}\int_0^x \xi \, d\xi. \tag{33}$$

$$\tag{34}$$

The function f taking on only positive values throughout the interval of integration, and $f(0)$ having the value 3, we can rewrite this as

$$\ln f(\xi)\Big|_0^x = \frac{1}{4}\xi^2\Big|_0^x; \tag{35}$$

$$\ln f(x) - \ln f(0) = \frac{1}{4}x^2; \tag{36}$$

$$\ln f(x) = \frac{1}{4}x^2 + \ln 3; \tag{37}$$

$$f(x) = 3e^{x^2/4}. \tag{38}$$

Thus,

$$f(0.2) = 3e^{0.01} \sim 3.03015. \tag{39}$$

5 Problem 5

5.1 Part a

See Figure 2.

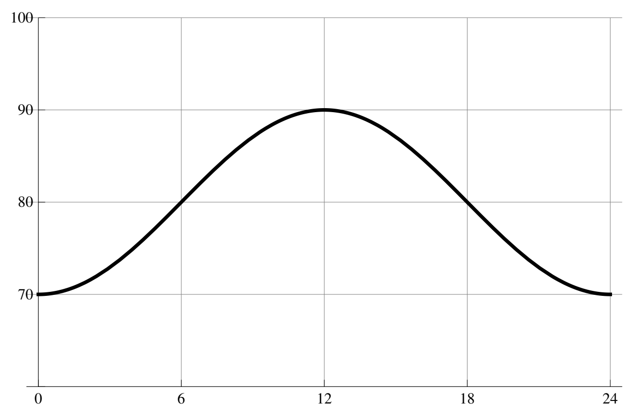

Figure 2: Problem 5, Part a

5.2 Part b

The average temperature is

$$\frac{1}{14-6}\int_6^{14}\left(80-10\cos\frac{\pi t}{12}\right)\,dt = \frac{1}{8}\left(80t-\frac{120}{\pi}\sin\frac{\pi t}{12}\right)\Bigg|_6^{14} \tag{40}$$

$$= \frac{1}{2}\left(160+\frac{45}{\pi}\right) \sim 87.16197^\circ. \tag{41}$$

To the nearest degree, this is 87°.

5.3 Part c

Examination of Figure 2 shows that an approximate answer is $5 \le t \le 19$. Numerical solution of the equation

$$78 = \left(80-10\cos\frac{\pi t}{12}\right) \tag{42}$$

yields $t \sim 5.24087$ for the left-hand solution, and $t \sim 18.76913$ for the right-hand solution. We conclude that the air condition ran when $5.24087 \le t \le 18.76913$. (Although, in this context, accuracy of more than a single digit to the right of the decimals is probably silly.)

5.4 Part d

The approximate total cost is

$$0.05\int_{5.24087}^{18.76913}\left(2-10\cos\frac{\pi t}{12}\right)\,dt \sim 5.09637. \tag{43}$$

We have evaluated the integral numerically, because we know the limits of integration only approximately and there is little point it trying for an "exact" integral.

6 Problem 6

6.1 Part a

If

$$x'(t) = \frac{1}{\sqrt{2t+1}}, \tag{44}$$

and $x(0) = -4$,then, by the Fundamental Theorem of Calculus,

$$x(t) = x(0) + \int_0^t x'(\tau)\, d\tau \tag{45}$$

$$= -4 + \int_0^t \frac{d\tau}{\sqrt{2\tau + 1}} \tag{46}$$

$$= -4 + \sqrt{2\tau + 1}\Big|_0^t \tag{47}$$

$$= -4 + \sqrt{2t + 1} - 1 = \sqrt{2t + 1} - 5. \tag{48}$$

6.2 Part b

$$y(t) = [x(t)]^3 - 3x(t), \text{ so} \tag{49}$$

$$y'(t) = 3[x(t)]^2 x'(t) - 3x'(t) = 3\left([x(t)]^2 - 1\right) x'(t). \tag{50}$$

It now follows that

$$y'(t) = \frac{3\left[\left(\sqrt{2t+1} - 5\right)^2 + 1\right]}{\sqrt{2t+1}} \tag{51}$$

6.3 Part c

Location when $t = 4$ is given by,

$$x(4) = \sqrt{2 \cdot 4 - 1} - 5 = -2, \tag{52}$$

$$y(4) = x^3 - 3x = (-2)^3 - 3 \cdot (-2) = -2. \tag{53}$$

Also,

$$x'(4) = \frac{1}{\sqrt{2 \cdot 4 + 1}} = \frac{1}{3}, \tag{54}$$

while

$$y'(4) = 3\left([x(4)]^2 - 1\right) x'(4) = 3 \cdot (4 - 1) \cdot \frac{1}{3} = 3. \tag{55}$$

Speed at $t = 4$ is then

$$\sqrt{[x'(4)]^2 + [y'(4)]^2} = \sqrt{\frac{1}{9} + 9} = \frac{\sqrt{82}}{3} \sim 3.01846. \tag{56}$$