AP Calculus 1999 BC FRQ Solutions

Louis A. Talman, Ph.D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

September 17, 2017

1 Problem 1

1.1 Part a

See Figure 1

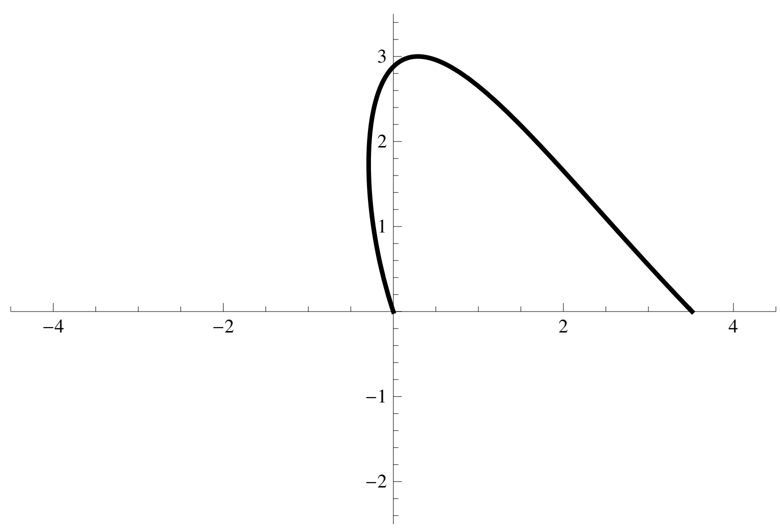

Figure 1: Problem 1, Part a

When $t = 0$, we have $x = 0$ and $y = 0$, so as t increases from 0 to π, the curve is traced out from the origin upward to the left, and around back down to its terminal point near $(7/2, 0)$.

1.2 Part b

$$x(t) = \frac{t^2}{2} - \ln(1+t), \text{ so} \tag{1}$$
$$x'(t) = t - \frac{1}{1+t}. \tag{2}$$

Thus, $x'(t)$ is defined for all $t \in (0, \pi)$. The equation $x'(t) = 0$ becomes

$$t - \frac{1}{1+t} = 0, \text{ or} \tag{3}$$
$$t^2 + t - 1 = 0. \tag{4}$$

By the Quadratic Formula, $x'(t) = 0$ for $t \in (0, \pi)$ only when $t = \frac{\sqrt{5}-1}{2} \sim 0.61803$, where it is easily checked that $x(t) < 0$. Because $x(0) = 0$ and $x(\pi) > 0$, the minimum value for $x(t)$ on $[0, \pi]$ occurs at this only critical point. We have

$$x\left(\frac{\sqrt{5}-1}{2}\right) \sim -0.09924 \text{ and} \tag{5}$$
$$y\left(\frac{\sqrt{5}-1}{2}\right) \sim 1.73830. \tag{6}$$

1.3 Part c

The particle is on the y-axis for $0 < T < \pi$ when $x(T) = 0$, or

$$\frac{T^2}{2} - \ln(1+T) = 0. \tag{7}$$

Numerical solution of this equation give $T \sim 1.28589$.

2 Problem 2

2.1 Part a

The area of the pictured region is

$$\int_{-2}^{2} \left(4 - x^2\right)\, dx = \left(4x - \frac{1}{3}x^3\right)\Big|_{-2}^{2} = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = \frac{32}{3}. \tag{8}$$

2.2 Part b

Revolving the pictured region about the x-axis produces a solid whose volume is

$$\pi \int_{-2}^{2} \left(16 - x^4\right) \, dx = \pi \left(16x - \frac{1}{5}x^5\right) \Big|_{-2}^{2} \tag{9}$$

$$= \pi \left(32 - \frac{32}{5}\right) - \pi \left(-32 + \frac{32}{5}\right) = \frac{256}{5}\pi \sim 160.84954. \tag{10}$$

2.3 Part c

The required equation is

$$\pi \int_{-2}^{2} \left[\left(k - x^2\right)^2 - (k-4)^2\right] \, dx = \frac{256}{5}\pi. \tag{11}$$

Solution is not required. However, a tedious integration reduces the equation to

$$\frac{64}{3}k - \frac{256}{5} = \frac{256}{5}, \tag{12}$$

which easily gives $k = 24/5$.

3 Problem 3

3.1 Part a

The midpoint Riemann sum with 4 subdivisions of equal length gives

$$\int_{0}^{24} R(t) \, dt \sim 10.4 \times 6 + 11.2 \times 6 + 11.3 \times 6 + 10.2 \times 6 = 258.6. \tag{13}$$

This means that approximately 258.6 gallons of water flows out of the pipe during the time interval $0 \leq t \leq 24$.

3.2 Part b

The function R is given differentiable on $[0, 24]$, so it must also be continuous on that interval. Moreover, $R(0) = 9.6 = R(24)$. Thus, R meets the requirements of Rolle's Theorem, and we may conclude that the must be a time, t, with $0 < t < 24$, such that $R'(t) = 0$.

3.3 Part c

The average rate of flow is approximately

$$\frac{1}{24}\int_0^{24}\frac{1}{79}\left(768+23t-t^2\right)\,dt=\frac{1}{24}\cdot\frac{1}{79}\left(768t+\frac{23}{2}t^2-\frac{1}{3}t^3\right)\Bigg|_0^{24}=\frac{852}{79}\text{ gallons/hour.}\tag{14}$$

4 Problem 4

4.1 Part a

The third degree Taylor polyomial T_3 about $x=2$ for f is

$$\begin{aligned}T_3(x)&=f(0)+f'(0)(x-2)+\frac{1}{2}f''(0)(x-2)^2+\frac{1}{6}f'''(0)(x-2)^3 &(15)\\ &=-3+5(x-2)+\frac{3}{2}(x-2)^2-\frac{4}{3}(x-2)^3. &(16)\end{aligned}$$

Thus,

$$F(1.5)\sim T_3(1.5)=-4.95833.\tag{17}$$

4.2 Part b

The Lagrange estimate for the error in this approximation is

$$|f(1.5)-T_3(1.5)|\leq\frac{M}{4!}|1.5-2|^4,\tag{18}$$

where M is any number such that $|f^{(4)}(x)|\leq M$ throughout the interval $[1.5,2]$. Therefore, it being given that $\left|f^{(4)}(x)\right|\leq 3$ throughout $[1.5,2]$,

$$|-4.95833-f(1.5)|\leq\frac{3}{24}|1.5-2|^4=0.0078125.\tag{19}$$

But

$$|-4.95833-(-5)|=0.04167>0.0078125,\tag{20}$$

so that $f(1.5)=-5$ is not possible.

4.3 Part c

$P_4(x)$, the fourth degree Taylor polynomial about $x = 0$ for $g(x) = f(x^2 + 2)$, can be obtained by expanding and truncating $T_3(x^2 + 1)$. Thus,

$$P_4(x) = -3 + 5x^2 + \frac{3}{2}x^4. \tag{21}$$

The coefficient of x in $Q_4(x)$ is $f'(0)$, and the coefficient of x^2 in $Q_4(x)$ is half of $f''(0)$. Hence, $f'(0) = 0$ and $f''(0) = 3/2 > 0$. By the Second Derivative Test, f must have a local minimum at $x = 0$.

5 Problem 5

5.1 Part a

On the interval $[2, 4]$, the graph is symmetric about the point $(3, 0)$, so the integral over $[2, 4]$ is zero. Consequently,

$$\int_1^4 f(t)\,dt = \int_1^2 f(t)\,dt, \tag{22}$$

and the latter integral is the area of the trapezoid whose corners are $(1, 0)$, $(2, 0)$, $(2, 1)$, and $(1, 4)$, or

$$\frac{4+1}{2} \cdot 1 = \frac{5}{2}. \tag{23}$$

Thus,

$$g(4) = \frac{5}{2}, \text{ and} \tag{24}$$

$$g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^{1} f(t)\,dt \tag{25}$$

is the negative of the area of a triangle of base 3, height 4, or -6.

5.2 Part b

By the Fundamental Theorem of Calculus,

$$g'(x) = \frac{d}{dx}\int_1^x f(t)\,dt = f(x). \tag{26}$$

Hence $f'(1) = f(1) = 4$.

5.3 Part c

The absolute minimum of $g(x)$ for $-2 \leq x \leq 4$ is to be found at either, on the one hand, at one of the points $x = -2$ or $x = 4$, or, on the other hand, at a value of x where $-2 < x < 4$ and $g'(x) = 0$. As we have seen in Part a, above, $g(-2) = -6$, and $g(4) = 5/2$. If $g'(x) = 0$, then by our first observation in Part b, above, $f(x) = 0$. This happens only at $x = 3$. But f, which is g' undergoes a change of sign from positive to negative at $x = 3$, so, by the First Derivative Test, g must have a local maximum—which, because f is not a constant function, cannot also be an absolute minimum for f—at $x = 3$. We see, thus, that the absolute minimum for g on $[-2, 4]$ is $g(-2) = -6$.

5.4 Part d

If g is to have an inflection point at a point, then g' must change from increasing to decreasing or from decreasing to increasing at that point. We can see from the graph that g' changes from increasing to decreasing at $x = 1$, but g' does not change its monotonicity at $x = 2$. So g has an inflection point at just one of the two points in question.

6 Problem 6

6.1 Part a

An equation for the required tangent line is

$$y = 6 + \frac{1 + e^3}{9}(x - 3). \tag{27}$$

Substitution of 3.1 for x gives $y \sim 6.23425$.

6.2 Part b

The Euler's Method recursion for this problem is

$$x_0 = 3; \tag{28}$$

$$y_0 = 6; \tag{29}$$

$$x_k = x_{k-1} + 0.05; \tag{30}$$

$$y_k = y_{k-1} + \frac{1 + e^{x_{k-1}}}{x_{k-1}^2} 0.05. \tag{31}$$

Applying this recursion twice in succession, we obtain

$$x_1 = 3.05; \tag{32}$$

$$y_1 = 6 + \frac{1+e^3}{9} \cdot 0.05 \sim 6.11714; \tag{33}$$

$$x_2 = 3.10; \tag{34}$$

$$y_2 = 6.11714 + \frac{1+e^{3.05}}{9.3025} \cdot 0.05 \sim 6.23601. \tag{35}$$

We have

$$f''(x) = \frac{d}{dx}\frac{1+e^x}{x^2} = \frac{xe^x - 2e^x - 2}{x^3}, \tag{36}$$

so $f''(x)$ is surely positive when $3 \leq x \leq 4$.. This means that tangent lines at points of the curve in the interval $3 \leq x \leq 3.1$ lie below the curve (locally, of course, and except at the point of tangency). For this reason, Euler's method underestimates each $y(x_k)$ when $k = 1, 2, \ldots$. Thus the value y_2 we computed above is smaller than the value at 3.1 of the actual solution to the initial value problem.

6.3 Part c

By the Fundamental Theorem of Calculus,

$$f(3.1) = f(3) + \int_3^{3.1} f'(t)\,dt \tag{37}$$

$$= 6 + \int_2^{3.1} \frac{1+e^t}{t^2}\,dt. \tag{38}$$

It isn't possible to evaluate this definite integral in terms of elementary function. Numerical integration gives $f(3.1) \sim 6.23777$.