AP Calculus 2000 BC FRQ Solutions

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1 Problem 1

1.1 Part a

We must first locate the first intersection of the curves $y = e^{-x^2}$ and $y = 1 - \cos x$ to the right of the y-axis. Solving numerically for b, we find that the smallest positive value of b for which $e^{-b^2} = 1 - \cos b$ is $b \sim 0.94194$. A numerical integration then gives the area of the region R as

$$\int_0^b \left[e^{-x^2} - (1 - \cos x)\right]\, dx \sim 0.59096. \tag{1}$$

1.2 Part b

By the method of washers and a numerical integration, the volume generated when R is revolved about the x-axis is

$$\pi \int_0^b \left[e^{-2x^2} - (1 - \cos x)^2\right]\, dx \sim 1.74661. \tag{2}$$

1.3 Part c

Another numerical integration gives this volume as

$$\pi \int_0^b \left[e^{-x^2} - (1 - \cos x)\right]^2\, dx \sim 0.46106. \tag{3}$$

2 Problem 2

2.1 Part a

From the graph of runner A's velocity, we see that her velocity at time $t = 2$ is $\frac{20}{3}$ meters per second. Runner B's velocity at time t is given as $\frac{24t}{2t+3}$, so runner B's velocity at time $t = 2$ is $\frac{24 \cdot 2}{2 \cdot 2 + 3} = \frac{48}{7}$ meters per second.

2.2 Part b

Acceleration is the derivative, taken with respect to time, of velocity. In the case of runner A, the slope of the velocity curve at time $t = 2$ is $\frac{10}{3}$, so her acceleration at time $t = 2$ is $\frac{10}{3}$ meters per second per second. For runner B, we have

$$\frac{d}{dt}\left(\frac{24t}{2t+3}\right)\bigg|_{t=2} = \frac{24(2t+3) - 24t \cdot 2}{(2t+3)^2}\bigg|_{t=2} = \frac{72}{(2t+3)^2}\bigg|_{t=2} = \frac{72}{49} \text{ meters/sec/sec.} \quad (4)$$

2.3 Part c

When velocity is non-negative, as in the circumstances of this problem, distance traveled is the integral of velocity. Hence, reasoning from the graph of runner A's velocity, we find that the distance, in meters, runner A covered during $0 \le t \le 10$ is the sum of the area of a triangle of base 3, altitude 10 and the area of a rectangle of base 7, altitude 10, or

$$\frac{1}{2} \cdot 3 \cdot 10 + 7 \cdot 10 = 85 \text{ meters} \quad (5)$$

During the same time interval, runner B covered

$$\int_0^{10} \frac{24t}{2t+3}\,dt = \int_0^{10} \left[\frac{24t+36}{2t+3} - \frac{36}{2t+3}\right] dt \quad (6)$$

$$= \int_0^{10} \left[12 - \frac{36}{2t+3}\right] dt \quad (7)$$

$$= \left[12t - 18\ln(2t+3)\right]\bigg|_0^{10} = 120 + 18\ln\frac{3}{23} \sim 83.33612 \text{ meters.} \quad (8)$$

3 Problem 3

We are given

1. The Taylor series for f about $x = 5$ converges to $f(x)$ on an unspecified interval of convergence.
2. [For all non-negative integers n]
$$f^{(n)}(5) = \frac{(-1)^n n!}{2^n(n+2)}. \tag{9}$$
3.
$$f(5) = \frac{1}{2}. \tag{10}$$
(This is given explicitly, even though it is a consequence of equation (9).)

3.1 Part a

The third-degree Taylor polynomial, $T_3(x)$ for f about 5 is

$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(5)}{n!}(x-5)^n \tag{11}$$
$$= \frac{1}{2} - \frac{1}{6}(x-5) + \frac{1}{16}(x-5)^2 - \frac{1}{40}(x-5)^3 \tag{12}$$

3.2 Part b

Writing $\sum_{k=0}^{\infty} a_k(x-5)^k$ for the Taylor series of f about $x = 5$, we know that

$$a_k = \frac{f^{(k)}(5)}{k!} = \frac{(-1)^k}{2^k(k+2)} \text{ for } k = 0, 1, \ldots. \tag{13}$$

Thus,

$$\lim_{k\to\infty} \left| \frac{a_{k+1}(x-5)^{k+1}}{a_k(x-5)^k} \right| = \lim_{k\to\infty} \frac{k+2}{2(k+3)}|x-5| = \frac{1}{2}|x-5|. \tag{14}$$

By the Ratio Test, the series converges absolutely when this limit is less than 1 and diverges when this limit is greater than 1. We conclude that the desired radius of convergence is 2.

3.3 Part c

When $x = 6$, the Taylor series becomes

$$\sum_{k=0}^{\infty} a_k(x-5)^k = \frac{1}{2} - \frac{1}{6} + \frac{1}{16} - \frac{1}{40} + \cdots + \frac{(-1)^k}{2^k(k+2)} + \cdots . \tag{15}$$

As k increases, both 2^k and $k+2$ increase, so the quotient $\frac{1}{2^k(k+2)}$ decreases monotonically to zero as $k \to \infty$. By the Alternating Series Test, the error in replacing $f(6)$ with the sum of the first seven terms of the series (that is, the degree six Taylor polynomial) is at most

$$\frac{1}{2^7(7+2)} = \frac{1}{128 \cdot 9} = \frac{1}{1152} < \frac{1}{1000}. \tag{16}$$

4 Problem 4

4.1 Part a

The acceleration vector is the derivative of the velocity vector, taken with respect to time:

$$\mathbf{a}(t) = \frac{d}{dt}\mathbf{v}(t) = \frac{d}{dt}\left\langle 1 - \frac{1}{t^2}, 2 + \frac{1}{t^2} \right\rangle \tag{17}$$

$$= \left\langle \frac{2}{t^3}, -\frac{2}{t^3} \right\rangle. \tag{18}$$

Thus,

$$\mathbf{a}(3) = \left\langle \frac{2}{27}, -\frac{2}{27} \right\rangle. \tag{19}$$

4.2 Part b

Position $\mathbf{r}(t)$ at time t is

$$\begin{aligned}
\mathbf{r}(t) &= \langle 2,6\rangle + \int_1^t \mathbf{v}(\tau)\,d\tau && (20)\\
&= \langle 2,6\rangle + \int_1^t \left\langle 1-\frac{1}{\tau^2}, 2+\frac{1}{\tau^2}\right\rangle d\tau && (21)\\
&= \langle 2,6\rangle + \left\langle \tau+\frac{1}{\tau}, 2\tau-\frac{1}{\tau}\right\rangle\Bigg|_1^t && (22)\\
&= \langle 2,6\rangle + \left\langle t+\frac{1}{t}, 2t-\frac{1}{t}\right\rangle - \langle 2,1\rangle && (23)\\
&= \left\langle t+\frac{1}{t}, 5+2t-\frac{1}{t}\right\rangle. && (24)
\end{aligned}$$

Thus,

$$\mathbf{r}(3) = \left\langle \frac{10}{3}, \frac{32}{3}\right\rangle. \tag{25}$$

4.3 Part c

We have $\mathbf{v}(t) = \langle x'(t), y'(t)\rangle$ so the slope $m(t)$ of the tangent line to the position curve at time t is

$$m(t) = \frac{y'(t)}{x'(t)} = \frac{2+t^{-2}}{1-t^{-2}} = \frac{2t^2+1}{t^2-1}. \tag{26}$$

For this to be 8 we must have $2t^2+1 = 8t^2-8$, or $6t^2 = 9$. The only positive solution for this equation is $t = \sqrt{3/2} \sim 1.22474$

4.4 Part d

From (25), position, $\mathbf{r}(t)$, is given by

$$\mathbf{r}(t) = \left\langle t+\frac{1}{t}, 5+2t-\frac{1}{t}\right\rangle \tag{27}$$

We let $\mathbf{P}(t) = \langle t, 5+2t\rangle$, and we note that

$$\|\mathbf{r}(t)-\mathbf{P}(t)\| = \left\|\left\langle \frac{1}{t}, -\frac{1}{t}\right\rangle\right\| = \frac{2}{|t|} \to 0 \text{ as } t \to \pm\infty. \tag{28}$$

The particle's path is therefore asymptotic to the path given by $\mathbf{P}(t)$ as $t \to \pm\infty$. But this path is the straight line through $(0,5)$ with slope 2, or the line with Cartesian equation $y = 2x + 5$.

5 Problem 5

5.1 Part a

If (x,y) is a point on the curve given by the equation $xy^2 - x^3y = 6$ and the equation defines y implicitly as a function of x near that point, then

$$\frac{d}{dx}\left(xy^2 - x^3y\right) = \frac{d}{dx}6; \tag{29}$$

$$y^2 + 2xy\frac{dy}{dx} - 3x^2y - x^3\frac{dy}{dx} = 0; \tag{30}$$

$$(2xy - x^3)\frac{dy}{dx} = 3x^2y - y^2; \tag{31}$$

$$\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}, \tag{32}$$

as required.

5.2 Part b

If the point $(1,y)$ lies on the curve, then

$$1 \cdot y^2 - 1^3 \cdot y = 6, \text{ or} \tag{33}$$

$$y^2 - y - 6 = 0; \tag{34}$$

$$(y-3)(y+2) = 0. \tag{35}$$

There are, consequently, two such points: $(1,3)$ and $(1,-2)$. We have

$$\left.\frac{dy}{dx}\right|_{(1,3)} = \frac{3 \cdot 1^2 \cdot 3 - 3^2}{2 \cdot 1 \cdot 3 - 1^3} = 0, \tag{36}$$

so that the line tangent to the curve at $(1,3)$ has equation $y = 3$.

At $(1,-2)$, we have

$$\left.\frac{dy}{dx}\right|_{(1,-2)} = \frac{3 \cdot 1^2 \cdot (-2) - (-2)^2}{2 \cdot 1 \cdot (-2) - (1)^3} = \frac{-10}{-5} = 2. \tag{37}$$

An equation for the line tangent to the curve at $(1, -2)$ is therefore $y = -2 + 2(x - 1)$, or $y = 2x - 4$.

5.3 Part c

At a point where the tangent to the curve is vertical, we can't expect that the equation defines y implicitly as a function of x, so differentiation with respect to x is meaningless. Therefore, we assume that the equation gives x as a function of y, and we carry out an implicit differentiation with respect to y:

$$\frac{d}{dy}\left(xy^2 - x^3y\right) = \frac{d}{dy}6; \tag{38}$$

$$2xy + y^2\frac{dx}{dy} - x^3 - 3x^2y\frac{dx}{dy} = 0; \tag{39}$$

$$\frac{dx}{dy} = \frac{x^3 - 2xy}{y^2 - 3x^2y}. \tag{40}$$

At a point with a vertical tangent, $\frac{dx}{dy}$ must vanish, so $x^3 - 2xy$ must be zero—that is $x = 0$ or $y = x^2/2$. But $xy^2 - x^3y = 6$, so $x = 0$ is not possible. if $y = x^2/2$, on the other hand, then

$$xy^2 - x^3y = 6 \text{ becomes} \tag{41}$$

$$x\left(\frac{x^2}{2}\right)^2 - x^3\left(\frac{x^2}{2}\right) = 6. \tag{42}$$

The only real solution for this equation is easily seen to be $x = -\sqrt[5]{24}$, and the corresponding point on the curve is $\left(-\sqrt[5]{24}, \sqrt[5]{18}\right)$.

We would like to conclude that the tangent line to the curve at the point $\left(-\sqrt[5]{24}, \sqrt[5]{18}\right)$ is vertical. Strictly speaking, we must check to be sure that the denominator on the right side of equation (40) doesn't vanish at this point before we may draw this conclusion, but the readers probably didn't care. (To see why this last step is necessary, consider the curve $y^2 = x^2$ at the origin.)

6 Problem 6

6.1 Part a

See Figure 1.

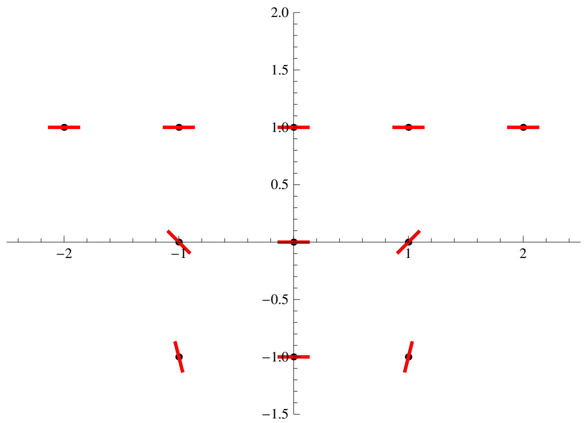

Figure 1: Problem 6, Part a

6.2 Part b

No solution could have the graph shown because the slope field requires any solution that passes through a point that lies on the line $y = 1$ to have zero slope at that point. The graph shown has non-zero slopes at the two points where it crosses the line $y = 1$.

6.3 Part c

If $y = f(x)$ is a solution to the differential equation $y' = x(y-1)^2$ for which $f(0) = -1$, Then

$$f'(\xi) = \xi[f(\xi) - 1]^2. \tag{43}$$

As the solution to a differential equation, f must be continuous at $\xi = 0$, so from $f(0) = -1$ it follows that $f(\xi) - 1 < 0$ for all ξ in some open interval I centered at $\xi = 0$. We may therefore divide both sides of (43) by $f(\xi) - 1$, and the resulting equation will be a statement about functions that are continuous, at least on the interval I. If x is any point in I, we may then write

$$\int_0^x \frac{f'(\xi)}{[f(\xi) - 1]^2}\, d\xi = \int_0^x \xi\, d\xi. \tag{44}$$

Equivalently,

$$-\frac{1}{f(\xi)-1}\bigg|_0^x = \frac{\xi^2}{2}\bigg|_0^x; \tag{45}$$

$$\frac{1}{1-f(x)} - \frac{1}{2} = \frac{x^2}{2}. \tag{46}$$

Solving for $f(x)$, we find that

$$f(x) = \frac{x^2-1}{x^2+1}. \tag{47}$$

6.4 Part d

We solve equation (43) for x^2 in terms of y, and we learn that

$$x^2 = \frac{1+y}{1-y}. \tag{48}$$

Now, whatever x may be, $x^2 \geq 0$, so there can be a y corresponding to a real number x if, and only if, the quantity $\frac{1+y}{1-y}$ is non-negative. For this to be so, either $y = -1$, or the quantities $1+y$ and $1-y$ have the same sign. Both can't be negative, because $1+y < 0 \Rightarrow y < -1$, while $1-y < 0 \Rightarrow 1 < y$, and no number is simultaneously smaller than -1 and larger then 1. On the other hand, both are positive when $-1 < y < 1$, so the range of this solution is $\{y\colon -1 \leq y < 1\}$.