AP Calculus 2001 BC FRQ Solutions

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August 27, 2017

1 Problem 1

1.1 Part a

The object is at $(4,5)$ when $t=2$ so the slope of the line tangent to the curve when $t=2$ is the value of

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \tag{1}$$

when $t=2$, or

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{3\sin 2^2}{\cos 2^3} = 3\sin 4 \sec 8. \tag{2}$$

An equation for the required tangent line is thus $y = 5 + 3\sin 4\sec 8(x-4)$.

1.2 Part b

The speed σ of the object is given by

$$\sigma(t) = \sqrt{[x'(t)]^2 + [y'(t)]^2}. \tag{3}$$

Thus

$$\sigma(2) = \sqrt{9\sin^2 4 + \cos^2 8} \tag{4}$$

1.3 Part c

Distance traveled during the interval $0 \le t \le 1$ is

$$\int_0^1 \sigma(t)\,dt = \int_0^1 \sqrt{9\sin^2 t^2 + \cos^2 t^3}\,dt \sim 2.37638, \tag{5}$$

where we have carried out the integration numerically.

1.4 Part d

The position of the object at t is given by

$$\begin{aligned}(x(t), y(t)) &= \left(x(2) + \int_2^t x'(\tau)\,d\tau, y(2) + \int_2^t y'(\tau)\,d\tau\right) &(6)\\ &= \left(4 + \int_2^t \cos\tau^3\,d\tau, 5 + 3\int_2^t \sin\tau^2\,d\tau\right). &(7)\end{aligned}$$

Thus

$$\big(x(3), y(3)\big) = \left(4 + \int_2^3 \cos\tau^3\,d\tau, 5 + 3\int_2^3 \sin\tau^2\,d\tau\right) \sim (3.95350, 4.90636), \tag{8}$$

where, again, we have integrated numerically.

2 Problem 2

2.1 Part a

We have

$$W'(12) \sim \frac{W(15) - W(9)}{15 - 9} = \frac{21 - 24}{6} = -\frac{1}{2}\text{ degrees C/day}. \tag{9}$$

2.2 Part b

The required trapezoidal approximation to the average value is

$$\frac{1}{15 - 0} \cdot \frac{20 + 2\cdot 31 + 2\cdot 28 + 2\cdot 24 + 2\cdot 22 + 21}{2} \cdot 3 = \frac{251}{10}. \tag{10}$$

2.3 Part c

If P is given by

$$P(t) = 20 + 10te^{-t/3}, \tag{11}$$

then

$$P'(t) = 10e^{-t/3} - \frac{10}{3}te^{-t/3}, \tag{12}$$

and

$$P'(12) = -30e^{-4} \sim -0.54947. \tag{13}$$

Ar the beginning of the twelfth day, the water temperature is decreasing[1] at a rate of about 0.54947 degrees Celsius per day.

2.4 Part d

The required average value is

$$\frac{1}{15}\int_0^{15} P(t)\,dt \sim 25.75743 \text{ degrees Celsius.} \tag{14}$$

3 Problem 3

3.1 Part a

When t is near 2, the graph shows that acceleration is near 15 ft/sec^2. This is a positive number, so velocity is increasing[2] in the vicinity of $t = 2$.

3.2 Part b

The portion of the acceleration curve on the interval $6 \leq t \leq 12$ is symmetric, about the point $(6, 0)$, with the portion of the acceleration curve on the interval $(0, 6)$. Consequently, the integral of acceleration from 0 to 12 (which is total change in velocity over that interval) is zero. Thus, velocity at $t = 12$ is 55 feet per second.

[1] But see the note to Problem 3, Part a.

[2] We have phrased our answer this way because the phrase "increasing at $t = 2$" is not defined in most calculus textbooks. In this context, the term "increasing" applies only to functions known on intervals.

3.3 Part c

The car's absolute maximum velocity for $0 \leq t \leq 18$ is 115 ft/sec, which is the velocity it attains when $t = 6$. Thereafter it decreases as long as acceleration is negative—that is, while $6 \leq t \leq 14$. Finally, it increases again while $14 \leq t \leq 18$. However, the area under the acceleration curve on the latter interval is smaller than the area between the acceleration curve and the t-axis on the interval $6 \leq t \leq 14$, so the total increase in velocity that accrues while $14 \leq t \leq 18$ does not balance out the total decrease that accrued while $6 \leq t \leq 14$.

This means that velocity attains its absolute maximum for $0 \leq t \leq 18$ when $t = 6$. We calculate this maximum value by finding the area of the trapezoid over the interval $0 \leq t \leq 6$, which is

$$\frac{2+6}{2} \cdot 15 = 60, \tag{15}$$

and adding it to the initial velocity, 55, to obtain a maximal velocity of 115 ft/sec.

3.4 Part d

The car never reaches a velocity of 0 ft/sec. In fact, the absolute minimum velocity attained by the car occurs when $t = 16$, and this velocity is the sum of 55 ft/sec, the area of the region above the t-axis in the interval $[0, 6]$, and the negative of the area of the region below the t-axis in the interval $[6, 16]$, or $55 + 60 - 105 = 10$ ft/sec.

4 Problem 4

4.1 Part a

If

$$h'(x) = \frac{x^2 - 2}{x} = \frac{\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)}{x}, \tag{16}$$

then $h'(x) = 0$ when $x = \pm\sqrt{2}$, so the graph of h has a horizontal tangent when $x = \pm\sqrt{2}$. We note that

- $h'(x) < 0$ for $x < -\sqrt{2}$;
- $h'(x) > 0$ for $-\sqrt{2} < x < 0$;

- $h'(x) < 0$ for $0 < x < \sqrt{2}$;
- $h'(x) > 0$ for $\sqrt{2} < x$.

Thus, by the First Derivative Test, h has a local minimum at $x = -\sqrt{2}$, and h has a local minimum at $x = \sqrt{2}$.

Note: The quantity $h'(0)$ is undefined, but $x = 0$ fails to be a critical point for h. This is because h itself need not be defined at $x = 0$.

4.2 Part b

We have

$$h''(x) = \frac{d}{dx}\left[x - 2x^{-1}\right] = 1 + \frac{2}{x^2}, \tag{17}$$

which is always positive—except, of course, when $x = 0$. Hence h is concave upward on $(-\infty, 0)$ and on $(0, \infty)$. Whether we should list the closures of these intervals depends very much on which of several inequivalent definitions for concavity we use; the decision does not affect grading.

4.3 Part c

The equation of the line tangent to the graph of h at $x = 4$ is

$$6 = h(4) + h'(4)(x - 4), \text{ or} \tag{18}$$

$$y = (-3) + \frac{4^2 - 2}{4}(x - 4). \tag{19}$$

This can be rewritten as

$$y = \frac{7}{2}x - 17. \tag{20}$$

4.4 Part d

We have $h''(x) = 1 + 2x^{-2}$, so that $h''(x) > 1$ for all $x \neq 0$. Thus, h' is increasing on $[4, \infty)$, and $h'(x) > h(4) = 7/2$ for all $x > 4$. Consequently,

$$h(x) - x(4) = \int_4^x h'(\xi)\, d\xi > \int_4^x \frac{7}{2}\, d\xi = \frac{7}{2}(x - 4), \tag{21}$$

again for all $x > 4$. Thus, when $x > 4$, we have

$$h(x) > \frac{7}{2}(x-4) + h(4) = \frac{7}{2}x - 17. \tag{22}$$

But the right-hand side of (22) is just the right-hand side of the equation of the tangent line to h at $(4, -3)$ as given in (20). Thus, the line tangent to the graph of $y = h(x)$ at $x = 4$ lies below the graph of h for $x > 4$.

5 Problem 5

5.1 Part a

If $f'(x) = -3xf(x)$, with $f(1) = 4$ and $\lim_{x\to\infty} f(x) = 0$, then

$$\begin{aligned} \int_1^\infty [-3xf(x)]\; dx &= \lim_{T\to\infty} \int_1^T f'(x)\, dx && (23) \\ &= \lim_{T\to\infty} [f(T) - f(1)] && (24) \\ &= 0 - 4 = -4. && (25) \end{aligned}$$

5.2 Part b

We have $f'(1) = -3 \cdot 1 \cdot 4 = -12$, so the linearization of f at the point $(1, 4)$ is

$$L_1(x) = 4 - 12(x-1). \tag{26}$$

Putting $x = 3/2$, we obtain

$$L_1\left(\frac{3}{2}\right) = 4 - 6 = -2, \tag{27}$$

and we take this as the approximate value of $f\left(\frac{3}{2}\right)$. Then the approximate slope of f when $x = \frac{3}{2}$ is

$$f'\left(\frac{3}{2}\right) = -3\left(\frac{3}{2}\right) f\left(\frac{3}{2}\right) \sim 9, \tag{28}$$

so the linearization corresponding to $x = \frac{3}{2}$ is

$$L_{3/2}(x) = -2 + 9\left(x - \frac{3}{2}\right). \tag{29}$$

Thus,

$$f(2) \sim L_{3/2}(2) = \frac{5}{2}. \tag{30}$$

5.3 Part c

If $f'(x) = -3xf(x)$, with $f(1) = 4$, then, f being a solution to a differential equation near $x = 1$, f must be a continuous function taking on positive values on some open interval centered at $x = 1$. In particular, f does not vanish in such an interval. Thus, for any x sufficiently close to 1, we may write

$$\int_1^x \frac{f'(\xi)}{f(\xi)}\, d\xi = -3 \int_1^x \xi\, d\xi, \tag{31}$$

or

$$\ln f(\xi)\bigg|_1^x = -\frac{3}{2}\xi^2\bigg|_1^x, \tag{32}$$

or

$$\ln f(x) - \ln 4 = \frac{3}{2} - \frac{3}{2}x^2. \tag{33}$$

Solving, we find that

$$f(x) = 4\exp\left[\frac{3}{2}(1-x^2)\right], \tag{34}$$

where we have used the common notation $\exp u$ for e^u.

6 Problem 6

6.1 Part a

The solution to this part of Problem 6 will follow from later work, and will be given then.

6.2 Part b

We have

$$f(x) - \frac{1}{3} = \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots, \tag{35}$$

so the series expansion for the quotient is

$$\frac{f(x) - 1/3}{x} = \frac{2}{3^2} + \frac{3}{3^3}x + \cdots + \frac{n+1}{3^{n+1}}x^{n-1} + \cdots \tag{36}$$

Thus

$$\lim_{x\to 0} \frac{f(x) - 1/3}{x} = \frac{2}{9}. \tag{37}$$

6.3 Parts a, c, & d

Integrating term by term, we have

$$\int_0^x f(\xi)\, d\xi = \frac{1}{3}x + \frac{1}{3^2}x^2 + \frac{1}{3^3}x^3 + \cdots + \frac{1}{3^{n+1}}x^{n+1} + \cdots \tag{38}$$

This latter is a geometric series with common ratio $x/3$, which converges when $|x| < 3$. Thus,

$$\int_0^x f(\xi)\, d\xi = \frac{x}{3} \cdot \frac{1}{1 - x/3} = \frac{x}{3 - x} \tag{39}$$

for $-3 < x < 3$.

In particular,

$$\int_0^1 f(\xi)\, d\xi = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n+1}} + \cdots = \frac{1}{3-1} = \frac{1}{2}, \tag{40}$$

and this is the solution to Parts c & d.

The interior of the interval of convergence for the integrated series must be the same as that for the original series. We note that this is (almost) the solution to Part a, above.

It remains to determine endpoint behavior of the original series. When $x = 3$, the original series becomes

$$\sum_{n=1}^{\infty} \frac{n+1}{3^{n+1}} 3^n = \frac{1}{3} \sum_{n=1}^{\infty} (n+1). \tag{41}$$

The terms of this series become infinite, so it diverges.

When $x = -3$ the series becomes

$$\sum_{n=1}^{\infty} \frac{n+1}{3^{n+1}} 3^n = \frac{1}{3} \sum_{n=1}^{\infty} (-1)^n (n+1), \tag{42}$$

and this series diverges for the same reason. The solution to Part a is that the interval of convergence for this series is $(-3, 3)$.