AP Calculus 2002 BC (Form B) FRQ Solutions

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1 Problem 1

1.1 Part a

See Figure 1.

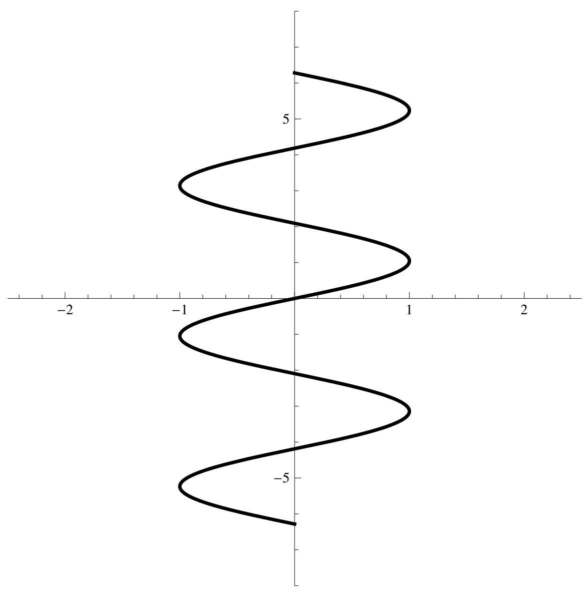

Figure 1: Problem 1, Part a

1.2 Part b

The range of $x(t) = \sin 3t$ is $(-1 \leq x \leq 1$. The range of $y(t) = 2t$ is $-2\pi \leq y \leq 2\pi$.

1.3 Part c

$x(t) = \sin 3t$ reaches its positive maximum of 1 at $t = -\pi/2$, at $t = \pi/6$, and at $t = 5\pi/6$ in the interval $-\pi \leq t \leq \pi$. Thus, the smallest positive value of t for which $x(t)$ is maximal is $t = \pi/6$.

Speed at time t, $\sigma(t)$, is the length of the velocity vector at that time:

$$\sigma(t) = \sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{9\cos^2 3t + 4}. \tag{1}$$

Thus,

$$\sigma\left(\frac{\pi}{6}\right) = \sqrt{9\cos^2 \frac{\pi}{2} + 4} = 2. \tag{2}$$

1.4 Part d

We obtain distance traveled during the time interval $a \leq t \leq b$ by integrating speed over $[a, b]$. Speed, found above, is given by $\sigma(t) = \sqrt{4 + 9\cos^2 3t}$. Thus, the distance traveled over $[-\pi, \pi]$ is, by numerical integration,

$$\int_{-\pi}^{\pi} \sqrt{4 + 9\cos^2 3t}\, dt \sim 17.97343. \tag{3}$$

But $5\pi \sim 15.71$, so the distance the particle travels when $-\pi \leq t \leq \pi$ is more than 5π.

2 Problem 2

2.1 Part a

If $P'(t) = 1 - 3e^{-0.2\sqrt{t}}$, then P' is continuous on its domain, and $P'(9) \sim -0.64643 < 0$. This means that $P(t)$, the amount of pollutant in the lake at time t, is decreasing in the vicinity of $t = 0$.

Note: We phrase our answer this way because the phrase "increasing at a point" is almost never defined in elementary calculus texts. It is possible, in fact, for a function to have a derivative that is positive at a point but for the function to be increasing in no open interval centered at that point. This can't happen, however, if the derivative is continuous at the point in question.

2.2 Part b

The exponential $e^{-0.2\sqrt{t}}$ is decreasing on the interval $[0, \infty)$, so P' is an increasing function on that interval. Moreover, $P'(0) = -2$, while $\lim_{t\to\infty} P'(t) = 1$. It follows that P' has exactly one zero, and that P has exactly one critical point in the interval—where, moreover, P' changes sign from negative to positive. Consequently, P has an absolute minimum at this critical point. We find the critical point by solving $1 - 3e^{-0.2\sqrt{t_0}} = 0$ to obtain $t_0 = 25(\ln 3)^2 \sim 30.17372$.

The amount of pollutant in the lake is at a minimum when $t = t_0 \sim 30.17372$ days.

2.3 Part c

By the Fundamental Theorem of Calculus, the amount $P(t)$ of pollutant in the lake at time t is given by

$$\begin{aligned} P(t) &= P(0) + \int_0^t P'(\tau)\, d\tau && (4) \\ &= 50 + \int_0^t \left[1 - 3e^{-0.2\sqrt{\tau}}\right] d\tau. && (5) \end{aligned}$$

Integrating numerically, we find that

$$\begin{aligned} P(t_0) &= 50 + \int_0^{25(\ln 3)^2} \left[1 - 3e^{-0.2\sqrt{\tau}}\right] d\tau && (6) \\ &\sim 35.10434 \text{ gallons.} && (7) \end{aligned}$$

If the lake is considered safe when $P(t) \leq 40$, the lake is safe when the amount of pollutant reaches its minimum value of about 35.10424.

2.4 Part d

The linearization L at $t = 0$ has equation

$$L(t) = 50 + P'(0)t \quad (8)$$
$$= 50 - 2t. \quad (9)$$

Thus, the linearization predicts the arrival of safety when $50 - 2t = 40$, or when $t = 5$.

Note: Newton's Method, together with repeated numerical integration, gives the arrival of safety at $t \sim 10.16000$. The lake then remains safe until $t \sim 56.47974$, after which an unsafe condition will prevail.

3 Problem 3

3.1 Part a

Solving numerically, we find that $\frac{3}{4}x = 4x - x^3 + 1$ and $x > 0$ when $x \sim 1.94045$. Thus, the intersection of the two curves is at $x = b$, where $b \sim 1.94945$. A_R, he area of the region R, is thus given by

$$A_R = \int_0^b \left[(4x - x^3 + 1) - \frac{3}{4}x \right] dx = \int_0^b \left[1 + \frac{13}{4}x - x^3 \right] dx \sim 4.51468. \quad (10)$$

The integral is elementary, but we know the upper limit of the integral only approximately, so we integrated numerically.

3.2 Part b

The volume V of the solid generated by revolving the region R about the x-axis is, by the method of washers,

$$V = \pi \int_0^b \left[(4x - x^3 + 1)^2 - \left(\frac{3}{4}x \right)^2 \right] dx \sim 57.46237. \quad (11)$$

This integral, too, is elementary, but as in Part a, we have carried out the integration numerically.

3.3 Part c

The perimeter, P_R, of the region R is

$$P_R = 1 + \int_0^b \sqrt{1 + \left(\frac{3}{4}\right)^2}\,dx + \int_0^b \sqrt{1 + (4 - 3x^2)^2}dx \tag{12}$$

$$= 1 + \frac{5}{4}b + \int_0^b \sqrt{9x^4 - 24x^2 + 17}\,dx. \tag{13}$$

Note: The integral is not elementary. Evaluation is not required, but numerical integration gives

$$P_R \sim 9.60048. \tag{14}$$

4 Problem 4

4.1 Part a

If

$$g(x) = 5 + \int_6^x f(t)\,dt, \tag{15}$$

then $g(6) = 5$, because $\int_6^6 g(t)\,dt = 0$. By the Fundamental Theorem of Calculus, $g'(x) = f(x)$, so, according to the graph given in the statement of the problem, $g'(6) = 3$.

Also, $g''(x) = f'(x)$, and because it is given that the line tangent to the curve $y = f(x)$ at $x = 6$ is horizontal, we know that $f'(6) = 0$. Hence, $g''(6) = 0$.

4.2 Part b

We know that $g'(x) = f(x)$, and we also know that g is decreasing on any interval where g' is negative. Moreover, if a continuous function is decreasing on an open interval it is also decreasing on the closure of that interval. From the graph given, we see that $f(x) < 0$ on $[-3, 0)$ and on $(12, 15]$. We conclude that g is decreasing on $[-3, 0]$ and that g is decreasing on $[12, 15]$.

Notes:

- In recent history, the readers have not cared about the endpoints of intervals of monotonicity.

- We must not conclude that g is decreasing on $[-3,0]\cup[12,15]$. In fact, values of $g(u)$ will be negative when $u\in[-3,0]$, and consequently smaller than any of the positive values that $g(v)$ assumes for each $v\in[12,15]$.

4.3 Part c

A function is concave downward on any open interval where its derivative is decreasing. But $g'(x)=f(x)$ is decreasing on $[6,15]$, and nowhere else. So g is concave downward on $(6,15)$. Whether to include the endpoints of this interval is highly dependent upon the definition one chooses for the term "concave downward". Several distinct definitions appear in different textbooks, so the choice should not affect scoring.

4.4 Part d

The required trapezoidal approximation is

$$\int_3^{15} f(t)\,dt \sim \frac{3}{2}\big[f(-3)+2f(0)+2f(3)+2f(6)+2f(9)+2f(12)+f(15)\big] \tag{16}$$

$$\sim \frac{3}{2}\big[(-1)+0+2+6+2+0+(-1)\big]=12. \tag{17}$$

5 Problem 5

5.1 Part a

If the line $y=-2$ is tangent to the solution curve $y=f(x)$ to the differential equation

$$y'=\frac{3-x}{y}, \tag{18}$$

then at the point of tangency we must have $y'=0$, whence $3-x=0$, which means that $x=3$. Because the y-coordinate of the point of tangency is $y=-2$ and solutions of differential equations are continuous on their domains, there is then an open interval I centered at $x=3$ and throughout which $f(x)<0$. We may assume that I does not contain zero. If $x\in I$ and $x<3$, then $y'=(3-x)/y<0$ because $3-x>0$ and $y<0$. If, on the other hand, $x\in I$ and $x>3$, then $y'=(3-x)/y>0$ because $3-x<0$ and $y<0$. Consequently, $x=3$ gives a critical point for f, and $f'(x)<0$ for x immediately to the left of $x=3$ but $f'(x)>0$ for x immediately to the right of $x=3$. It follows from the First Derivative Test that f has a local minimum at $x=3$.

5.2 Part b

If $y = g(x)$ is a solution to $y' = (3 - x)/y$ for which $g(6) = -4$, then

$$g'(x) = \frac{3 - x}{g(x)}, \text{ so} \tag{19}$$

$$g(x)g'(x) = 3 - x. \tag{20}$$

Integrating both sides of this latter equation from 6 to x, we have

$$\int_6^x g(\xi)g'(\xi)\,d\xi = \int_6^x (3 - \xi)\,d\xi; \tag{21}$$

$$\frac{1}{2}\big[g(\xi)\big]^2\Big|_6^x = -\frac{1}{2}(3 - \xi)^2\Big|_6^x; \tag{22}$$

$$\big[g(x)\big]^2 - \big[-4\big]^2 = -(3 - x)^2 + 9; \tag{23}$$

$$\big[g(x)\big]^2 = 16 + 6x - x^2. \tag{24}$$

Now $g(6) = -4$, and, again by continuity, $g(x)$ must have constant sign throughout some neighborhood of $x = 6$. Consequently, we choose the negative square root, and we write the solution:

$$g(x) = -\sqrt{16 + 6x - x^2}. \tag{25}$$

Note: We can solve this differential equation, but with the initial condition $f(3) = -2$, in the same way, leading to $f(x) = -\sqrt{-(x^2 - 6x + 5)}$. This gives

$$f'(x) = \frac{x - 3}{\sqrt{-x^2 + 6x - 5}}. \tag{26}$$

Then we can solve Part a of this problem by applying either the First Derivative Test or the Second Derivative Test at the critical point $x = 3$. The First Derivative Test is more efficient.

6 Problem 6

6.1 Part a

We are given that, for $-1 \le u < 1$,

$$\sum_{k=1}^{\infty} \frac{u^k}{k} = \ln \frac{1}{1 - u}. \tag{27}$$

Thus,

$$\ln \frac{1}{1+3x} = \ln \frac{1}{1-(-3x)} = \sum_{k-1}^{\infty} \frac{(-3x)^k}{k} \tag{28}$$

when $-1 \leq -3x < 1$, or, equivalently, when $-\frac{1}{3} < x \leq \frac{1}{3}$. Thus, the interval of convergence for the series of equation (28) is $-\frac{1}{3} < x \leq \frac{1}{3}$.

6.2 Part b

By (27),

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln \frac{1}{1-(-1)} = \ln \frac{1}{2} = -\ln 2. \tag{29}$$

6.3 Part c

We know that the series $\sum_{k=1}^{\infty} \frac{1}{k^q}$ converges if $q > 1$, but diverges if $q \leq 1$. Consequently, for $\sum_{k=1}^{\infty} \frac{1}{k^{2p}}$ to diverge, we need $2p \leq 1$. On the other hand, $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p}$ will be a convergent alternating series if the magnitudes of the terms decrease to zero—which will be so if $p > 0$. Thus, $p = \frac{1}{2}$ will meet the requirements of the problem—as will any p such that $0 < p \leq \frac{1}{2}$.

6.4 Part d

Reasoning as in Part c, above, we need to have $p \leq 1$ so that $\sum_{k-1}^{\infty} \frac{1}{k^p}$ diverges, but we want $2p > 1$ so that $\sum_{k-1}^{\infty} \frac{1}{k^{2p}}$ converges. We may therefore take p to be 1—or, in fact, any value $\frac{1}{2} < p \leq 1$.