AP Calculus 2002 BC FRQ Solutions

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1 Problem 1

1.1 Part a

See Figure 1 for a plot of the region given in this problem.

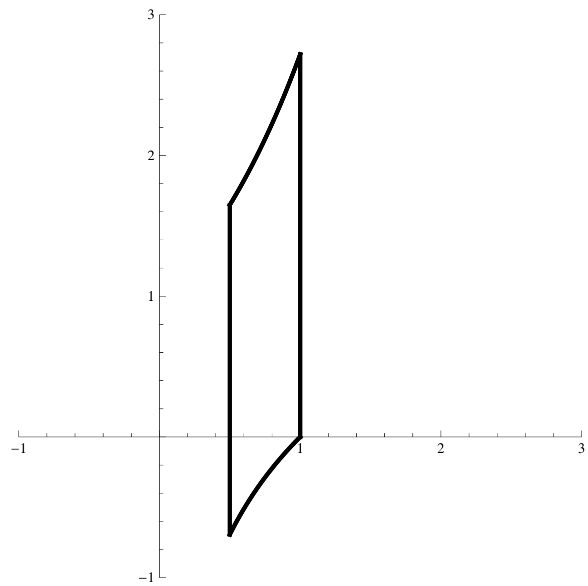

Figure 1: The region of Problem 1, Part a

The area of this region is

$$\int_{1/2}^{1} (e^x - \ln x)\ dx = (e^x + x - x\ln x)\Big|_{1/2}^{1} \tag{1}$$

$$= (e + 1 - 1\cdot 0) - \left(e^{1/2} + \frac{1}{2} - \frac{1}{2}\ln\frac{1}{2}\right) \tag{2}$$

$$= e - e^{1/2} + \frac{1}{2} - \ln\sqrt{2} \sim 1.22299. \tag{3}$$

1.2 Part b

We find, using the method of washers, that the required volume is

$$\pi \int_{1/2}^{1} \left[(4 - \ln x)^2 - (4 - e^x)^2\right] dx \sim 23.60949. \tag{4}$$

The integral is elementary, but tedious—and it requires repeated integrations by parts. We have saved time by carrying out the integration numerically.

1.3 Part c

First, we seek the critical points of $h(x) = e^x - \ln x$ in $(1/2, 2)$ together with the singularities of h' in the same interval. Now $h'(x) = e^x - x^{-1}$, which is defined for all $x \in (0, 1)$. Numeric solution of the equation $e^x - x^{-1} = 0$ gives one critical point in $(1/2, 1)$ at $x = x_1 \sim 0.56714$.

We know that the absolute extrema of $h(x)$ in $[1/2, 1]$ occur at one of the points $x = 1/2$, $x = x_1$, or $x = 1$. We have

$$h(1/2) \sim 2.34187; \tag{5}$$

$$h(x_1) \sim 2.33037; \tag{6}$$

$$h(1) = e \sim 2.71828. \tag{7}$$

$$\tag{8}$$

The absolute minimum value of $h(x)$ on $[1/2, 1]$ is $h(x_1) \sim 2.33037$, and the absolute maximum value of $h(x)$ on $[1/2, 1]$ is $h(1) = e$.

2 Problem 2

2.1 Part a

The number of people who have entered the park by the time $t = 17$ is

$$\int_9^{17} \frac{15600\,dt}{t^2 - 24t + 160} \sim 6004.27032. \tag{9}$$

To the nearest whole number, this is 6004.

Note: The integral is elementary:

$$\int \frac{dt}{t^2 - 24t + 160} = \int \frac{dt}{(t-12)^2 + 16} \tag{10}$$

$$= \frac{1}{4} \arctan \frac{t-12}{4} \Big|_9^{17} \tag{11}$$

$$\tag{12}$$

2.2 Part b

Revenue is given by

$$15 \int_9^{17} \frac{15600\,dt}{t^2 - 24t + 160} + 11 \int_{17}^{23} \frac{15600\,dt}{t^2 - 24t + 160} \sim 104048.16523 \tag{13}$$

To the nearest dollar, this is $\$104,048$.

2.3 Part c

By the Fundamental Theorem of Calculus,

$$H'(t) = \frac{15600}{t^2 - 24t + 160} - \frac{9890}{t^2 - 38t + 370}, \text{ so that} \tag{14}$$

$$H'(17) = -\frac{202690}{533} \sim 380.28143 \tag{15}$$

$H(t)$ gives the number of people in the partk at time t, where $9 \le t \le 23$. Thus, $H(17) \sim 3725$ gives the number of people in the park at 5:00 pm. $H'(T)$ gives the rate at which the number of people in the park is increasing at time t, again for $9 \le t \le 23$. $H'(17) \sim -380$ means that at 5:00 pm the number of people in the partk is decreasing at the rate of 380 people per hour.

2.4 Part d

As we have seen in Part c, above,

$$H'(t) = \frac{15600}{t^2 - 24t + 160} - \frac{9890}{t^2 - 38t + 370}. \tag{16}$$

See Figure 2 for a plot of $H'(t)$.

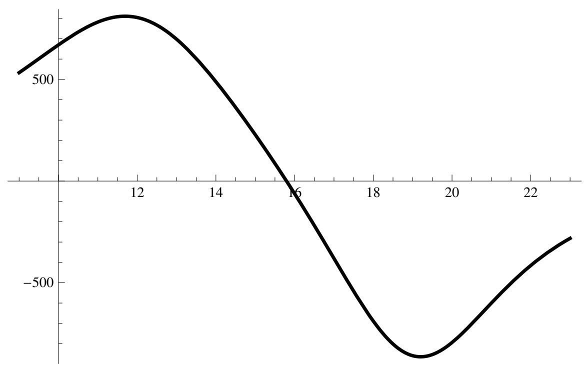

Figure 2: Problem 2, Part d: Plot of $H'(t)$

We see from the plot that there is a value t_0 near $t = 16$ for which $H'(t_0) = 0$, that $H'(t) > 0$ for $t < t_0$, and that $H'(t) < 0$ for $t > t_0$. So $H(t_0)$ must be the absolute maximum for $H(t)$, because $H(t)$ is increasing on $[9, t_0]$ and decreasing on $[t_0, 23]$. Setting $H(t) = 0$ and solving numerically gives $t_0 \sim 15.79481$.

We conclude that the model predicts the maximal number of people in the park just before 4:00 pm—when $t \sim 15.79481$, which is about 3:48 pm.

3 Problem 3

3.1 Part a

The slope m of a curve given parametrically by $x = x(t)$, $y = y(t)$ at the point corresponding to $t = t_0$ is

$$m(t_0) = \frac{y'(t_0)}{x'(t_0)}. \tag{17}$$

Thus,

$$m(2) = \frac{y'(2)}{x'(2)} \tag{18}$$

$$= \frac{18\sin 2 + \cos 2 - 1}{10 + 4\cos 2} \sim 1.79370. \tag{19}$$

3.2 Part b

Acceleration, $\mathbf{a}(t)$, at time t along the curve $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ is given by $\mathbf{a}(t) = \langle x''(t), y''(t)\rangle$. When $x(T) = 140$, we have (solving numerically) $T \sim 13.64708$. Then

$$\mathbf{a}(T) = \langle -4\sin T, -\sin T + (20 - T)\cos T - \sin T\rangle \tag{20}$$

$$\sim \langle -3.52917, 1.22573\rangle. \tag{21}$$

3.3 Part c

From the graph supplied with the statement of the problem, it is evident that the maximal value of y occurs at the first non-zero critical point for $y(t)$. But

$$y'(t) = -(1 - \cos t) + (20 - t)\sin t. \tag{22}$$

Equating this latter to 0 and solving numerically, we find that the desired critical point lies at $t = t_0 \sim 3.02393$. The speed at this instant, $\sigma(t_0)$, is given by

$$\sigma(t_0) = \sqrt{[x'(t_0)]^2 + [y'(t_0)]^2} \tag{23}$$

$$= x'(t_0) = 10 + 4\cos[t_0) \sim 6.02766. \tag{24}$$

3.4 Part d

$y(t) = (20 - t)(1 - \cos t)$, so $y(t)$ vanishes in $(0, 18)$ only where $\cos t = 1$. This happens only at $t_1 = 2\pi$ and at $t_2 = 4\pi$. The average speed, $\overline{\sigma}$, over the interval $t_1 \leq t \leq t_2$ is

$$\overline{\sigma} = \frac{1}{t_2 - t_1}\int_{2\pi}^{4\pi} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2}\, d\tau \tag{25}$$

$$= \frac{1}{2\pi}\int_{2\pi}^{4\pi} \sqrt{(10 + 4\cos\tau)^2 + [(20 - \tau)\sin\tau + \cos\tau - 1]^2}\, d\tau \tag{26}$$

Evaluation is not required. However, numerical integration gives

$$\overline{\sigma} \sim 12.50440 \tag{27}$$

4 Problem 4

4.1 Part a

The function g is given by

$$g(x) = \int_0^x F(t)\,dt \tag{28}$$

Thus, $g(-1)$ is the negative of the area of a triangle of base 1 and height 3, or $\frac{3}{2}$.

The value $g'(-1)$ is, by the Fundamental Theorem of Calculus, $f(-1) = 0$.

The value $g''(-1)$ is, by the Fundamental Theorem of Calculus again, $f'(-1)$. But in the vicinity of $x = -1$, the graph of the function f is a straight line of slope 3, so $g''(-1) = f'(-1) = 3$.

4.2 Part b

The function g is increasing on the closures of intervals where $g'(x) = f(x) > 0$. Thus g is increasing on $[-1, 1]$.

Note: It can be shown that a function which is continuous on an interval $[a, b]$ and increasing on (a, b) must also be increasing on $[a, b]$, so even though $g'(-1) = 0 = g'(1)$, the function g is nevertheless increasing on $[-1, 1]$. In the past, the readers have nevertheless accepted $(-1, 1)$ for the answer to a question such as this one.

4.3 Part c

As we have indicated above, $g''(x) = f'(x)$. We know that

$$f'(x) = \begin{cases} 3 & \text{for } -2 < x < 0 \\ \\ -3 & \text{for } 0 < x < 2 \end{cases} \tag{29}$$

Thus g is concave downward on the interval $(0, 2)$, where $g''(x) = f'(x) < 0$.

Note: Calculus textbooks vary in which of several inequivalent definitions of concavity they give. The question of whether to include endpoints of these intervals depends on which of these definitions we choose—and upon a careful reading of the definition we pick.

4.4 Part d

A pair of integrations shows that

$$g(x) = \begin{cases} 3x + \dfrac{3}{2}x^2 & \text{for } -2 \le x \le 0 \\ \\ 3x - \dfrac{3}{2}x^2 & \text{for } 0 < x \le 2. \end{cases} \tag{30}$$

See Figure 3 for the required graph. (We could have solved Parts a–c using this alternate description of g.)

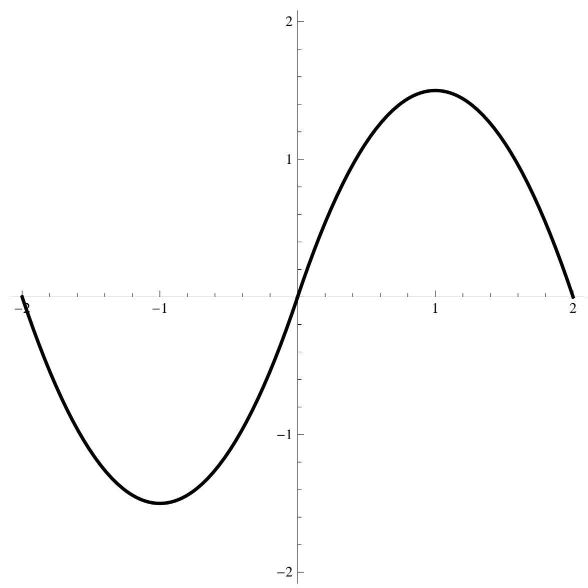

Figure 3: Graph for Problem 4, part d

5 Problem 5

5.1 Part a

See Figure 4.

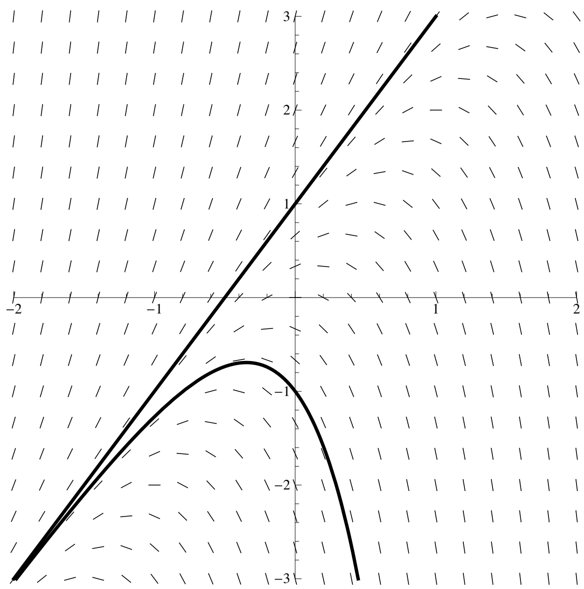

Figure 4: Problem 5, Part a

5.2 Part b

If $f(0) = 1$, then $f'(0) = 2 \cdot 1 - 4 \cdot 0 = 2$, so

$$f(0.1) \sim f(0) + f'(0) \cdot (0.1) = 1 + 2 \cdot (0.1) = 1.2 \tag{31}$$

But then

$$f'(0.1) \sim 2 \cdot f(0.1) - 4 \cdot (0.1) \sim 2 \cdot (1.2) - 4 \cdot (0.1) = 2, \tag{32}$$

so that

$$f(0.2) \sim f(0.1) + f'(0.1) \cdot (0.1) \sim 1.2 + 2 \cdot (0.1) = 1.4. \tag{33}$$

5.3 Part c

If $f(x) = 2x + b$ is to be a solution of $y' = 2y - 4x$, we must have

$$\begin{aligned} 2 &= \frac{d}{dx}(2x + b) && (34)\\ &= f'(x) && (35)\\ &= 2f(x) - 4x && (36)\\ &= 2(2x + b) - 4x && (37)\\ &= 2b, && (38) \end{aligned}$$

and we see that $b = 1$.

5.4 Part d

If g is a function that satisfies the equation $y' = 2y - 4x$ with initial condition $g(0) = 0$, then $g'(0) = 2 \cdot 0 - 4 \cdot 0 = 0$. Thus, g has a critical point at $x = 0$. But

$$\begin{aligned} g''(x) &= \frac{d}{dx}\left[g'(x)\right] && (39)\\ &= \frac{d}{dx}\left[2g(x) - 4x\right] && (40)\\ &= 2g'(x) - 4, \text{ so that} && (41)\\ g''(0) &= 2g'(0) - 4 = -4 < 0. && (42) \end{aligned}$$

By the Second Derivative Test, g has a local minimum at the point $(0, 0)$.

Note: If $y = \varphi(x)$ is a solution of $y' = 2y - 4x$, then

$$\begin{aligned} \varphi'(x) - 2\varphi(x) &= -4x; && (43)\\ e^{-2x}\varphi'(x) - 2e^{-2x}\varphi(x) &= -4xe^{-2x}; && (44)\\ \frac{d}{dx}\left[e^{-2x}\varphi(x)\right] &= -4xe^{-2x}. && (45) \end{aligned}$$

It follows that

$$\int_0^x \frac{d}{d\xi}\left[e^{-2\xi}\varphi(\xi)\right] d\xi = -4\int_0^x \xi e^{-2\xi}\, d\xi, \tag{46}$$

or

$$\left[e^{-2\xi}\varphi(\xi)\right]\Big|_0^x = e^{-2\xi}(2\xi + 1)\Big|_0^x \tag{47}$$

and

$$e^{-2x}\varphi(x) - \varphi(0) = e^{-2x}(2x+1) - 1. \tag{48}$$

If it is also required that $\varphi(0) = a$ for some constant a, then $\varphi(x) = (a-1)e^{2x}+2x+1$.

6 Problem 6

6.1 Parts a & b

If

$$f(x) = \sum_{n=0}^{\infty} \frac{(2x)^{n+1}}{n+1} \tag{49}$$

then term by term differentiation gives

$$f'(x) = 2\sum_{n=0}^{\infty}(2x)^n \tag{50}$$

$$= 2 + 4x + 8x^3 + 16x^2 + \cdots + 2^{n+1}x^n + \cdots . \tag{51}$$

(This is Part b.)

Both series have the same radius of convergence. But the latter series is a geometric series with common ratio $2x$, so it converges where $|2x| < 1$, or throughout the interval $-\frac{1}{2} < x < \frac{1}{2}$, and the radius of convergence for both series is $\frac{1}{2}$.

It remains, then, to decide what happens to the original series when $x = \pm\frac{1}{2}$.

When $x = \frac{1}{2}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n+1}. \tag{52}$$

This is the harmonic series, which diverges.

When $x = -\frac{1}{2}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}. \tag{53}$$

This is the alternating harmonic series, which converges.

We conclude that the interval of convergence is $\left[-\frac{1}{2}, \frac{1}{2}\right)$.

6.2 Part c

We have seen above that

$$f'(x) = 2\sum_{n=0}^{\infty}(2x)^n \tag{54}$$

is a geometric series with common ratio $2x$. Consequently, on $\left(-\frac{1}{2}, \frac{1}{2}\right)$,

$$f'(x) = \frac{2}{1-2x}, \text{so that} \tag{55}$$

$$f'\left(-\frac{1}{3}\right) = \frac{2}{1-2\cdot\left(-\frac{1}{3}\right)} = \frac{6}{5}. \tag{56}$$