AP Calculus 2003 BC (Form B) FRQ Solutions

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1 Problem 1

1.1 Part a

If $f(x) = 4x^2 - x^3$ and $g(x) = 18 - 3x$, then the curves have an intersection in the first quadrant where $x = 3$. Setting $f(x) = g(x)$ we find that $x^3 - 4x^2 - 3x + 18 = 0$. We note that $x = 3$ is a solution of this equation, and that $f(3) = g(3) = 9$. Moreover, $f'(x) = 8x - 3x^2$, and thus, $f'(3) = -3$, which is precisely the slope of the line $y = 18 - 3x = g(x)$. It follows that the line $y = 18 - 3x$ is the tangent line to the graph of $y = f(x)$ at the point $x = 3$.

1.2 Part b

The solutions of the equation $f(x) = 0$ are $x = 0$ and $x = 4$. The solution of the equation $18 - 3x = 0$ is $x = 6$

The region R extends horizontally from $x = 3$ on the left to $x = 6$ on the right, so the area, A_R of R is given by

$$A_R = \int_3^4 \left[(18 - 3x) - \left(4x^2 - x^3\right)\right] dx + \int_4^6 (18 - 3x)\, dx \tag{1}$$

Now

$$\int_3^4 \left[(18-3x)-\left(4x^2-x^3\right)\right] dx = \left[18x - \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4\right]\Bigg|_3^4 \tag{2}$$

$$= \left(72 - 24 - \frac{256}{3} + 64\right) - \left(54 - \frac{27}{2} - 36 + \frac{81}{4}\right) \tag{3}$$

$$= \frac{80}{3} - \frac{99}{4} = \frac{23}{12} \sim 1.91667, \tag{4}$$

while

$$\int_4^6 (18-3x)\, dx = \left(18x - \frac{3}{2}x^2\right)\Bigg|_4^6 \tag{5}$$

$$= (108-54) - (72-24) = 54 - 48 = 6. \tag{6}$$

Thus

$$A_R = \frac{23}{12} + 6 = \frac{95}{12} \sim 7.91667. \tag{7}$$

1.3 Part c

The curve $y = 4x^2 - x^3$ intersects the x-axis, as we have seen in Part a, above, at $x = 0$ and at $x = 4$. thus, the volume generated when the region R is revolved about the x-axis is

$$\pi \int_0^4 \left(4x^2 - x^3\right)^2 dx = \pi \int_0^4 \left(16x^4 - 8x^5 + x^6\right) dx \tag{8}$$

$$= \pi \left(\frac{16}{5}x^5 - \frac{4}{3}x^6 + \frac{1}{7}x^7\right)\Bigg|_0^4 = \frac{16384}{105}\pi \sim 490.20813. \tag{9}$$

Remark: This is a calculator-active problem, and we can save time by doing the integrations of Parts b and c numerically.

2 Problem 2

2.1 Part a

The circle of radius $\sqrt{2}$ has equation $y = \sqrt{2-x^2}$ in the first quadrant and extends over the interval $0 \leq x \leq \sqrt{2}$ there. The first-quadrant portion of the circle of radius 1 has equation $y = \sqrt{1-(x-1)^2}$ and extends over the interval $0 \le x \le 2$ there. Using the fact that the two semi-circles intersect at $x = 1$, the area, A_R, of the region R is therefore given by

$$A_R = \int_0^1 \sqrt{1-(x-1)^2}\,dx + \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx. \tag{10}$$

2.2 Part b

The circle of radius $\sqrt{2}$ has equation $x = \sqrt{2-y^2}$ in the first quadrant and extends over the interval $0 \le y \le \sqrt{2}$ there. The first-quadrant portion of the circle of radius 1 that lies inside the larger circle has equation $x = 1 - \sqrt{1-y^2}$ and extends over the interval $0 \le y \le 1$. Thus,

$$A_R = \int_0^1 \left[\sqrt{2-y^2} - 1 + \sqrt{1-y^2}\right] dy. \tag{11}$$

2.3 Part c

The segment that connects the origin to the first-quadrant intersection point of the two circles, which is $(1,1)$ in cartesian coordinates, lies in the ray $\theta = \pi/4$. The area of the sector of the large circle that lies below this segment but above the positive x-axis is a part of R and has area

$$\frac{1}{2}\int_0^{\pi/4} r^2\,d\theta = \int_0^{\pi/4} d\theta. \tag{12}$$

What remains of R after this sector is removed has bounding curve $r = 2\cos\theta$, where $\dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{2}$. The area of this region is therefore

$$\frac{1}{2}\int_{\pi/4}^{\pi/2} r^2\,d\theta = 2\int_{\pi/4}^{\pi/2} \cos^2\theta\,d\theta. \tag{13}$$

It now follows that the area A_R, being the sum of these two areas is given by

$$A_R = \int_0^{\pi/4} d\theta + 2\int_{\pi/4}^{\pi/2} \cos^2\theta\,d\theta \tag{14}$$

Note: Evaluation is not required for any of these integrals. But, of the expressions on the right-hand sides of equations (10), (11), and (14), the last one is most easily evaluated. we have

$$\int_0^{\pi/4} d\theta + 2\int_{\pi/4}^{\pi/2} \cos^2\theta \, d\theta = \int_0^{\pi/4} d\theta + \int_{\pi/4}^{\pi/2} (1+\cos 2\theta)\, d\theta \tag{15}$$

$$= \theta\Big|_0^{\pi/4} + \left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{\pi/4}^{\pi/2} = \frac{1}{2}(\pi - 1). \tag{16}$$

3 Problem 3

3.1 Part a

Average radius is $\displaystyle\frac{1}{720}\int_0^{360} B(x)\, dx$.

3.2 Part b

The required midpoint Riemann sum is

$$\frac{1}{720}\big[20\cdot(120-0) + 30\cdot(240-120) + 24\cdot(360-240)\big] = 14 \tag{17}$$

3.3 Part c

The integral $\displaystyle\pi\int_{125}^{275}\left[\frac{B(x)}{2}\right]^2 dx$ gives the volume, in cubic centimeters, of the segment of the blood vessel that extends from $x = 125$ mm to $x = 275$ mm.

3.4 Part d

The function B is given twice differentiable, so if $0 \le a < b \le 360$, then B is continuous on $[a,b]$ and differentiable on (a,b). Also, B' is continuous on $[a,b]$ and differentiable on (a,b). (We interpret continuity and differentiability at an endpoint of an interval in the appropriate one-sided sense.) Consequently, Rolle's Theorem can be applied to B on any interval, $0 \le a < b \le 360$ for which $B(a) = B(b)$. Similarly for B'.

We have $B(60) = B(180)$. By Rolle's Theorem, then, there is a number $\xi_1 \in (60, 180)$ for which $B'(\xi_1) = 0$. We also have $B(240) = B(360)$, so—again by Rolle's Theorem—there is a number $\xi_2 \in (240, 360)$ such that $B'(\xi_2) = 0$. It is clear that $\xi_1 < \xi_2$ because we know that $\xi_1 < 180 < 240 < \xi_2$.

Now $B'(\xi_1) = 0 = B'(\xi_2)$, and a third application of Rolle's Theorem, this time to B' on the interval $[\xi_1, \xi_2]$, yields a number $\eta \in (\xi_1, \xi_2)$ such that $B''(\eta) = 0$. Noting that $0 < \xi_1 < \eta < \xi_2 < 360$, we conclude that we have found $\eta \in (0, 360)$ such that $B''(\eta) = 0$

4 Problem 4

4.1 Part a

We are given $x(t) = 2e^{3t} + e^{-7t}$ and $y(t) = 3e^{3t} - e^{-2t}$. The velocity vector, $\mathbf{v}(t)$, at time t is

$$\begin{aligned} \mathbf{v}(t) &= \left\langle x'(t), y'(t) \right\rangle && (18) \\ &= \left\langle 6e^{3t} - 7e^{-7t}, 9e^{3t} + 2e^{-2t} \right\rangle && (19) \end{aligned}$$

Thus

$$\mathbf{v}(0) = \langle -1, 11 \rangle, \tag{20}$$

and speed at time $t = 0$ is

$$|\mathbf{v}(0)| = \sqrt{(-1)^2 + (11^2)} = \sqrt{122}. \tag{21}$$

4.2 Part b

$$\begin{aligned} \frac{dy}{dx} &= \frac{y'(t)}{x'(t)} && (22) \\ &= \frac{9e^{3t} + 2e^{-2t}}{6e^{3t} - 7e^{-7t}} && (23) \\ &= \frac{9 + 2e^{-5t}}{6 - 7e^{-10t}} \to \frac{3}{2} \text{ as } t \to \infty. && (24) \end{aligned}$$

4.3 Part c

The tangent line is horizontal when $\frac{dy}{dx}$ is zero. By Part b, above, this can happen only when $9e^{3t} + 2e^{-2t} = 0$. But both exponentials are always positive, so $\frac{dy}{dx}$ never vanishes and there are no horizontal tangent lines to this curve.

4.4 Part d

We have seen, in Part c, above, that the tangent vector to this curve never vanishes, because its second component never vanishes). But if the tangent vector never vanishes, the tangent line is vertical precisely when the tangent vector is vertical, and this happens when $6e^{3t} - 7e^{-7t} = 0$. This is equivalent to $t = -\frac{1}{10}\ln\frac{6}{7}$, so the tangent line to the curve is horizontal at the point that corresponds to $t = \dfrac{1}{10}\ln\dfrac{7}{6}$.

Note: The analysis becomes considerably more difficult if the velocity vector ever vanishes.

5 Problem 5

5.1 Part a

$g(3)$ is the sum of the area of a rectangle of height 2, base 1, with the area of a triangle of height 2, base 1, which is $2+1 = 3$. By the Fundamental Theorem of Calculus, $g'(x) = f(x)$, so $g'(3) = f(3) = 2$. In the interval $[2, 4]$, the Fundamental Theorem of Calculus tells us that

$$g'(x) = f(x) = f(4) + \frac{f(4) - f(2)}{4 - 2}(x - 4) \tag{25}$$

$$= 0 + \frac{0 - 4}{4 - 2}(x - 4) = -2(x - 4) \tag{26}$$

$$= 8 - 2x, \tag{27}$$

so

$$g''(x) = -2 \tag{28}$$

on $[2, 4]$. Hence, $g''(3) = -2$.

5.2 Part b

The average rate of change of g on $[0, 3]$ is

$$\frac{g(3) - g(0)}{3 - 0} = \frac{3 - (-4)}{3} = \frac{7}{3}. \tag{29}$$

See Part a, above, for the calculation of $g(3)$. To find $g(0)$ we simply observe that

$$g(0) = \int_2^0 f(t)\,dt, \tag{30}$$

which is the negative of the area of a triangle of base 2, height 4, which is -4.

5.3 Part c

By the Fundamental Theorem of Calculus, $g'(x) = f(x)$. Thus, on the interval $(0, 3)$, the function $g'(x)$ takes on its average value $\frac{7}{3}$ just twice—where the horizontal line $y = \frac{7}{3}$ intersects the graph of f.

One intersection lies in the interval $[0, 2]$ where $f(x) = 2x$. Thus, this intersection is at $x = \frac{7}{6}$. The other intersection lies in the interval $[2, 4]$, where $f(x) = 8 - 2x$, as we have seen in Part a, above. This intersection must therefore be at $x = \frac{17}{6}$.

We conclude that the only such points lie at $x = \frac{7}{6}$ and at $x = \frac{17}{6}$.

5.4 Part d

Inflection points occur where the montonicity of the derivative changes from increasing to decreasing, or vice versa. There are two such points for $g'(x) = f(x)$ (which equality we know from the Fundamental Theorem of Calculus). They are at $x = 2$ and at $x = 5$.

6 Problem 6

6.1 Part a

The Tayor series at $x = 2$ for f has general term $a_k(-2)^k$, where

$$a_k = \frac{f^{(k)}(2)}{k!} = \frac{(k+1)!}{3^k \cdot k!} = \frac{k+1}{3^k}. \tag{31}$$

Thus, in the interval of convergence,

$$f(x) = 1 + \frac{2}{3}(x-2) + \frac{1}{3}(x-2)^2 + \frac{4}{27}(x-2)^3 + \ldots + \frac{k+1}{3^k}(x-2)^k + \ldots . \tag{32}$$

6.2 Part b

We have

$$\lim_{k\to\infty}\left(\left|\frac{k+2}{3^{k+1}}(x-2)^{k+1}\right| / \left|\frac{k+1}{3^k}(x-2)^k\right|\right) = \frac{1}{3}|x-2|\lim_{k\to\infty}\frac{1+\frac{2}{k}}{1+\frac{1}{k}} \tag{33}$$

$$= \frac{1}{3}|x-2|, \tag{34}$$

This limit is less than one when $|x-2| < 3$, so, by the Ratio Test, the radius of convergence for the series is 3.

6.3 Part c

By the Fundamental Theorem of Calculus, we have

$$g(x) = g(2) + \int_2^x f(\xi)\,d\xi \tag{35}$$

$$= 3 + \int_2^x \left[1 + \frac{2}{3}(\xi-2) + \frac{1}{3}(\xi-2)^2 + \frac{4}{27}(\xi-2)^3 + \cdots + \frac{k+1}{3^k}(x-2)^k + \cdots\right] d\xi. \tag{36}$$

We may integrate the series term by term as long as $|x-2| < 3$, and, doing so, we find that

$$g(x) = 3 + (x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{3^2}(x-2)^3 + \cdots + \frac{1}{3^{k-1}}(x-2)^k + \cdots . \tag{37}$$

6.4 Part d

The radius of convergence of a power series obtain by integrating a power series is identical to that of the original series. If $x = -2$, then $|x-2| = |-4| = 4 > 3$, so $x = -2$ lies outside the interval of convergence for the original series—and therefore for the series for g. The Taylor series for g that we obtained in Part c, above, therefore diverges at $x = -2$.

Note: There are other approaches to Part d. We can observe that when $x = -2$, the k-th term of the series in (37) becomes

$$\frac{1}{3^{k-1}}(x-2)^k = -4\left(-\frac{4}{3}\right)^{k-1}, \tag{38}$$

which doesn't go to zero as k becomes infinite.

Still another possibility is to note that the series of (37) is a geometric series whose common ratio is $\dfrac{x-2}{3}$, and that the magnitude of this ratio exceeds 1 when $x = -2$. The observation that this series is geometric gives us, not only an alternate way to determine the radius of convergence for both series, but a way to sum the series when $|x-2| < 3$:

$$g(x) = 3\sum_{k=0}^{\infty}\left(\frac{x-2}{3}\right)^k = \frac{3}{1-\frac{x-2}{3}} = \frac{9}{5-x}. \tag{39}$$

From this, it is immediate that

$$f(x) = g'(x) = \frac{9}{(5-x)^2}. \tag{40}$$