AP Calculus 2003 BC FRQ Solutions

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1 Problem 1

1.1 Part a

The two curves intersect when $x = a$, where $\sqrt{a} = e^{-3a}$. Solving numerically, we find that $a \sim 0.23873$. Thus, we find (after a numerical integration) that the area of the region R is

$$\int_a^1 \left(\sqrt{x} - e^{-3x}\right)\, dx \sim 0.44263. \tag{1}$$

Note: The exact integral is

$$\int_a^1 \left(\sqrt{x} - r^{-3x}\right)\, dx = \frac{1}{3}\left[2x^{3/2} + e^{-3x}\right]\Big|_a^1 \tag{2}$$

$$= \frac{1}{3}\left(2 + e^{-3}\right) - \frac{1}{3}\left(2a^{3/2} + e^{-3a}\right). \tag{3}$$

However, we know a only approximately, so "exact" integration is misleading.

1.2 Part b

This problem is most easily solved using the method of washers. The required volume, V, is

$$V = \pi \int_a^1 \left[\left(1 - e^{-3x}\right)^2 - \left(1 - \sqrt{x}\right)^2\right] dx \tag{4}$$

$$\sim 1.42356. \tag{5}$$

It is also possible—but probably not wise—to use the method of shells:

$$V = 2\pi \int_{e^{-3}}^{\sqrt{a}} (1-y)\left(1+\frac{1}{3}\ln y\right)\,dy + 2\pi \int_{\sqrt{a}}^{1} (1-y)(1-y^2)\,dy. \tag{6}$$

Note: For the sake of completeness (See the Note to Part a, above), we record the "exact" value:

$$V = \frac{1}{6}\pi\left(-8a^{3/2} + 3a^2 + e^{-6a} - 4e^{-3a} + 4e^{-3} - e^{-6} + 5\right). \tag{7}$$

1.3 Part c

The area $A(h)$ of the cross section meeting the x-axis at $x = h$ is

$$A(h) = 5\left(\sqrt{h} - e^{-3h}\right)^2 \tag{8}$$

The required volume is therefore

$$\int_a^1 A(x)\,dx \sim 1.55435. \tag{9}$$

The integral is not elementary, and we have carried out the integration numerically.

2 Problem 2

2.1 Part a

The quantity

$$x'(t) = -9\cos\frac{\pi t}{6}\sin\frac{\pi\sqrt{t+1}}{2} \tag{10}$$

is negative on both of the intervals $(0,3)$ and $(3,8)$. (This is because $\pi t/6$ lies between 0 and $\pi/2$ when $0 < t < 3$, but between $\pi/2$ and $3\pi/2$ when $3 < t < 9$, making the cosine factor in $x'(t)$ positive on $(0,3)$ but negative on $(3,9)$, while $\pi\sqrt{t+1}/2$ lies between $\pi/2$ and π when t is in $(0,3)$ but between π and $\sqrt{10}\pi/2 < 2\pi$ when t is in $(3,9)$—making the sine factor positive on $(0,3)$ and negative on $(3,9)$.) Thus, $x'(t) < 0$ on $(0,3)$ and on $(3,9)$, which means that $x(t)$ is a decreasing function on $[0,9]$.. Furthermore, $x'(3) = 0$.

We also know that the slope of the tangent line to the curve at a point $(x(t_0), y(t_0))$ is $y'(t_0)/x'(t_0)$.

This shows that the point B must have coordinates $(x(3), y(3))$, because B is the only point on the curve where $0 < t < 9$ and the slope of the tangent line is undefined. It follows that the point C, corresponds to some value, t_1 of t in $(3, 9)$ where $x'(t)$ is negative. Moreover, the slope of the tangent line to the curve at C is positive. That slope being $y'(t_1)/x'(t_1)$, we conclude that if t_1 is such that C is the point with coordinates $(x(t_1), y(t_1))$, then $y'(t_1) < 0$ because, as we have seen above, $x'(t_1) < 0$.

2.2 Part b

We have shown in Part a, above, that the particle is at point B when $t = 3$.

2.3 Part c

The slope of the tangent line to the curve at the point $(x(8), y(8))$, $y = \frac{5}{9}x - 2$, is $\frac{5}{9}$, Hence,

$$\frac{y'(8)}{x'(8)} = \frac{5}{9}, \text{ or} \tag{11}$$

$$y'(8) = \frac{5}{9}x'(8) \tag{12}$$

$$= \frac{5}{9}\left[-9\cos\frac{4\pi}{3}\sin\frac{3\pi}{2}\right] \tag{13}$$

$$= \frac{5}{\not{9}}\left[-\frac{\not{9}}{2}\right] \tag{14}$$

$$= -\frac{5}{2}, \tag{15}$$

The velocity vector $\mathbf{v}(8)$ is thus

$$\mathbf{v}(8) = \langle x'(8), y'(8)\rangle \tag{16}$$

$$= \left\langle -\frac{9}{2}, -\frac{5}{2}\right\rangle. \tag{17}$$

Speed at time $t = 8$ is then

$$\sqrt{|\mathbf{v}(8)|} = \sqrt{[x'(8)]^2 + [y'(8)]^2} = \sqrt{\frac{81}{4} + \frac{25}{4}} = \sqrt{\frac{53}{2}} \sim 5.14582. \tag{18}$$

2.4 Part d

Distance from A to D is

$$|x(9) - x(0)| = \left|\int_0^9 x'(t)\,dt\right| = 9\left|\int_0^9 \cos\frac{\pi t}{6}\sin\frac{\pi\sqrt{t+1}}{2}\,dt\right| \sim 39.25537, \tag{19}$$

where we have carried out the integration numerically.

3 Problem 3

3.1 Part a

At the point P we must have

$$\frac{5}{3}y = \sqrt{1+y^2}, \tag{20}$$

which implies that $25y^2 = 9 + 9y^2$, or $y^2 = 9/16$. Rejecting the extraneous negative solution for y, we obtain $y = 3/4$. P lies on the curve C, where $x = \sqrt{1+y^2}$. Thus, at P we have

$$x = \sqrt{1+y^2} = \sqrt{1+\frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}. \tag{21}$$

The coordinates of the point P are thus $\left(\frac{5}{4}, \frac{3}{4}\right)$.

Because $x = \sqrt{1+y^2}$ on the curve C,

$$\left.\frac{dx}{dy}\right|_P = \left.\frac{y}{\sqrt{1+y^2}}\right|_P = \frac{3}{5}. \tag{22}$$

3.2 Part b

The area of the region S is

$$\int_0^{3/4}\left(\sqrt{1+y^2} - \frac{5}{3}y\right)dy = \int_0^{3/4}\sqrt{1+y^2}\,dy - \frac{5}{3}\int_0^{3/4} y\,dy \tag{23}$$

$$\sim 0.81532 - 0.46875 \sim 0.34657, \tag{24}$$

by numerical integration.

Note: The integrations are elementary and can be carried out as follows:

The second integral on the right side of (23) is easy.

$$\frac{5}{3}\int_0^{3/4} y\,dy = \frac{5}{6}y^2\bigg|_0^{3/4} = \frac{15}{32}. \tag{25}$$

To evaluate the first integral, we make the substitution $y = \sinh u$. Then

$$dy = \cosh u\,du; \tag{26}$$

$$y = 0 \Rightarrow u = 0; \tag{27}$$

$$y = \frac{3}{4} \Rightarrow u = \sinh^{-1}\frac{3}{4}. \tag{28}$$

Therefore

$$\int_0^{3/4}\sqrt{1+y^2}\,dy = \int_0^{\sinh^{-1}3/4}\sqrt{1+\sinh^2 u}\cosh u\,du = \int_0^{\sinh^{-1}3/4}\cosh^2 u\,du \tag{29}$$

$$= \frac{1}{2}\int_0^{\sinh^{-1}3/4}(1+\cosh 2u)\,du = \frac{1}{2}\left[u + \frac{1}{2}\sinh 2u\right]\bigg|_0^{\sinh^{-1}3/4} \tag{30}$$

$$= \frac{1}{2}\left[u + \sinh u\cosh u\right]\bigg|_0^{\sinh^{-1}3/4} \quad \text{(because } \sinh 2\alpha \equiv 2\sinh\alpha\cosh\alpha\text{)} \tag{31}$$

$$= \frac{1}{2}\left[\sinh^{-1}\frac{3}{4} + \frac{3}{4}\cosh\left(\sinh^{-1}\frac{3}{4}\right)\right] = \frac{1}{2}\left[\sinh^{-1}\frac{3}{4} + \frac{15}{16}\right], \tag{32}$$

where we have used the relation $\cosh\alpha \equiv \sqrt{1+\sinh^2\alpha}$ to make the last transformation.

This latter integral can also be obtained by making the substitution $x = \tan\theta$, but this leads to the somewhat more difficult $\int \sec^3\theta\,d\theta$.

3.3 Part c

The relations for the transformation from rectangular to polar coordinates are $x = r\cos\theta$ and $y = r\sin\theta$. We substitute these relations for x and y,

$$x^2 - y^2 = 1 \text{ becomes} \tag{33}$$

$$r^2\cos^2\theta - r^2\sin^2\theta = 1; \tag{34}$$

$$r^2(\cos^2\theta - \sin^2\theta) = 1; \tag{35}$$

$$r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}. \tag{36}$$

3.4 Part d

The line $3x = 5y$ can be written as $3r\cos\theta = 5r\sin\theta$, which is equivalent to $\tan\theta = \frac{3}{5}$ or $\theta = \tan^{-1} 3/5$.

Using (36) now leads to the integral

$$\frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta = \frac{1}{2}\int_0^{\tan^{-1} 3/5} \frac{d\theta}{\cos^2\theta - \sin^2\theta} \tag{37}$$

Note: Evaluation of this integral is not required, but it easier than the one we evaluated in Part b, above.

$$\frac{1}{2}\int_0^{\tan^{-1} 3/5} \frac{d\theta}{\cos^2\theta - \sin^2\theta} = \frac{1}{2}\int_0^{\tan^{-1} 3/5} \frac{d\theta}{\cos 2\theta} = \frac{1}{2}\int_0^{\tan^{-1} 3/5} \sec 2\theta \, d\theta \tag{38}$$

$$= \frac{1}{4}\ln|\sec 2\theta + \tan 2\theta| \Big|_0^{\tan^{-1} 3/5} \tag{39}$$

$$= \frac{1}{4}\ln\left|\sqrt{1+\tan^2 2\theta} + \tan 2\theta\right| \Bigg|_0^{\tan^{-1} 3/5}. \tag{40}$$

But

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}, \tag{41}$$

so

$$\tan\left(2\tan^{-1}\frac{3}{5}\right) = \frac{2\cdot(3/5)}{1-(3/5)^2} = \frac{6}{5}\cdot\frac{25}{16} = \frac{15}{8}. \tag{42}$$

Substituting this latter into (40) and simplifying, we find that

$$\frac{1}{2}\int_0^{\tan^{-1} 3/5} \frac{d\theta}{\cos^2\theta - \sin^2\theta} = \frac{1}{4}\ln 4 = \ln\sqrt{2} \sim 0.34657. \tag{43}$$

It is reassuring to find that the two integrals give the same value.

4 Problem 4

4.1 Part a

The graph of $y = f'(x)$, as given, lies above the x-axis only on the interval $[-3,-2)$, so f is increasing precisely on the interval $[-3,-2]$.

Note: Positivity of the derivative on $[-3,-2)$ guarantees that f is increasing on $[-3,-2)$. It is easily shown that a continuous function that is increasing on $[a,b)$, or, in fact, on (a,b), must be increasing on $[a,b]$. However, the readers have ignored this subtlety in the past.

4.2 Part b

Inflection points can be found at places where the derivative changes from increasing to decreasing, or vice versa. For the function f, we see from the graph of f' that one of these things happens at $x=0$ and at $x=2$.

4.3 Part c

We have $f'(0) = -2$, so the tangent line to $y = f(x)$ at the point with coordinates $(0,3)$ is

$$y = 3 - 2x \tag{44}$$

4.4 Part d

The Fundamental Theorem of Calculus assures us that

$$f(x) = 3 + \int_0^x f'(\xi)\,d\xi, \tag{45}$$

so

$$f(-3) = 3 + \int_0^{-3} f'(\xi)\,d\xi \tag{46}$$

Now $\int_{-3}^0 f(\xi)\,d\xi = -\int_0^{-3} f(\xi)\,d\xi$ is the area of a triangle of base 1 and height 1 minus the area of a triangle of base 2 and height 2, or $\frac{1}{2} - 2 = -\frac{3}{2}$. So

$$f(-3) = 3 + \frac{3}{2} = \frac{9}{2}. \tag{47}$$

On the other hand,

$$f(4) = 3 + \int_0^4 f(t)\,dt, \tag{48}$$

and this integral is the negative of the area that remains when a semicircle of radius 2 is removed from a rectangle of base 4 and height 2, or $8 - 2\pi$. Thus,

$$f(4) = 3 - (8 - 2\pi) = 2\pi - 5. \tag{49}$$

5 Problem 5

5.1 Part a

We have

$$V = \pi r^2 h = 25\pi h, \tag{50}$$

so

$$\frac{dV}{dt} = 25\pi \frac{dh}{dt}. \tag{51}$$

But it is given that

$$\frac{dV}{dt} = -5\pi\sqrt{h}. \tag{52}$$

Therefore

$$25\pi \frac{dh}{dt} = -5\pi\sqrt{h}, \tag{53}$$

and, dividing by 25π, we obtain

$$\frac{dh}{dt} = -\frac{\sqrt{h}}{5}. \tag{54}$$

5.2 Part b

Let $h = f(t)$ be the solution of the differential equation $5h' = -\sqrt{h}$ for which $h = 17$ when $t = 0$. Then f, being the solution of a differential equation with a positive initial value at $t = 0$, is a continuous function, remains positive over some interval centered at $t = 0$. We can therefore choose t so that $f(\tau)$ doesn't vanish for any value of τ that lies in the closed interval whose endpoints are 0 and t. For such values of τ we see that from

$$f'(\tau) = -\frac{\sqrt{f(\tau)}}{5}, \tag{55}$$

it follows that

$$\int_0^t \frac{f'(\tau)}{\sqrt{f(\tau)}}\, d\tau = -\frac{1}{5}\int_0^t d\tau. \tag{56}$$

Integrating, we obtain

$$2\sqrt{f(t)}\Big|_0^t = -\frac{1}{5}\tau\Big|_0^t, \tag{57}$$

or

$$2\sqrt{f(t)} - 2\sqrt{f(0)} = -\frac{t}{5}. \tag{58}$$

But $f(0) = 17$, so

$$\sqrt{f(t)} = \sqrt{17} - \frac{t}{10}, \tag{59}$$

and we conclude that

$$f(t) = \frac{1}{100}t^2 - \frac{\sqrt{17}}{5}t + 17. \tag{60}$$

The solution we seek is thus $h = f(t) = \frac{1}{100}t^2 - \frac{\sqrt{17}}{5}t + 17$.

5.3 Part c

The coffee pot is empty when $(\sqrt{17} - t/10)^2 = 0$, or when $t = 10\sqrt{17}$ seconds.

6 Problem 6

6.1 Part a

When

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{61}$$

it follows that

$$a_k = \frac{f^{(k)}(0)}{k!}, \text{ k = 0, 1, 2, \dots.} \tag{62}$$

Thus,

$$f'(0) = a_1 = 0, \text{ and} \tag{63}$$

$$f''(0) = 2a_2 = -\frac{1}{3}. \tag{64}$$

Consequently, f has a critical point at $x = 0$, with $f''(0) < 0$. By the Second Derivative Test, f has a local maximum at $x = 0$.

6.2 Part b

We have

$$f(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots . \tag{65}$$

The denominators increase as we move to the right in the series, and the series is alternating. Hence, by the Alternating Series Test, the error in approximating $f(1)$ by $1 - 1/(3!)$ is no larger that $1/(5!) = 1/120 < 1/100$.

6.3 Part c

We have

$$xf(x) = x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^k x^{2k}}{(2k+1)!} + \cdots\right) \tag{66}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots \tag{67}$$

$$= \sin x, \tag{68}$$

and it follows that

$$f(x) = \frac{\sin x}{x} \tag{69}$$

extended through the origin by continuity. Substituting (69) for y in the expression $xy' + y$ then gives

$$xy' + y = x\left(\frac{x\cos x - \sin x}{x^2}\right) + \frac{\sin x}{x} \tag{70}$$

$$= x\left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right) + \frac{\sin x}{x} \tag{71}$$

$$= \cos x - \frac{\sin x}{x} + \frac{\sin x}{x} = \cos x. \tag{72}$$