AP Calculus 2004 BC (Form B) FRQ Solutions

Louis A. Talman, Ph.D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

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1 Problem 1

1.1 Part a

If a particle's position at time t is given by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, and its velocity $\mathbf{v}(t)$ is given by

$$\mathbf{v}(t) = \mathbf{r}'(t) = \frac{d}{dt} \langle x(t), y(t) \rangle = \left\langle \sqrt{t^4 + 9}, 2e^t + 5e^{-t} \right\rangle, \tag{1}$$

then its speed $\sigma(t)$ at time t is given by

$$\sigma(t) = |\mathbf{v}(t)| = \sqrt{\mathbf{v}(t) \cdot \mathbf{v}(t)} = \sqrt{(t^4 + 9) + (2e^t + 5e^{-t})^2}, \text{ so} \tag{2}$$

$$\sigma(0) = \sqrt{58} \sim 7.61577. \tag{3}$$

Its acceleration $\mathbf{a}(t)$ at time t is

$$\mathbf{a}(t) = \mathbf{v}'(t) \tag{4}$$

$$= \frac{d}{dt} \left\langle \sqrt{t^4 + 9}, 2e^t + 5e^{-t} \right\rangle \tag{5}$$

$$= \left\langle \frac{2t}{\sqrt{t^4 + 9}}, 2e^t - 5e^{-t} \right\rangle, \text{ and} \tag{6}$$

$$\mathbf{a}(0) = \langle 0, -3 \rangle. \tag{7}$$

1.2 Part b

The tangent vector $\mathbf{T}(t)$ is

$$\mathbf{T}(t) = \mathbf{v}(t), \text{ so that} \tag{8}$$
$$\mathbf{T}(0) = \langle 3, 7 \rangle. \tag{9}$$

The slope of a line parallel to the tangent vector is $\frac{7}{3}$, so an equation for the line tangent to the curve at $\mathbf{r}(0) = \langle 4, 1 \rangle$ is

$$y = 1 + \frac{7}{3}(x - 4). \tag{10}$$

The equation can also be written in vector notation:

$$\mathbf{r}(t) = \langle 4, 1 \rangle + t\langle 3, 7 \rangle = \langle 4 + 3t, 1 + 7t \rangle. \tag{11}$$

Alternately, we can put $\mathbf{R} = \langle x, y, 0 \rangle$ and write

$$\big(\mathbf{R} - \langle 4, 1, 0 \rangle\big) \times \langle 3, 7, 0 \rangle = \mathbf{0}, \tag{12}$$

where "$\times$" denotes the vector cross-product and $\mathbf{0} = \langle 0, 0, 0 \rangle$.

1.3 Part c

Total distance traveled over the interval $[0, 3]$ is the integral of speed over that interval:

$$\int_0^3 \sigma(\tau)\, d\tau = \int_0^3 \sqrt{(\tau^4 + 9) + (2e^{\tau} + 5e^{-\tau})^2}\, d\tau \sim 45.22682, \tag{13}$$

where we have carried out the integration numerically.

1.4 Part d

The x-coordinate of the particle at time $t = 3$ is

$$x(t) = x(0) + \int_0^3 x'(\tau)\, d\tau \tag{14}$$
$$= 4 + \int_0^3 \sqrt{\tau^4 + 9}\, d\tau \sim 17.93079 \tag{15}$$

2 Problem 2

2.1 Part a

If $T_n(x) = \sum_{k=0}^{n} a_k(x-a)^k$ is a Taylor polynomial for the function f, then

$$a_k = \frac{f^{(k)}(a)}{k!}, \text{ for } k = 1, \ldots, n. \tag{16}$$

From what is given here, we deduce that $f(2) = 7$ and $f''(2) = -18$.

2.2 Part b

Reasoning as in Part a, above, we find that $f'(2) = 0$, so f has a critical point at $x = 3$. Because $f''(2) = -18$, the Second Derivative Test allows us to conclude that f has a local maximum at $x = 2$.

2.3 Part c

$$f(0) \sim 7 - 9 \cdot (-2)^2 - 3 \cdot (-2)^3 = -5. \tag{17}$$

There is not enough information to determine whether f has a critical point at $x = 0$. This is because the third degree Taylor Polynomial carries no information about derivatives at any point other than the point about which the expansion has been done; it is determined solely by the values of the function and its first three derivatives at that point.

2.4 Part d

The Lagrange Remainder, R_3, for the third degree Taylor polynomial of t at $x = 2$ has the form

$$R_3 = \frac{f^{(4)}(\xi)}{4!}(x-2)^4, \tag{18}$$

where ξ is some unknown number in the interval whose endpoints are x and 2. Thus,

$$f(0) = T_3(0) + \frac{1}{24} f^{(4)}(\xi)(0-2)^4 \tag{19}$$

for a certain $\xi \in [0,2]$. But $|f^{(4)}(x)| \leq 6$ for all $x \in [0,2]$, so

$$|f(0) - T(0)| \leq \frac{6}{24} \cdot 16 = 4. \tag{20}$$

We have seen in part c, above, that $T_3(0) = -5$. Hence

$$-4 \leq f(0) - (-5) \leq 4, \tag{21}$$

whence

$$-9 \leq f(0) \leq -1, \tag{22}$$

so that $f(0)$ must be negative.

3 Problem 3

3.1 Part a

The Midpoint Rule with four subintervals of equal length gives

$$\int_0^{40} v(t)\,dt \sim v(5) \cdot (10-0) + v(15) \cdot (20-10) + v(25) \cdot (30-20) + v(35) \cdot (40-30) \tag{23}$$

$$\sim (9.2 + 7.0 + 2.4 + 4.3) \cdot 10 = 229 \tag{24}$$

The integral gives miles the plane traveled during the time interval $0 \leq t \leq 40$.

3.2 Part b

By Rolle's Theorem, acceleration—which is $v'(t)$—must be zero at least once in the interval $0 \leq t \leq 15$ because $v(0) = v(15)$. Similarly, $v'(t)$ must be zero at least once in the interval $25 \leq t \leq 30$, because $v(25) = v(30)$. Thus, acceleration must vanish at least twice in the interval $0 \leq t \leq 40$.

3.3 Part c

If the plane's velocity is given by

$$f(t) = 6 + \cos\frac{t}{10} + 3\sin\frac{7t}{40}, \tag{25}$$

then

$$f'(t) = \frac{21}{40}\cos\frac{7t}{40} - \frac{1}{10}\sin\frac{t}{10} \tag{26}$$

gives acceleration. At $t = 23$, this gives acceleration as

$$f'(23) = \frac{21}{40}\cos\frac{161}{40} - \frac{1}{10}\sin\frac{23}{10}\ \text{miles/min}^2 \tag{27}$$

$$\sim -0.40769\ \text{miles/min}^2. \tag{28}$$

3.4 Part d

Average velocity over $0 \le t \le 40$ is

$$\frac{1}{40}\int_0^{40}\left(6 + \cos\frac{t}{10} + 3\sin\frac{7t}{40}\right)dt = \frac{1}{40}\left[6t + 10\sin\frac{t}{10} - \frac{120}{7}\cos\frac{7t}{40}\right]\Bigg|_0^{40} \tag{29}$$

$$= \frac{1}{40}\left[240 + 10\sin 4 - \frac{120}{7}\cos 7\right] - \frac{1}{40}\left[\frac{120}{7}\right] \tag{30}$$

$$\sim 5.91627\text{miles/min}. \tag{31}$$

4 Problem 4

4.1 Part a

Inflection points are to be found where f'' changes sign—that is, where the slope of f' changes from positive to negative or vice versa. Consequently, the function f whose derivative is pictured has inflection points at $x = 1$ and at $x = 3$.

4.2 Part b

the function f is decreasing on the interval $[-1, 4]$ and increasing on the interval $[4, 5]$ because f' is non-positive, with only isolated zeros, on the first of these intervals and non-negative, with only an isolated zero on the second.

The absolute maximum vale of f must fall at one of the points $x = -1$ or $x = 5$. (There can be no absolute maximum for f at any point interior to $(-1, 5)$ because f' does not change signs from positive to negative anywhere in that interval.) The (unsigned) area bounded by f and the x-axis on the interval $[-1, 4]$ is clearly larger than the area between f and the x-axis on the interval $[4, 5]$, so

$$-\int_{-1}^{4} f'(t)\,dt = f(-1) - f(4) > f(5) - f(4) = \int_{4}^{5} f'(t)\,dt, \tag{32}$$

whence

$$f(-1) > f(5), \tag{33}$$

so the absolute maximum value taken on in the interval $[-1, 5]$ is $f(-1)$.

4.3 Part c

We are given that $g(x) = xf(x)$, so

$$g'(2) = f(2) + 2f'(2) = 6 + 2 \cdot (-1) = 4. \tag{34}$$

Also

$$g(2) = 2f(2) = 12. \tag{35}$$

An equation for the line tangent to the graph at $x = 2$ is therefore

$$y = 12 + 4(x - 2), \text{ or} \tag{36}$$

$$y = 4x + 4. \tag{37}$$

5 Problem 5

5.1 Part a

The average value of $g(x) = x^{-1/2}$ on the interval $[1, 4]$ is

$$\frac{1}{4-1}\int_{1}^{4} \frac{dx}{\sqrt{x}} = \frac{2}{3}\sqrt{x}\Big|_{1}^{4} = \frac{2}{3}\sqrt{4} - \frac{2}{3}\sqrt{1} = \frac{2}{3}. \tag{38}$$

5.2 Part b

The volume of the solid generated when the region bounded by the graph of $y = g(x)$, the vertical lines $x = 1$ and $x = 4$, and the x-axis is revolved about the x-axis is

$$\pi\int_{1}^{4} \frac{dx}{x} = \pi \ln x\Big|_{1}^{4} = \pi \ln 4. \tag{39}$$

5.3 Part c

The average value of the areas of the cross sections perpendicular to the x-axis is

$$\frac{\pi}{4-1}\int_1^4 \frac{dx}{x} = \frac{\pi}{3}\ln 4. \tag{40}$$

5.4 Part d

We have

$$\begin{aligned}
\int_4^\infty g(x)\,dx &= \lim_{T\to\infty}\int_4^T \frac{dx}{\sqrt{x}} &(41)\\
&= \lim_{T\to\infty} 2x^{1/2}\Big|_4^T &(42)\\
&= 2\lim_{T\to\infty}\left[\sqrt{T}-2\right], &(43)
\end{aligned}$$

which does not exist. Consequently, the improper integral $\int_4^\infty g(x)\,dx$ diverges.

However,

$$\begin{aligned}
\lim_{b\to\infty}\left[\frac{1}{b-4}\int_4^b \frac{dx}{\sqrt{x}}\right] &= 2\lim_{b\to\infty}\frac{\sqrt{b}-2}{b-4} &(44)\\
&= 2\lim_{b\to\infty}\frac{\cancel{\sqrt{b}-2}}{\cancel{(\sqrt{b}-2)}(\sqrt{b}+2)} = 0. &(45)
\end{aligned}$$

The average value is not only finite, it's zero!

6 Problem 6

6.1 Part a

If $n > 1$, then

$$\int_0^1 x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1^{n+1}}{n+1} - \frac{0^{n+1}}{n+1} = \frac{1}{n+1}. \tag{46}$$

6.2 Part b

If $n > 1$ and $y = x^n$, then

$$y' = nx^{n-1}, \text{ so} \tag{47}$$

$$y'\Big|_{x=1} = n. \tag{48}$$

It follows that the equation of the line tangent to $y = x^n$ at $(1, 1)$ is

$$y = 1 + n(x - 1). \tag{49}$$

This line crosses the x-axis at $x = 1 - \dfrac{1}{n}$, so that the base of the triangle T has length $\dfrac{1}{n}$. The altitude of T is one, so the area of T is $\dfrac{1}{2n}$.

6.3 Part c

From what we have seen in Parts a and b, above, the area, $A(n)$ of the region S, as a function of n, is

$$A(n) = \frac{1}{n+1} - \frac{1}{2n} = \frac{n-1}{2n^2 + 2n}. \tag{50}$$

Thus,

$$A'(n) = \frac{(2n^2 + 2n) - (n-1)(4n+2)}{4n^2(n+1)^2} \tag{51}$$

$$= -\frac{n^2 - 2n - 1}{2n^2(n+1)^2}. \tag{52}$$

When $n > 0$, we see that $A'(n) = 0$ only for $n = 1 + \sqrt{2}$, by the Quadratic Formula. Noting that $A'(n) > 0$ for $1 \leq n < 1 + \sqrt{2}$ but that $A'(n) < 0$ for $1 + \sqrt{2} < n$, we conclude, by the First Derivative Test, that the maximal area occurs when $n = 1 + \sqrt{2}$.