AP Calculus 2004 BC FRQ Solutions

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1 Problem 1

1.1 Part a

The function $F(t) = 82 + 4\sin(t/2)$ gives the rate, in cars per minute, at which cars pass through the intersection. Thus, the total number of cars that pass through the intersection in the period $0 \leq t \leq 30$ is

$$\int_0^{30} F(t)\,dt = \int_0^{30} \left[82 + 4\sin\frac{t}{2}\right]\,dt \tag{1}$$

$$= \left[82t - 8\cos\frac{t}{2}\right]\Bigg|_0^{30} \tag{2}$$

$$= [2460 - 8\cos 15] - [0 - 8] \sim 2474.07750, \tag{3}$$

or 2474 to the nearest whole number.

1.2 Part b

$$F'(t) = 2\cos\frac{t}{2}, \text{ so} \tag{4}$$

$$F'(7) = 2\cos\frac{7}{2} \sim -1.87291 < 0, \tag{5}$$

and, F' being a continuous function, we conclude that traffic flow is decreasing near $t = 7$ because $F'(7) < 0$ and F' is continuous near $t = 7$. (We have phrased our answer this way because the terms "increasing" and "decreasing" are almost always defined only for intervals, and not at individual points.)

1.3 Part c

The average value, in cars per minute, of traffic flow over the interval $10 \leq t \leq 15$ is

$$\frac{1}{15-10}\int_{10}^{15} F(t)\,dt = \frac{1}{5}\left[82t - 8\cos\frac{t}{2}\right]\Bigg|_{10}^{15} \tag{6}$$

$$= \frac{1}{5}\left(410 + 8\cos 5 - 8\cos\frac{15}{2}\right) \tag{7}$$

$$\sim 81.89924 \text{ cars per minute.} \tag{8}$$

1.4 Part d

The average rate of change of the traffic flow over the interval $10 \leq t \leq 15$ is

$$\frac{F(15) - F(10)}{15-10} = \frac{4\sin(15/2) - 4\sin 5}{5} \text{ cars per minute per minute} \tag{9}$$

$$\sim 1.51754 \text{ cars per minute per minute.} \tag{10}$$

2 Problem 2

Throughout this problem we understand that

$$f(x) = 2x(1-x) \text{ and} \tag{11}$$

$$g(x) = 3(x-1)\sqrt{x} \tag{12}$$

for $0 \leq x \leq 1$.

2.1 Part a

The graphs of the curves $y = f(x)$ and $y - g(x)$ intersect on the x-axis at $x = 0$ and at $x = 1$. Thus, the area between the two curves is

$$\int_0^1 [f(x) - g(x)]\,dx = \int_0^1 \left[2x(1-x) - 3(x-1)\sqrt{x}\right] dx \tag{13}$$

$$= \int_0^1 \left[3x^{1/2} + 2x - 3x^{3/2} - 2x^2\right] dx \tag{14}$$

$$= \left[2x^{3/2} + x^2 - \frac{6}{5}x^{5/2} - \frac{2}{3}x^3\right]\Bigg|_0^1 \tag{15}$$

$$= \left[2 + 1 - \frac{6}{5} - \frac{2}{3}\right] - 0 = \frac{17}{15}. \tag{16}$$

2.2 Part b

The volume of the solid generated by rotating the shaded region about the horizontal line $y = 2$ is

$$\int_0^1 \left[\pi[2 - g(x)]^2 - \pi[2 - f(x)]^2\right] dx \tag{17}$$

$$= \pi \int_0^1 \left(4x^4 - 17x^3 + 30x^2 + 12x^{3/2} - 17x - 12x^{1/2}\right) dx \tag{18}$$

$$= \pi \left(8x^{3/2} + \frac{17}{2}x^2 - \frac{24}{5}x^{5/2} - 10x^3 + \frac{17}{4}x^4 - \frac{4}{5}x^5\right)\Bigg|_0^1 \tag{19}$$

$$= \frac{103}{20}\pi \sim 16.17920. \tag{20}$$

2.3 Part c

The volume of the solid given is

$$\int_0^1 [h(x) - g(x)]^2\,dx = \int_0^1 \left[kx(1-x) - 3(x-1)\sqrt{x}\right]^2 dx \tag{21}$$

Thus, the desired equation is

$$\int_0^1 \left[kx(1-x) - 3(x-1)\sqrt{x}\right]^2 dx = 15. \tag{22}$$

Note: Solving equation (22) is not required, so evaluation of the integral is also not necessary. However,

$$\int_0^1 \left[kx(1-x) - 3(x-1)\sqrt{x}\right]^2 dx = \frac{1}{30}k^2 + \frac{32}{105}k + \frac{3}{4}, \tag{23}$$

and solution of the resulting quadratic equation for $k > 0$ gives

$$k = \frac{\sqrt{87886} - 64}{14} \sim 16.60398. \tag{24}$$

3 Problem 3

Throughout this problem, we have

$$\frac{dx}{dt} = 3 + \cos t^2; \tag{25}$$

$$x(2) = 1; \tag{26}$$

$$y(2) = 8. \tag{27}$$

3.1 Part a

By the Fundamental Theorem of Calculus,

$$x(4) = x(2) + \int_2^4 x'(t)\, dt = 1 + \int_2^4 (3 + \cos t^2)\, dt. \tag{28}$$

Numerical integration gives $x(2) \sim 7.13200$.

3.2 Part b

If we assume that we can solve the parametric equations, at least locally, near $x = 2$ for y as function of x, the Chain Rule yields

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}, \text{ or} \tag{29}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{30}$$

But

$$\left.\frac{dy}{dt}\right|_{t=2} = -7, \text{ and} \tag{31}$$

$$\left.\frac{dx}{dt}\right|_{t=2} = 3 + \cos t^2\Big|_{t=2} = 3 + \cos 4. \tag{32}$$

Thus,

$$\left.\frac{dy}{dx}\right|_{t=2} = -\frac{7}{3 + \cos 4} \sim -2.98335 \tag{33}$$

An equation for the line tangent to the curve at $\big(x(2), y(2)\big)$ is therefore

$$y = 8 - \frac{7}{3 + \cos 4}(x - 1). \tag{34}$$

3.3 Part c

Speed $\sigma(t)$ at time t is given by

$$\sigma(t) = |v(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}. \tag{35}$$

Therefore

$$\sigma(2) = \sqrt{[x'(2)]^2 + [y'(2)]^2} \tag{36}$$

$$= \sqrt{(-7)^2 + (3 + \cos 4)^2} \sim 7.38278. \tag{37}$$

3.4 Part d

Let us suppose that the slope of the tangent line at $\big(x(t), y(t)\big)$ is $(2t+1)$ when $t \geq 3$. From our observations in Part b, above, we have

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \tag{38}$$

$$= (2t + 1)(3 + \cos t^2) \tag{39}$$

when $t \geq 3$. Therefore

$$\frac{d^2x}{dt^2} = -2t \sin t^2 \text{ and} \tag{40}$$

$$\frac{d^2y}{dt^2} = 2(3 + \cos t^2) + (2t + 1) \cdot (-2t \sin t^2). \tag{41}$$

When $t = 4$, this gives the acceleration vector $\mathbf{a}(4)$ as

$$\mathbf{a}(4) = \big\langle -8 \sin 16, 6 + 2 \cos 16 - 72 \sin 16 \big\rangle \sim \langle 2.30323, 28.81372 \rangle. \tag{42}$$

4 Problem 4

4.1 Part a

From

$$x^2 + 4y^2 = 7 + 3xy \tag{43}$$

we obtain, by implicit differentiation with respect to x, treating y as (locally) a function of x,

$$2x + 8yy' = 3y + 3xy', \tag{44}$$

so that

$$8yy' - 3xy' = 3y - 2x, \tag{45}$$

or

$$\frac{dy}{dx} = y' = \frac{3y - 2x}{8y - 3x}. \tag{46}$$

4.2 Part b

If we are to have $y' = 0$ in Part a, above, then we must have, from (46),

$$0 = y' = \frac{3y - 2x}{8y - 3x}, \tag{47}$$

and from this we conclude that $3y - 2x = 0$. But we are given that $x = 3$, and so $y = 2$. These values for x and y give

$$x^2 + 4y^2 = 3^2 + 4 \cdot 2^2 = 9 + 16 = 25 = 7 + 18 = 7 + 3 \cdot 3 \cdot 2 = 7 + 3xy, \tag{48}$$

showing that the point $(3, 2)$ lies on the curve. The point $P = (3, 2)$ thus meets our requirements.

4.3 Part c

From Part a, above, we have

$$(8y - 3x)y' = 3y - 2x. \tag{49}$$

Another implicit differentiation with respect to x then gives

$$(8y' - 3)y' + (8y - 3x)y'' = 3y' - 2. \tag{50}$$

At $(3, 2)$, as we have seen above, we have $y' = 0$. Substituting these values for x, y, and y' in equation (50) gives

$$(8 \cdot 0 - 3) \cdot 0 + (8 \cdot 2 - 3 \cdot 3)y'' = 3 \cdot 0 - 2, \tag{51}$$

whence

$$y''\Big|_{(3,2)} = -\frac{2}{7} < 0. \tag{52}$$

We conclude, from the Second Derivative Test, that the curve has a local maximum at $(3, 2)$.

5 Problem 5

In this problem, we are given that

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right). \tag{53}$$

5.1 Part a

Equilibrium solutions are $P(t) \equiv 0$ and $P(t) \equiv 12$. For $0 < P < 12$, $P'(t) > 0$, while for $12 < P$, $P'(t) < 0$. Hence, any solution whose initial value is positive will be asymptotic to the equilibrium solution $P(t) \equiv 12$. (Here we interpret a horizontal line as its own horizontal asymptote.) Both of the required limits are therefore 12.

5.2 Part b

$P(t)$ grows fastest when $P'(t)$ is maximal. This can happen only when $P = 3$ (the end-point of the interval under consideration) or or at a critical point for P'. But by (53),

$$\frac{d^2P}{dt^2} = \frac{1}{5}\left(1 - \frac{P}{12}\right) - \frac{P}{60} = \frac{1}{5} - \frac{P}{30}, \tag{54}$$

and this vanishes when $P = 6$.

When $P = 3$, $P' = 9/20$, and when $P = 6$, $P' = 3/5$. The latter is the larger, so P grows fastest when $P = 6$.

5.3 Part c

If

$$Y'(t) = \frac{1}{5}Y(t)\left(1 - \frac{t}{12}\right) \tag{55}$$

and

$$Y(0) = 3 \tag{56}$$

then

$$\frac{Y'(t)}{Y(t)} = \frac{1}{5}\left(1 - \frac{t}{12}\right). \tag{57}$$

Because Y is the solution of a differential equation and $Y(0) = 3$, Y is a continuous function and there is an positive number δ such that $Y(t) > 0$ for all $t \in (-\delta, \delta)$. If we choose $t \in (-\delta, \delta)$ then

$$\int_0^t \frac{Y'(\tau)}{Y(\tau)}\, d\tau = \frac{1}{5}\int_0^t \left[1 - \frac{\tau}{12}\right] d\tau, \tag{58}$$

or

$$\ln Y(\tau)\Big|_0^t = \frac{1}{5}\left(\tau - \frac{\tau^2}{24}\right)\Big|_0^t; \tag{59}$$

$$\ln Y(t) - \ln Y(0) = \frac{1}{5}\left(t - \frac{t^2}{24}\right). \tag{60}$$

But $Y(0) = 3$, so this is

$$\ln Y(t) = \ln 3 + \frac{1}{5}\left(t - \frac{t^2}{24}\right). \tag{61}$$

Finally,

$$Y(t) = 3\exp\left[\frac{1}{5}\left(t - \frac{t^2}{24}\right)\right], \tag{62}$$

where we have taken $\exp u$ to mean e^u.

5.4 Part d

As $t \to \infty$, $\frac{1}{5}\left(t - \frac{t^2}{24}\right) \to -\infty$. Thus, $\lim_{t\to\infty} Y(t) = 0$.

6 Problem 6

6.1 Part a

If f is given by

$$f(x) = \sin\left(5x + \frac{\pi}{4}\right), \tag{63}$$

and $P_3(x)$ is the third degree Taylor polynomial for f about $x = 0$, then

$$P_3(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(0)x^3. \tag{64}$$

Now

$$f'(x) = 5\cos\left(5x + \frac{\pi}{4}\right); \tag{65}$$

$$f''(x) = -25\sin\left(5x + \frac{\pi}{4}\right); \tag{66}$$

$$f'''(x) = -125\cos\left(5x + \frac{\pi}{4}\right), \tag{67}$$

and, indeed,

$$f^{(4k)}(x) = 5^{4k}\sin\left(5x + \frac{\pi}{4}\right), \tag{68}$$

$$f^{(4k+1)}(x) = 5^{4k+1}\cos\left(5x + \frac{\pi}{4}\right), \tag{69}$$

$$f^{(4k+2)}(x) = -\left(5^{4k+2}\right)\sin\left(5x + \frac{\pi}{4}\right), \tag{70}$$

$$f^{(4k+3)}(x) = -\left(5^{4k+3}\right)\cos\left(5x + \frac{\pi}{4}\right) \tag{71}$$

for $k = 0, 1, 2, \ldots$.

Thus,

$$P_3(x) = \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}x - \frac{25\sqrt{2}}{4}x^2 - \frac{125\sqrt{2}}{12}x^3. \tag{72}$$

6.2 Part b

Using what we have seen in Part a, above, we find that the coefficient of x^{22} in the Taylor series for f about $x = 0$ is

$$\frac{f^{(22)}(0)}{22!} = \frac{f^{(4\cdot 5+2)}(0)}{22!} = -\frac{5^{22}\sqrt{2}}{2\cdot 22!}. \tag{73}$$

6.3 Part c

The Lagrange Remainder, R_3 for the third degree Taylor series at $x = 0$ has the form

$$R_3 = \frac{f^{(4)}(\xi)}{24} x^4, \tag{74}$$

where ξ is some number that lies between 0 and x. Now,

$$f^{(4)}(x) = 5^4 \sin\left(5x + \frac{\pi}{4}\right). \tag{75}$$

Thus,

$$f\left(\frac{1}{10}\right) = P_3\left(\frac{1}{10}\right) + \frac{5^4 \sin(5\xi + \pi/4)}{24} \cdot \left(\frac{1}{10}\right)^4 \tag{76}$$

for some $\xi \in \left(0, \frac{1}{10}\right)$. Hence

$$\left| f\left(\frac{1}{10}\right) - P_3\left(\frac{1}{10}\right) \right| = \left| \frac{\cancel{5^4} \sin(5\xi + \pi/4)}{24 \cdot (\cancel{5} \cdot 2)^4} \right| \le \frac{1}{2^4 \cdot 24} = \frac{1}{384} < \frac{1}{100}, \tag{77}$$

where we have made use of the fact that $|\sin u| \le 1$ for all real u.

6.4 Part d

We obtain the third degree Taylor polynomial, $T_3(x)$, about $x = 0$ for

$$G(x) = \int_0^x f(t)\, dt \tag{78}$$

by integrating the second degree Taylor polynomial for f from 0 to x.

$$\begin{aligned}
T_3(x) &= \int_0^x P_2(\xi)\, d\xi & (79)\\
&= \int_0^x \left[\frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}\xi - \frac{25\sqrt{2}}{4}\xi^2\right] d\xi & (80)\\
&= \left[\frac{\sqrt{2}}{2}\xi + \frac{5\sqrt{2}}{4}\xi^2 - \frac{25\sqrt{2}}{12}\xi^3\right]\Bigg|_0^x & (81)\\
&= \frac{\sqrt{2}}{2}x + \frac{5\sqrt{2}}{4}x^2 - \frac{25\sqrt{2}}{12}x^3. & (82)
\end{aligned}$$