AP Calculus 2005 BC FRQ Solutions

Louis A. Talman, Ph.D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

July 15, 2017

1 Problem 1

1.1 Part a

The area of the region R is $\int_0^a \left[4^{-x} - \frac{1}{4} - \sin \pi x\right]\, dx$, where a is the smallest positive solution of the equation

$$1 + 4\sin \pi x = 4^{1-x}. \tag{1}$$

Numerical solution, and then a numerical integration, give

$$a \sim 0.17823, \text{ and} \tag{2}$$

$$\int_0^a \left[4^{-x} - \frac{1}{4} - \sin \pi x\right]\, dx \sim 0.06475. \tag{3}$$

1.2 Part b

The second smallest positive solution, b, of equation (1) is easily seen to be $b = 1$. The area of the region S is therefore given by

$$\int_a^1 \left[4^{-x} - \frac{1}{4} - \sin \pi x\right]\, dx \sim 0.41036, \tag{4}$$

where we have again carried out the integration numerically.

1.3 Part c

The volume of the solid generated when S is revolved about the horizontal ine $y = -1$ is (integrating numerically one more time)

$$\pi \int_a^1 \left[\left(\frac{1}{4} + \sin \pi x + 1 \right)^2 - \left(4^{-x} + 1 \right)^2 \right] dx \sim 4.55876. \tag{5}$$

2 Problem 2

2.1 Part a

The required area is

$$\frac{1}{2} \int_0^\pi [r(\theta)]^2 \; d\theta = \frac{1}{2} \int_0^\pi [\theta + \sin 2\theta]^2 \; d\theta \sim 4.38231 \tag{6}$$

The integral is elementary but tedious, so we carried out the integration numerically. Exact integration gives the value $\frac{\pi}{12} \left(2\pi^2 - 3 \right)$.

2.2 Part b

If $x = -2$, then $r \cos \theta = -2$, or $(\theta + \sin 2\theta) \cos \theta = -2$. Solving this equation numerically, we find that $\theta \sim 2.78696$.

2.3 Part c

When $r'(\theta) < 0$, r decreases as θ increases. This means that as θ ranges upward from $\pi/3$ to $2\pi/3$, the corresponding point on the curve gets closer to the pole.

2.4 Part d

We seek to find the absolute maximum of $r(\theta)$ on the interval $[0, \pi/2]$. This will be found at a point where $\theta = 0$, where $\theta = \pi/2$, or where $r'(\theta) = 0$. Now $r(0) = 0$ and $r(\pi/2) = \pi/2 \sim 1.57080$. Also,

$$r'(\theta) = 1 + 2\cos(2\theta) = 0 \tag{7}$$

when $\theta = \pi/3$. We have

$$r\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{\pi}{3} \sim 1.91322. \tag{8}$$

We conclude that the maximum distance from the pole to the curve is about 1.91322, and that it is to be found when $\theta = \sqrt{3}/2 + \pi/3 \sim 1.91322$.

3 Problem 3

3.1 Part a

$$T'(7) \sim \frac{T(8) - T(6)}{8 - 6} = \frac{55 - 62}{8 - 6} = -\frac{7}{2}. \tag{9}$$

3.2 Part b

The average temperature of the wire is

$$\begin{aligned} \frac{1}{8}\int_0^8 T(x)\,dx &\sim \frac{1}{8}\left[\frac{100+93}{2} + \frac{93+70}{2}(5-1) + \frac{70+62}{2} + \frac{62+55}{2}(8-6)\right] && (10)\\ &\sim \frac{1211}{16} \text{ degrees Celsius.} && (11)\end{aligned}$$

3.3 Part c

We are given that T is twice differentiable—though we are not told where. We take the statement to mean that T is twice differentiable, and, consequently that T' is continuous, on a domain that includes $[0, 8]$, so that the problem is meaningful. By the Fundamental Theorem of Calculus,

$$\int_0^8 T'(t)\,dt = T(8) - T(0) = -45 \text{ degrees Celsius.} \tag{12}$$

The integrand, $T'(x)$, is the (instantaneous) rate at which $T(x)$ changes per unit length at each point of the interval $[0, 8]$, and the integral gives net temperature change over the same interval.

3.4 Part d

By hypothesis, T is continuous on $[1,5]$ and differentiable on $(1,5)$, so the Mean Value Theorem guarantees that there is a point $\xi \in (1,5)$ such that

$$T'(\xi) = \frac{T(5) - T(1)}{5 - 1} = -\frac{23}{4}. \tag{13}$$

By the same reasoning, there is a point $\eta \in (5,6)$ for which

$$T'(\eta) = \frac{T(6) - T(5)}{6 - 5} = -8. \tag{14}$$

We note that, necessarily, $0 < \xi < \eta < 8$. We apply the Mean Value Theorem still a third time, now on the interval $[\xi, \eta]$, and we obtain $\zeta \in (\xi, \eta)$ such that

$$T''(\zeta) = \frac{T'(\xi) - T'(\eta)}{\xi - \eta} = \frac{-8 + (23/4)}{\xi - \eta} = \frac{-9}{4(\xi - \eta)} < 0. \tag{15}$$

Thus, the data in the table are not consistent with the assertion that $T''(x) > 0$ throughout $0, 8)$.

4 Problem 4

4.1 Part a

See Figure 1.

4.2 Part b

At a local minimum on a differentiable curve, we must have $y' = 0$, and because we are on a solution curve, we know that $0 = 2x - y = 2\ln(3/2) - y$. Hence, the y-coordinate of this minimum is $y = \ln(9/4)$.

4.3 Part c

The recursion for Euler's method with step-size $h = -0.2$ to approximate the solution to the problem

$$y' = f(x, y) = 2x - y \tag{16}$$

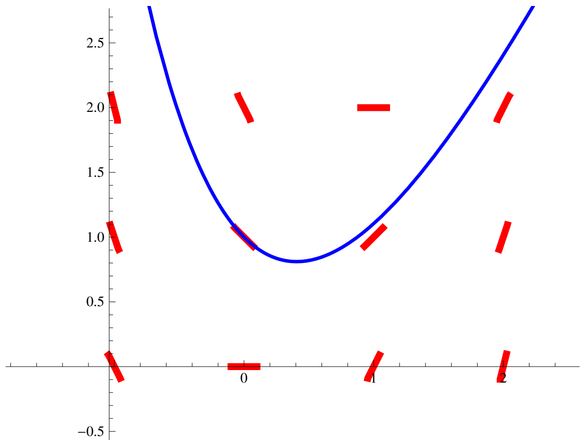

Figure 1: Problem 4, Part a

given that $y = 1$ when $x = 0$ is

$$x_0 = 0; \tag{17}$$
$$y_0 = 1; \tag{18}$$
$$x_k = x_{k-1} + h = x_{k-1} - 0.2; \tag{19}$$
$$y_k = y_{k-1} + hf(x_{k-1}, y_{k-1}) = y_{k-1} - 0.2(2x_{k-1} - y_{k-1}) = 1.2y_{k-1} - 0.4x_{k-1}. \tag{20}$$

Thus

$$x_1 = -0.2; \tag{21}$$
$$y_1 = 1.2(1) - 0.4(0) = 1.2; \tag{22}$$
$$x_2 = -0.4; \tag{23}$$
$$y_2 = 1.2(1.2) - 0.4(-0.2) = 1.52. \tag{24}$$

We conclude that if $y = f(x)$ is the solution in question, then $f(-0.4) \sim 1.52$.

4.4 Part d

If $y' = 2x - y$, then $y'' = 2 - y' = 2 - (2x - y) = y - 2x + 2$. This latter quantity is positive when $x \leq 0$ and $y > 0$—as they surely are for our solution. Hence the solution must be concave upward in the region where we are examining it. Consequently, tangent lines to the solution lie below the curve throughout the interval $(-0.4, 0)$. Because Euler's Method proceeds by replacing the curve with tangent lines, the method underestimates y in this region.

Note: Let $y = f(x)$ be the solution of Part a. Then $f'(x) = 2x - f(x)$, or

$$f'(x) + f(x) = 2x, \text{ and} \tag{25}$$
$$f'(x)e^x + f(x)e^x = 2xe^x. \text{ Thus,} \tag{26}$$
$$\frac{d}{dx}\left[e^x f(x)\right] = 2xe^x. \tag{27}$$

It now follows that

$$\left[e^\xi f(\xi)\right]\Big|_0^x = 2\int_0^x \xi e^\xi \, d\xi = 2\left[\xi e^\xi - e^\xi\right]\Big|_0^x; \tag{28}$$
$$e^x f(x) - f(0) = (2xe^x - 2e^x) - 2(0-1); \tag{29}$$
$$e^x f(x) = 2xe^x - 2e^x + 3; \tag{30}$$
$$f(x) = 2x - 2 + 3e^{-x}, \tag{31}$$

and the solution is $f(x) = 2x - 2 + 3e^{-x}$.

5 Problem 5

5.1 Part a

$$\int_0^{24} v(t)\, dt = \frac{1}{2}(4-0)\cdot 20 + (16-4)\cdot 20 + \frac{1}{2}(34-16)\cdot 20 = 360 \text{ meters.} \tag{32}$$

The integral gives the distance, in meters, that the car travels during the time period $0 \leq t \leq 24$.

5.2 Part b

The definition of $v'(t_0)$ is

$$v'(t_0) = \lim_{h\to 9} \frac{v(t_0+h) - v(t_0)}{h}. \tag{33}$$

For the piecewise linear function given,

$$\lim_{h\to 0^-} \frac{v(4+h) - v(4)}{h} = 5, \text{ while} \tag{34}$$
$$\lim_{h\to 0^+} \frac{v(4+h) - v(4)}{h} = 0. \tag{35}$$

These one-sided limits are distinct, so the two-sided limit, which would be $v'(4)$, doesn't exist.

On the other hand, $v'(20) = -5/2$.

5.3 Part c

Acceleration, $a(t)$, is given by

$$a(t) = \begin{cases} t & \text{when } 0 < t < 4 \\ 0 & \text{when } 4 < t < 16 \\ -\dfrac{5}{2} & \text{when } 16 < t < 24. \end{cases} \tag{36}$$

5.4 Part d

The average rate of change of v over $8 \leq t \leq 20$ is

$$\frac{v(20) - v(8)}{20 - 8} = \frac{10 - 20}{20 - 8} = -\frac{5}{6}. \tag{37}$$

The hypotheses of the Mean Value Theorem require that a function be differentiable at every point of the interior of the interval on which we wish to apply the theorem, so we may not apply the Mean Value Theorem to the function v on the interval $[8, 20]$, because $v'(16)$ does not exist.

6 Problem 6

6.1 Part a

The degree-six Taylor polynomial , $T_6(x)$, about $x = 2$ for this function is

$$T_6(x) = \sum_{k=0}^{6} \frac{f^{(k)}(2)}{k!}(x-2)^k \tag{38}$$

$$= 7 + \frac{1}{18}(x-2)^2 + \frac{1}{324}(x-2)^4 + \frac{1}{4374}(x-2)^6. \tag{39}$$

6.2 Part b

The coefficient a_{2n} of $(x-2)^{2n}$ in the Taylor expansion of this function about $x = 2$ is $\dfrac{1}{(2n)\cdot 3^{2n}}$.

6.3 Part c

We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{2n+2}|x-2|^{2n+2}}{a_{2n}|x-2|^{2n}} &= \lim_{n\to\infty} \frac{(2n)3^{2n}|x-2|^{2n+2}}{(2n+2)3^{2n+2}|x-2|^{2n}} &(40)\\
&= \frac{|x-2|^2}{9} \lim_{n\to\infty} \frac{n}{n+1} &(41)\\
&= \frac{|x-2|^2}{9} \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{|x-2|^2}{9}, &(42)
\end{aligned}$$

and this limit is less than one when $-1 < x < 5$. The Ratio Test tells us that the series converges in the interval $(-1, 5)$ and diverges in $(-\infty, -1) \cup (5, \infty)$. When $x = -1$, the series becomes

$$7 + \sum_{n=1}^{\infty} a_{2n}(x-2)^{2n} = 7 + \sum_{n=1}^{\infty} \frac{(-3)^{2n}}{(2n)3^{2n}} = 7 + \sum_{n=1} \frac{1}{2n} = 7 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}, \tag{43}$$

This is the divergent harmonic series.

An altogether similar calculation at $x = 5$ also produces the divergent harmonic series. We conclude that the interval of convergence for this series is $(-1, 5)$.