AP Calculus 2006 BC (Form B) FRQ Solutions

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1 Problem 1

1.1 Part a

We must first find the intersection nearest the y-axis of the curve $y = f(x)$ with the negative x-axis. In order to do so, we solve the equation

$$0 = \frac{a^3}{4} - \frac{a^2}{3} - \frac{a}{2} + 3\cos a. \tag{1}$$

Solving numerically for a, we obtain $a \sim -1.37312$.

The area of the region R is thus

$$\int_a^0 f(x)\,dx \sim 2.90309, \tag{2}$$

where we have evaluated the integral numerically.

1.2 Part b

Using the method of washers, the required volume is

$$y(2) = \pi \int_a^0 \left([(f(x) - (-2)]^2 - [0 - (-2)]^2 \right)\,dx \sim 59.36140, \tag{3}$$

where we have again integrated numerically.

1.3 Part c

$$f'(x) = -\frac{1}{2} - \frac{2}{3}x + \frac{3}{4}x^2 - 3\sin x, \text{ so} \tag{4}$$
$$f'(0) = -\frac{1}{2}. \tag{5}$$

Thus, an equation for the line tangent to the curve $y = f(x)$ at the point $(0,3)$ is $y = 3 - x/2$. Solving numerically, we find that this line also meets the the curve at $(b, f(b))$, where $f(b) = 3 - b/2$, or $b \sim 3.38987$. The required integral is therefore

$$\int_0^b \left[3 - \frac{1}{2}x - f(x)\right] dx \sim 6.98200 \tag{6}$$

Note: Evaluation is not required. Numerical integration gives the result shown.

2 Problem 2

2.1 Part a

At any point $\big(x(t_0), y(t_0)\big)$, the slope of the line tangent to the curve is $y'(t_0)/x'(t_0)$. Thus, the slope of the tangent to the solution at $(2, -3)$ is $\sec(e^{-1})/\tan(e^{-1}) = \csc(e^{-1})$, and the equation of the tangent line at $(2, -3)$ is

$$y = -3 + (x-2)\csc(e^{-1}). \tag{7}$$

2.2 Part b

The acceleration vector, $\mathbf{a}(t)$, is

$$\mathbf{a}(t) = \frac{d}{dt}\mathbf{v}(t) = \frac{d}{dt}\big\langle \tan(e^{-t}), \sec(e^{-t})\big\rangle \tag{8}$$
$$= \big\langle -e^{-t}\sec^2(e^{-t}), -e^{-t}\sec(e^{-t})\tan(e^{-t})\big\rangle. \tag{9}$$

Speed at time t, $\sigma(t)$, is given by

$$\sigma(t) = \sqrt{\mathbf{v}(t)\cdot\mathbf{v}(\mathbf{t})} = \sqrt{\tan^2(e^{-t}) + \sec^2(e^{-t})}. \tag{10}$$

Thus

$$\mathbf{a}(1) = \big\langle -e^{-1}\sec^2(e^{-1}), -e^{-1}\sec(e^{-1})\tan(e^{-1})\big\rangle \tag{11}$$

and

$$\sigma(1) = \sqrt{\tan^2(e^{-1}) + \sec^2(e^{-1})} \tag{12}$$

2.3 Part c

Total distance traveled over the time interval $1 \leq t \leq 2$ is

$$\int_1^2 \sqrt{\tan^2(e^{-t}) + \sec^2(e^{-t})}\, dt \sim 1.05934, \tag{13}$$

where we have carried out the integration numerically.

2.4 Part d

If $t \geq 0$, then $0 < e^{-t} \leq 1 < \pi/2$. Thus,

$$x'(t) = \tan(e^{-t}) > 0 \tag{14}$$

when $t \geq 0$. This guarantees that $x(t) > x(1) = 2$ when $t > 1$, because the positivity of the derivative on this interval means that $x(t)$ is an increasing function there. On the other hand,

$$x'(t) = \tan(e^{-t}) \leq \tan(1) < 1.56 \tag{15}$$

when $0 \leq t \leq 1$. It follows from this observation that for $0 \leq t \leq 1$ we must have

$$2 - (1.56 - 1.56t) < x(t), \tag{16}$$

or $0 < 0.44 + 1.56t < x(t)$. So $x(t) = 0$ is also impossible when $0 \leq t \leq 1$.

3 Problem 3

3.1 Part a

If $y = ax^2$, then $y' = 2ax$, so $x = 0$ gives both $y = 0$ and $y' = 0$, so that condition (i) is satisfied. However, if $x = 4$, then $1 = y' = 8a$, by condition (ii), so that $a = 1/8$. But then, using the other part of condition (ii), we find that $1 = y = (4)^2/8 = 2$, which is not possible. The curve $y = ax^2$ therefore can't satisfy condition (ii) for any choice of a.

3.2 Part b

Let $g(x) = cx^3 - \frac{1}{16}x^2$. Then condition (i) requires that

$$1 = g(4) = 64c - 1, \tag{17}$$

so that $c = 1/32$. Taking this value for c, we have

$$g(x) = \frac{1}{32}x^3 - \frac{1}{16}x^2; \tag{18}$$

$$g'(x) = \frac{3}{32}x^2 - \frac{1}{8}x; \tag{19}$$

$$g'(4) = \frac{3}{32} \cdot 4^2 - \frac{1}{8} \cdot 4 = \frac{3}{2} - \frac{1}{2} = 1. \tag{20}$$

Thus, $g(x) = \dfrac{1}{32}x^3 - \dfrac{1}{16}x$ satisfies condtion (ii).

3.3 Part c

If

$$g(x) = \frac{1}{32}x^3 - \frac{1}{16}x = \frac{1}{16}\left(\frac{1}{2}x^3 - x^2\right), \tag{21}$$

then

$$g'(x) = \frac{1}{16}\left(\frac{3}{2}x^2 - 2x\right) = \frac{x}{32}(3x - 4). \tag{22}$$

On the interval $(0, 4/3)$, $x > 0$ while $(3x - 4) < 0$. Consequently, $f'(x) < 0$ on $(0, 3/4)$, and it follows that f cannot be increasing on $(0, 4)$.

3.4 Part d

Let $h(x) = x^n/k$, where k is a nonzero constant and n is a positive integer. If h meets condition (ii), then $1 = h(4) = 4^n/k$, while $1 = h'(4) = n4^{n-1}/k$, too. Thus,

$$1 = \frac{k}{k} = \frac{4^n}{n4^{n-1}}, \tag{23}$$

so that $n = 4$. But if $n = 4$ and $1 = 4^4/k$, then $k = 4^4 = 256$. Thus, condition (ii), in conjunction with $h(x) = x^n/k$, forces $h(x) = x^4/256$. From $h(x) = x^4/256$, condition (i), $h(0) = 0 = h'(0)$ is immediate. Also, $h'(x) = x^3/64 > 0$ for all $x > 0$, and this means that h is increasing between $x = 0$ and $x = 4$. So condition (iii) is met.

4 Problem 4

4.1 Part a

The point $(22, f(22))$ lies at the midpoint of the segment whose endpoints are $(20, 15)$ and $(24, 3)$, and whose slope is $(15 - 3)/(20 - 24) = -3$. The segment is part of the tangent line to the curve $y = f(x)$ at the point $(22, f(22))$. Thus, $f'(22) = -3$ calories per minute per minute.

4.2 Part b

The function f is increasing only on the intervals $[0, 4]$ and $[12, 16]$. On the latter interval, its rate of increase is

$$\frac{15-9}{16-12} = \frac{3}{2} \text{ calories per minute per minute.} \tag{24}$$

When $0 \le t \le 4$, we have

$$f'(t) = -\frac{3}{4}t^2 + 3t, \text{ and} \tag{25}$$

$$f''(t) = -\frac{3}{2}t + 3. \tag{26}$$

Then $f''(t) = 0$ when $t = 2$; $f''(t) > 0$ when $0 < t < 2$, and $f''(t) < 0$ when $2 < t < 4$. Thus, f' is an increasing function on the interval $[0, 2]$ and a decreasing function on the interval $[2, 4]$. By the First Derivative Test, f' has a relative maximum at $x = 2$. The value of this maximum is $f'(2) = 3$, and this relative maximum must be an absolute maximum for the interval $[0, 4]$. This is larger than $f'(t)$ when $12 < t < 16$, so the maximal value of $f'(t)$, *i.e.* the maximal rate of increase for the rate at which calories are burned is 3 calories per minute per minute at time $t = 2$.

4.3 Part c

The total number of calories burned over the time interval $6 \le t \le 18$ is $\int_6^{18} f(t)\,dt$. We compute the areas of the relevant rectangles and the relevant trapezoid, and we find that

$$\int_6^{18} f(t)\,dt = 6 \cdot 9 + 4 \cdot \frac{15+9}{2} + 2 \cdot 15 = 132 \text{ calories.} \tag{27}$$

4.4 Part d

It is required that

$$\frac{1}{18-6}\int_6^{18} [f(t)+c]\; dt = 15. \tag{28}$$

But

$$\begin{aligned}\frac{1}{18-6}\int_6^{18} [f(t)+c]\; dt &= \frac{1}{12}\int_6^{18} f(t)\,dt + \frac{1}{12}\int_6^{18} c\,dt \qquad (29)\\ &= \frac{1}{12}\cdot 132 + \frac{1}{12} ct\Big|_6^{18} \qquad (30)\\ &= 11 + c, \qquad (31)\end{aligned}$$

and this means that $15 = 11 + c$, so that $c = 4$.

5 Problem 5

5.1 Part a

If $f'(x) = 2f(x)(3-x)$, then

$$\frac{f'(x)}{f(x)} = 2(3-x), \tag{32}$$

or $f(x) \equiv 0$. In the latter case, $f(4) = 1$ is impossible, so we can rule it out. Then we have

$$\int_4^x \frac{f'(\xi)}{f(\xi)}\,d\xi = \int_4^x (3-\xi)\,d\xi. \tag{33}$$

Because $f(4) = 1$, $f(x) < 0$, at least for values of x that lie near $x = 4$, so the integration yields

$$\begin{aligned}\ln f(\xi)\Big|_4^x &= 2\left(3\xi - \frac{1}{2}\xi^2\right)\Big|_4^x; \qquad (34)\\ \ln f(x) &= 2\left(3x - \frac{1}{2}x^2\right) - 2\left(3\cdot 4 - \frac{1}{2}\cdot 4^2\right); \qquad (35)\\ \ln f(x) &= 6x - x^2 - 8, \qquad (36)\end{aligned}$$

and it follows that $f(x) = e^{-x^2+6x-8}$.

5.2 Part b

Given that $g(4) = 1$, $\lim_{x\to\infty} g(x) = 3$ and $\lim_{x\to\infty} g'(x) = 0$.

5.3 Part c

If $y' = 2y(3 - y)$, then

$$\begin{aligned} y'' &= 2y'(3-y) - 2yy' && (37)\\ &= 6y' - 4yy' = 2y'(3-2y). && (38) \end{aligned}$$

We know that zeros of $y' = y(3 - y)$ are the asymptotes of solutions, and these are not eligible for consideration as ordinates of inflection points. Consequently, the inflection points of solutions are to be found at those values of y for which $(3-2y)$ changes sign—or when $y = 3/2$. At such points, we have

$$\begin{aligned} y' &= 2y(3-y) && (39)\\ &= 2 \cdot \frac{3}{2}\left(3 - \frac{3}{2}\right) = \frac{9}{2}. && (40) \end{aligned}$$

6 Problem 6

6.1 Part a

We can find the Maclaurin series for $f'(x)$ by differentiating that for $f(x)$ term by term. Thus, when

$$f(x) = \frac{1}{1+x^3} = 1 - x^3 + x^6 + x^9 + \cdots + (-1)^n x^{3n} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{3n}, \qquad (41)$$

$$\begin{aligned} f'(x) &= \frac{-3x^2}{(1+x^3)^2} = \sum_{n=1}^{\infty} 3n(-1)^n x^{3n-1} && (42)\\ &= -3x^2 + 6x^5 - 9x^8 + \cdots + 3n(-1)^n x^{3n-1} + \cdots . && (43) \end{aligned}$$

6.2 Part b

The interior of the interval of convergence for the series for $f'(x)$ is always identical with the interior of the interval where the Maclaurin series for $f(x)$ converges. Hence, the series of (42) converges to $f'(x)$ for all $x \in (-1, 1)$—and, in particular, for $x = 1/2$. But

$$f'(x) = \frac{-3x^2}{(1+x^3)^2}, \text{ so that} \tag{44}$$

$$-\frac{3}{2^2} + \frac{6}{2^5} - \frac{9}{2^8} + \cdots + (-1)^n \frac{3n}{2^{3n-1}} + \cdots = f'\left(\frac{1}{2}\right) = -\frac{(3/4)}{(9/8)^2} = -\frac{16}{27}. \tag{45}$$

6.3 Part c

We find the series for $\int_0^x f(t)\,dt$ by integrating the series for f term by term. Thus, the required series is

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1} = x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \cdots + (-1)^n \frac{x^{3n+1}}{3n+1} + \cdots. \tag{46}$$

6.4 Part d

We know that the Maclaurin series for $\int_0^x f(t)\,dt$ represents the function in the interval $(-1, 1)$. Hence, according to (46), when $|x| < 1$ we may write

$$\int_0^x f(t)\,dt \sim x - \frac{x^4}{4} + \frac{x^7}{7}, \tag{47}$$

where we have ignored all terms from the fourth on. When $x = 1/2$, this is a series whose n-th term alternates in sign and whose magnitude decreases monotonically to 0 as $n \to \infty$. According to the Alternating Series Test, the magnitude of the difference between the sum of the first three terms of the series and the sum of the series is at most the magnitude of the fourth term—which is

$$\frac{1}{10 \cdot 2^{10}} = \frac{1}{10240} < \frac{1}{10000}. \tag{48}$$

Consequently, the value of $\int_0^{1/2} f(t)\,dt$ is within $1/10000$ of

$$\frac{1}{2} - \frac{1}{2^4 \cdot 4} + \frac{1}{2^7 \cdot 7} = \frac{435}{896}. \tag{49}$$

Note: For the curious:

$$\int_0^{1/2} f(t)\,dt = \frac{\sqrt{3}\pi + \ln 27}{18}, \tag{50}$$

and the magnitude of the difference is

$$\left| \frac{\sqrt{3}\pi + \ln 27}{18} - \frac{435}{896} \right| \sim 0.000089129278. \tag{51}$$