AP Calculus 2007 BC (Form B) FRQ Solutions

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1 Problem 1

1.1 Part a

The curve $y = e^{2x-x^2}$ intersects the line $y = 2$ where $2x - x^2 = \ln 2$, or, where $x = 1 \pm \sqrt{1 - \ln 2}$. Thus, the area of the region R is

$$\int_{1-\sqrt{1-\ln 2}}^{1+\sqrt{1-\ln 2}} \left(e^{2x-x^2} - 2\right)\, dx \sim 0.51414, \tag{1}$$

where we have carried out the integration numerically.

1.2 Part b

The curve $y = 3^{2x-x^2}$ intersects the line $y = 1$ where $2x - x^2 = \ln 1 = 0$, or at $x = 0$ and at $x = 2$. Thus, the sum of the areas of the regions R and S is the integral of $e^{2x-x^2} - 1$ from zero to two. From this, we subtract the integral of Part a, above, to obtain the area of the region S. Once again, we have no choice but to carry the integration out numerically.

$$\int_0^2 \left(e^{2x-x^2} - 1\right)\, dx - \int_{1-\sqrt{1-\ln 2}}^{1+\sqrt{1-\ln 2}} \left(e^{2x-x^2} - 2\right)\, dx \sim 1.54601. \tag{2}$$

1.3 Part c

Using the method of washers, the volume of the solid generated by revolving the region R about the line $y = 1$ is

$$\pi \int_{1-\sqrt{1-\ln 2}}^{1+\sqrt{1+\ln 2}} \left[\left(e^{2x-x^2} - 1 \right)^2 - 1 \right] dx \sim 4.14661. \tag{3}$$

Note: The integration was not required, but we've done it numerically to satisfy the reader's curiosity.

2 Problem 2

2.1 Part a

For an object moving along a curve in the xy-plane so that its position at time t is $(x(t), y(t))$, speed is

$$\sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{\left[\arctan\left(\frac{t}{1+t} \right) \right]^2 + [\ln(1+t^2)]^2}. \tag{4}$$

Thus, speed when $t = 4$ is given by

$$\sqrt{[x'(4)]^2 + [y'(4)]^2} = \sqrt{\left[\arctan\left(\frac{4}{5} \right) \right]^2 + [\ln(17)]^2} \sim 2.91245. \tag{5}$$

2.2 Part b

Total distance traveled is the integral of speed with respect to time. We integrate numerically:

$$\int_0^4 \sqrt{\left[\arctan\left(\frac{t}{1+t} \right) \right]^2 + [\ln(1+t^2)]^2} \sim 6.42330 \tag{6}$$

2.3 Part c

To find $x(4)$, we note first that $x(0) = -3$. By the Fundamental Theorem of Calculus,

$$x(4) = x(0) + \int_0^4 x'(t)\,dt \tag{7}$$

$$= -3 + \int_0^4 \arctan\left(\frac{t}{1+t}\right)\,dt \sim -0.892059. \tag{8}$$

(We have again carried out the integration numerically.)

2.4 Part d

The slope of the tangent line to the curve $\big(x(t), y(t)\big)$ at the point corresponding to $t = t_0$ is $y'(t_0)/x'(t_0)$, This ratio must be 2 at the point we seek. Thus we must solve the equation

$$\frac{\ln(1+t^2)}{\arctan[t/(1+t)]} = 2 \tag{9}$$

for t. Numerical solution gives $t \sim 1.35766$.

3 Problem 3

3.1 Part a

We are given $W(v) = 55.6 - 22.1v^{0.16}$. So $W'(v) = -3.536v^{-0.84}$. Thus, $W'(20) = -0.28553$. This means that when the temperature is $32^\circ\ F$ and windspeed is 20 mph, the windchill is decreasing at the rate of $-0.28553^\circ F$ per mile per hour.

3.2 Part b

The average rate of change of W over the interval $[5, 60]$ is

$$\overline{W} \sim \frac{W(60) - W(5)}{60 - 5} \tag{10}$$

$$\sim \frac{13.05030281 - 27.00912318}{55} = -0.25380. \tag{11}$$

In order to obtain the value of v for which $W'(v) = \overline{W}$ we solve the equation $W'(v) = \overline{W}$ numerically, and we get

$$v \sim 23.011026 \tag{12}$$

Note: The precision of the numbers given in the statement of this problem makes this level of precision (or even the required level of three digits to the right of the decimal point) doubtful.

3.3 Part c

We put $v(t) = 20 + 5t$. Then

$$\frac{d}{dt}W[v(t)] = \frac{17.68}{(20+5t)^{0.84}}, \text{ so that} \tag{13}$$

$$\left.\frac{d}{dt}W[v(t)]\right|_{t=3} = -0.89220. \tag{14}$$

Thus, when $t = 3$, the rate of change of wind chill with respect to time is -0.89220 degrees Fahrenheit per hour.

4 Problem 4

4.1 Part a

Local maxima are to be found at points where the derivative undergoes a sign change from positive to negative. For this function, that happens twice: when $x = -3$ and when $x = 4$.

4.2 Part b

Any point where the derivative changes from increasing to decreasing, or *vice versa*, is an inflection point. For this function, we find such points at $x = -4$, $x = -1$, and $x = 2$.

4.3 Part c

If the derivative is both positive and increasing, then the function has positive slope and is concave upward. This function displays such behavior on the intervals $(-5,-5)$ and $(1,2)$. (When we should include any endpoints depends on which of several commonly used choices we have made for the definitions of upward and downward concavity.)

4.4 Part d

The absolute minimum value of $f(x)$ over the interval $[-5,5]$ is $f(1) = 3$. We know that $x = 1$ gives a local minimum because the derivative changes sign from negative to positive at $x = 1$. The only other possibilities for minima are at the endpoints of the interval, because we have already (Part b, above) identified the other critical points in the interval as the locations of relative maxima. We have

$$f(-5) = 3 + \int_1^{-5} f'(x)\,dx = 3 - \frac{\pi}{2} + 2\pi = 3 + \frac{3}{2}\pi > 3, \tag{15}$$

the integral being the sum of the signed areas of two triangles. On the other hand, we have

$$f(5) = 3 + \int_1^{5} f'(x)\,dx = 3 + \frac{1}{\cancel{2}} \cdot 3 \cdot \cancel{2} - \frac{1}{2} \cdot 1 \cdot 1 = \frac{11}{2} > 3, \tag{16}$$

this integral being the sum of the areas of two signed triangles.

It follows that the absolute minimum value of $f(x)$ over the interval $[-5,5]$ is to be found at $x = 1$. We have $f(1) = 3$.

5 Problem 5

5.1 Part a

If $y' = 3x + 2y + 1$, then $y'' = 3 + 2y' = 6x + 4y + 5$.

5.2 Part b

If $y = mx+b+e^{rx}$, then $y' = m+re^{rx}$. If this is to be a solution to the differential equation $y' = 3x + 2y + 1$, then, for all values of x, it must be true that

$$\begin{aligned} m + re^{rx} &= 3x + 2(mx + b + e^{rx}) + 1 && (17)\\ &= (3 + 2m)x + 2e^{rx} + (2b + 1). && (18) \end{aligned}$$

There are two cases: $r = 0$ and $r \neq 0$.

In the first case, equation (18) becomes $m = (3 + 2m)x + (2b + 3)$, which can be true for all x only if $2b + 3 = m$ and $3 + 2m = 0$. Simultaneous solution of these two equations leads to the conclusion that $r = 0$, $m = -3/2$, and $b = -9/4$.

In the second case, (18) holds for all x only if $3 + 2m = 0$, $r = 2$, and $2b + 1 = m$. Solving these last three equations simultaneously, we learn that $r = 2$, $m = -3/2$, and $b = -5/2$.

5.3 Part c

Taking $f(0) = -2$, we find that $f'(0) = 3 \cdot 0 + 2 \cdot (-2) + 1 = -3$. Hence, $f(1/2) \sim -2 + (-3)(1/2) = -7/2$. Then $f'(1/2) \sim 2 \cdot (1/2) + 2 \cdot (-7/2) + 1 = -9/2$, and this leads us to $f(1) \sim -7/2 + (-9/2) \cdot (1/2) = -23/4$.

5.4 Part d

The Euler's Method approximation for $g(1)$ with step-size 1 is

$$g(0) + [3 \cdot 0 + 2 \cdot g(0) + 1] \cdot 1 = k + [0 + 2k + 1]. \tag{19}$$

If this is to be 0, we must have $3k + 1 = 0$, or $k = -1/3$.

6 Problem 6

6.1 Part a

The Taylor expansion for e^u about $u = 0$ is

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots + \frac{u^n}{n!} + \cdots . \tag{20}$$

Consequently,

$$6e^{-x/3} = 6\left(1 - \frac{x}{3} + \frac{x^2}{9\cdot 2!} - \frac{x^3}{27\cdot 3!} + \cdots + (-1)^n \frac{x^n}{3^n\cdot n!} + \cdots\right) \tag{21}$$

$$= 6 - 2x + \frac{x^2}{3} - \frac{x^3}{54} + \cdots + (-1)^n \frac{2x^n}{3^{n-1}\cdot n!} + \cdots \tag{22}$$

6.2 Part b

We need only integrate the terms of the preceding series, term by term, to find the required series. Thus

$$\int_0^x f(t)\,dt = 6x - x^2 + \frac{x^3}{9} - \frac{x^4}{108} + \cdots + (-1)^n \frac{2x^{n+1}}{3^{n-1}(n+1)!}. \tag{23}$$

6.3 Part c

If $h(x) = \sum_{k=1}^{\infty} \frac{x^k}{k!}$, then $h(x) = e^x$. If $f(x) = 6e^{-x/3}$, then $f'(x) = -2e^{-x/3}$. Thus, $kf'(ax) = -2ke^{-ax/3}$, and the equation $h(x) = kf'(ax)$ becomes $e^x = -2ke^{-ax/3}$. This is possible only if $-2k = 1$ and $-a/3 = 1$. Thus $a = -3$ and $k = -1/2$.