AP Calculus 2007 BC FRQ Solutions

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1 Problem 1

1.1 Part a

The limits of the integral that gives the area are the solutions of the equation

$$\begin{aligned} 2 &= \frac{20}{1+x^2}; && (1)\\ 2+2x^2 &= 20; && (2)\\ x^2 &= 9, && (3) \end{aligned}$$

or $x = \pm 3$. The required area is therefore

$$\begin{aligned} \int_{-3}^{3} \left[\frac{20}{1+x^2} - 2\right]\,dx &= \left[20\arctan x + 2x\right]\Big|_{-3}^{3} && (4)\\ &= [20\arctan 3 - 6] - [20\arctan(-3) - (-6)] && (5)\\ &= 40\arctan 3 - 12. && (6) \end{aligned}$$

1.2 Part b

Using the method of washers, we find that the required volume is

$$\pi\int_{-3}^{3}\left[\left(\frac{20}{1+x^2}\right)^2 - 4\right]\,dx \sim 1871.19010, \tag{7}$$

where we have carried out the integration numerically.

Note: The integral is elementary, and can be done with a trigonometric substitution, but numerical integration saves time. For the curious, we find that

$$\int \frac{dx}{(1+x^2)^2} = \frac{1}{2}\left(\arctan x + \frac{x}{1+x^2}\right), \tag{8}$$

and the exact value of this volume turns out to be $\pi(96 + 400\arctan 3)$.

1.3 Part c

The diameter of the semicircle at $x = t$ is $\frac{20}{1+t^2} - 2$, so the radius is $\frac{10}{1+t^2} - 1$. Hence the area $A(t)$ of the cross section at $x = t$ is

$$A(t) = \frac{\pi}{2}\left(\frac{10}{1+t^2} - 1\right)^2. \tag{9}$$

The required volume is therefore

$$\frac{\pi}{2}\int_{-3}^{3}\left(\frac{10}{1+t^2} - 1\right)^2 dt \sim 174.26846, \tag{10}$$

where we have again integrated numerically to save time. We find that he exact volume is $\frac{\pi}{2}(36 + 60\arctan 3)$.

2 Problem 2

2.1 Part a

The amount of water that enters the tank during the time interval $0 \leq t \leq 7$ is

$$\int_0^7 f(t)\,dt = \int_0^7 100t^2 \sin\sqrt{t}\,dt. \tag{11}$$

Numerical integration gives $\int_0^7 f(t)\,dt \sim 8263.80654$, or, to the nearest gallon, 8264 gallons.

Note: The integration is elementary, but requires repeated integration by parts, and so consumes an unpleasant amount of time.

2.2 Part b

From the graph and what we are given about the intersection points of the curves, we see that the rate at which water leaves the tank exceeds that at which it leaves the tank on the intervals $[0, 1.617)$ and $(3, 5.076)$. It follows that the amount of water in the tank is decreasing on each of the intervals $[0, 1.617]$ and $[3.076]$.

Note: We include the endpoints because a continuous function that is decreasing on any interval must be decreasing on the closure of that interval. In the past, the readers have not paid any attention to this subtlety.

2.3 Part c

The rate at which the amount of water in the tank increases is, as we have seen in Part b, above, negative on $(3, 5.076)$, positive on $(1.617, 3)$. By the First Derivative Test, the amount of water in the tank has a local maximum at $t = 3$. By similar reasoning, we see that the critical point at $t = 1.627$ gives a local minimum, and we can therefore exclude that value of t from our search for an absolute maximum.

We must also consider the amount of water in the tank when $t = 0$, which is given as 5000 gallons, and when $t = 7$. Over the interval $[0, 7]$,

$$3 \times 250 + 4 \times 2000 = 8750 \text{ gallons} \tag{12}$$

have left the tank, while about 8263.806 gallons have entered the tank (see Part a, above). Thus, at time $t = 7$ there are about

$$5000 + 8263.806 - 8750 = 4513.806 \text{ gallons} \tag{13}$$

in the tank. The amount of water in the tank when $t = 3$ is

$$5000 + \int_0^3 \left[100t^2 \sin\sqrt{t} - 250\right] \, dt \sim 5126.59080 \text{ gallons.} \tag{14}$$

We see now that the absolute maximum occurs at $t = 3$ and is, to the nearest gallon, 5127 gallons.

3 Problem 3

3.1 Part a

The required area is

$$\frac{1}{2}\int_{-2\pi/3}^{2\pi/3} r^2\,d\theta + \frac{1}{2}\int_{2\pi/3}^{4\pi/3} r^2\,d\theta = 2\int_{-2\pi/3}^{2\pi/3} d\theta + \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (3+2\cos\theta)^2\,d\theta. \tag{15}$$

The first of these latter two integrals is easily seen to have the value $8\pi/3$. Integrating numerically, we find that the value of the other integral is about 1.99289. Consequently, the required area is about 10.37047.

Note: The second integration is elementary but tedious. Numerical integration saves time. The exact area is, in fact, $\frac{1}{6}(38\pi - 33\sqrt{3})$.

3.2 Part b

$r(\theta) = 3 + 2\cos\theta$, so $r'(\theta) = -2\sin\theta$. Thus, when $\theta = \pi/3$,

$$\frac{dr}{dt} = \frac{dr}{d\theta} = -\sqrt{3}. \tag{16}$$

This means that the particle traces out the curve in such a way that when it moves through the point whose polar coordinates are $r = 4$, $\theta = \pi/3$, the radial component of its velocity vector points toward the origin and has magnitude $\sqrt{3}$.

3.3 Part c

We have $y(\theta) = r(\theta)\sin\theta = (3 + 2\cos\theta)\sin\theta$, so

$$y'(\theta) = \cos\theta(3 + 2\cos\theta) - 2\sin^2\theta, \text{ and} \tag{17}$$

$$y'(\pi/3) = \frac{1}{2}. \tag{18}$$

Thus, at the instant when $\theta = \pi/3$, the vertical component of the particle's veloctiy vector points upward and has magnitude $1/2$.

4 Problem 4

4.1 Part a

If $f(e) = 2$ and $f'(x) = x^2 \ln x$, then $f'(e) = e^2 \ln e = e^2$.

4.2 Part b

Because $f'(x) = x^2 \ln x$, then $f'(x) = x(2\ln x + 1)$. When $x > 1$, $\ln x > 0$, so $f''(x) > 0$ on $(1, 3)$. Thus, the graph of the curve $y = f(x)$ is concave upward on $(1, 3)$.

4.3 Part c

We let $u = \ln x$, $dv = x^2\,dx$. Then $du = dx/x$, and we may take $v = x^3/3$. Then there is a constant c such that

$$\int x^2 \ln x\,dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2\,dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c. \tag{19}$$

But $f(e) = 2$, so

$$2 = \frac{1}{3}e^3 \ln e - \frac{1}{9}e^3 + c = \frac{2}{9}e^3 + c, \tag{20}$$

and $c = 2 - \frac{2}{9}e^3$. Thus,

$$f(x) = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + 2 - \frac{2}{9}e^3. \tag{21}$$

5 Problem 5

5.1 Part a

The linearization (*i.e.* the tangent line approximation) of r at $t = 5$ is the linear function

$$L(x) = f(5) + r'(5)(t - 5) = 30 + 2(t - 5). \tag{22}$$

Thus, an approximate value for $r(5.4)$ is

$$L(5.4) = 30 + 2(5.4 - 5) = 30.8 \text{ feet}. \tag{23}$$

The curve $r = r(t)$ is given to be concave downward on the interval $(0, 12)$, so the tangent line at each point in that interval lies on or above the curve. This means that the linearization estimate is an overestimate.

5.2 Part b

Because

$$V(t) = \frac{4}{3}\pi[r(t)]^3, \text{ we have} \tag{24}$$

$$V'(t) = 4\pi[r(t)]^2 r'(t). \text{ Thus} \tag{25}$$

$$V'(5) = 4\pi \cdot 30^2 \cdot 2 \sim 22,619.46711 \text{ ft}^3/\text{min}. \tag{26}$$

5.3 Part c

The right Riemann sum corresponding to the data given is

$$4.0(2-0) + 2.0(5-2) + 1.2(7-5) + 0.6(11-7) + 0.5(12-11) = 19.3 \text{ feet}. \tag{27}$$

By the Fundamental Theorem of Calculus, the integral

$$\int_0^{12} r'(t)\,dt = r(12) - r(0) \tag{28}$$

gives the change, in feet, in the radius from its value at $t = 0$ to its value at $t = 12$.

5.4 Part d

The function r is given concave down, so r' is a non-increasing function. Consequently, $r'(t) \geq r'(b)$ when t lies anywhere in an interval $[a, b]$. It follows that each term of a right Riemann sum is at most the area under the curve in the corresponding interval. The right Riemann sum therefore underestimates the integral.

6 Problem 6

6.1 Part a

In general

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots + \frac{u^2}{n!} + \cdots. \tag{29}$$

Thus,

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots. \tag{30}$$

6.2 Part b

We have

$$\lim_{x\to 0} \frac{1 - x^2 - f(x)}{x^4} = \lim_{x\to 0} \frac{1 - x^2 - \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right)}{x^4} \tag{31}$$

$$= \lim_{x\to 0} \left(-\frac{1}{2!} + \frac{x^2}{3!} - \frac{x^4}{4!} + \cdots\right) = -\frac{1}{2}. \tag{32}$$

6.3 Part c

Integrating the series from (30) term by term, we obtain

$$\int_0^x e^{-t^2}\, dt = x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots \tag{33}$$

Substituting $x = 1/2$ into the first two terms of this series, we obtain the approximation

$$\int_0^{1/2} e^{-t^2}\, dt \sim \frac{1}{2} - \frac{1}{24} = \frac{11}{24}. \tag{34}$$

6.4 Part d

As $n \to \infty$, the n-th term of the series (33) decreases to zero and the terms alternate in sign when $x = 1/2$. Thus, the Alternating Series Test guarantees that the magnitude of the error in our approximation doesn't exceed the magnitude of the first of all the terms we omitted. The bound on the error in that approximation is therefore

$$\frac{(1/2)^5}{5 \cdot 2!} = \frac{1}{320} < \frac{1}{200}. \tag{35}$$