AP Calculus 2008 AB (Form B) FRQ Solutions

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1 Problem 1

1.1 Part a

Acceleration is the time derivative of velocity, and velocity is given as $\langle\sqrt{3t}\,, 3\cos\frac{t^2}{2}\rangle$. Thus, acceleration is

$$\frac{d}{dt}\left\langle\sqrt{3t}\,, 3\cos\frac{t^2}{2}\right\rangle = \left\langle\frac{1}{2}\sqrt{\frac{3}{t}}, -3t\sin\frac{t^2}{2}\right\rangle. \tag{1}$$

When $t = 4$, acceleration is $\left\langle\dfrac{\sqrt{3}}{4}, -12\sin 8\right\rangle$.

1.2 Part b

By the Fundamental Theorem of Calculus, $y(0)$, the y-coordinate of position at time $t = 0$, satisfies

$$y(4) - y(0) = \int_0^4 y'(\tau)\,d\tau. \tag{2}$$

From what we have been given, we conclude that

$$y(0) = 5 - 3\int_0^4 \cos\frac{\tau^2}{2}\,d\tau \tag{3}$$

$$\sim 1.60060, \tag{4}$$

where we have carried out the integration numerically.

1.3 Part c

Speed is the magnitude of velocity, or

$$\sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{3t + 9\cos^2\frac{t^2}{2}}. \tag{5}$$

A calculator plot shows that the first time that speed reaches 3.5 is just larger than $t = 2$. Numerical solution gives $t \sim 2.22558$.

1.4 Part d

To find total distance, we integrate speed, which must be done numerically. We have

$$\int_0^4 \sqrt{3\tau + 9\cos^2\frac{\tau^2}{2}}\, d\tau \sim 13.18242. \tag{6}$$

2 Problem 2

2.1 Part a

We integrate speed, the magnitude of velocity, to obtain distance traveled. (The problem gives speed, but the given speed is never zero, and this guarantees that travel is unidirectional. We take the direction of travel to be the positive direction.) The required distance is

$$x(2) = \int_0^2 120\left(1 - e^{-10t^2}\right) dt \sim 206.37005 \text{ kilometers}. \tag{7}$$

Note that the integral must be evaluated numerically, which we have done.

2.2 Part b

We have $g(x) = 0.05\,x\left(1 - e^{-x/2}\right)$. We must find the value of

$$\frac{d}{dt} g[x(t)] = g'[x(t)]x'(t) \tag{8}$$

when $t = 2$. But

$$g'(x) = \frac{d}{dx}\left[0.05x\left(1 - e^{-x/2}\right)\right] \tag{9}$$

$$= 0.05\left(1 - e^{-x/2}\right) + 0.025xe^{-x/2}, \text{ while it is given that} \tag{10}$$

$$x'(t) = r(t) = 120(1 - e^{-10t^2}). \tag{11}$$

We have $x(2)$ from equation (7), while $x'(2) = r(2) = 120(1 - e^{-40})$. Thus,

$$\left.\frac{d}{dt}g[x(t)]\right|_{t=2} = \left[6\left(1 - e^{-x(2)/2}\right) + 3\,x(2)e^{-x(2)/2}\right](1 - e^{-40}) \tag{12}$$

$$\sim 6.00000 \tag{13}$$

The rate of change, taken with respect to time, of the number of liters of gasoline used by the car when $t = 2$ hours is approximately 6.00000 liters/hour.

2.3 Part c

We begin by solving the equation $120(1 - e^{-10t^2}) = 80$ for t. It is easy to see that the following are equivalent:

$$120(1 - e^{-10t^2}) = 80; \tag{14}$$

$$1 - e^{-10t^2} = \frac{2}{3}; \tag{15}$$

$$e^{10t^2} = 3; \tag{16}$$

$$10t^2 = \ln 3; \tag{17}$$

$$t^2 = \frac{\ln 3}{10}. \tag{18}$$

The only positive solution is

$$t = \sqrt{\frac{\ln 3}{10}} \sim 0.33145. \tag{19}$$

Thus, speed reaches 80 km/hr when $t \sim 0.33145$ hours. At that time, position is given by

$$x\left[\sqrt{\ln(3)/10}\right] \sim 120\int_0^{\ln(3)/\sqrt{10}}\left(1 - e^{-10\tau^2}\right)\,d\tau \sim 10.79410. \tag{20}$$

We carry out the required integration numerically again, and we find that the amount of fuel, in liters, consumed up to that time is

$$g\left(x\left[\sqrt{\ln(3)/10}\right]\right) \sim 0.53726. \tag{21}$$

3 Problem 3

3.1 Part a

The trapezoidal sum that approximates the area of the river cross section is

$$\frac{1}{2}\left[(0+7)\cdot 8+(7+8)\cdot 6+(8+2)\cdot 8+(2+0)\cdot 2\right]=115 \tag{22}$$

The area of the river cross section is about 115 square feet.

3.2 Part b

We integrate area times volumetric flow, with respect to time. Then we divide the result by the length of the time interval to obtain the average volumetric flow.

$$\frac{115}{120}\int_0^{120}\left(16+2\sin\sqrt{t+10}\right)\,dt \sim 1807.16972\ \text{ft}^3/\text{min}. \tag{23}$$

The integration is elementary, but numerical integration is faster, and that is the technique we have used.

3.3 Part c

Once again, we integrate depth from 0 to 24:

$$\int_0^{24} 8\sin\frac{\pi x}{24}\,dx = -\frac{192}{\pi}\cos\frac{\pi x}{24}\bigg|_0^{24} = -\frac{192}{\pi}\cos\pi+\frac{192}{\pi}\cos 0=\frac{384}{\pi}. \tag{24}$$

Based on this model, the area of the cross section is $384/\pi \sim 122.23100$ square feet.

3.4 Part d

We must again integrate area times volumetric flow, this time using the area found in Part c, above, and with t varying from 40 to 60. We integrate numerically again, and we obtain

$$\frac{384}{\pi}\cdot\frac{1}{20}\int_{40}^{60}\left(16+2\sin\sqrt{t+10}\right)\,dt \sim 2181.91265. \tag{25}$$

The average volumetric flow during the interval $40 \leq t \leq 60$ is about 2181.91265 cubic feet per minute. This value exceeds the given safety limit of 2100 cubic feet per minute and indicates that water must be diverted.

4 Problem 4

4.1 Part a

By the Fundamental Theorem of Calculus and the Chain Rule, from

$$f(x) = \int_0^{3x} \sqrt{4+t^2}\, dt, \tag{26}$$

we see that

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\int_0^{3x} \sqrt{4+t^2}\, dt\right] & (27)\\ &= \sqrt{4+9x^2} \cdot \frac{d}{dx}(3x) = 3\sqrt{4+9x^2}. & (28)\end{aligned}$$

Then from $g(x) = f(\sin x)$ it follows from what we have seen above and, again, the Chain Rule that

$$\begin{aligned} g'(x) &= f'(x)\cos x & (29)\\ &= 3\cos x\sqrt{4+9\sin^2 x}. & (30)\end{aligned}$$

4.2 Part b

The slope of the tangent line to $y = g(x)$ at $x = \pi$ is

$$g'(\pi) = 3\cos\pi\sqrt{4+9\sin^2\pi} = -6 \tag{31}$$

An equation for the tangent line to the curve $y = g(x)$ at the point corresponding to $x = \pi$ is therefore

$$\begin{aligned} &y = g(\pi) + g'(\pi)(\pi) = 0 - 6(x-\pi), \text{ or} & (32)\\ &y = 6(\pi - x). & (33)\end{aligned}$$

4.3 Part c

When $x > 0$, the value $f(x)$ is the integral of a positive quantity over an interval $[0, 3x]$, and from this it follows that f is an increasing function throughout the interval $[0, \infty)$. But the sine function carries the interval $[0, \pi]$ onto the interval $[0, 1]$, and the maximum value of $f(x)$ on this interval is $f(1) = f[\sin(\pi/2)]$. Therefore, the maximal value of $g(x)$ on $[0, \pi]$ is

$$g(\pi/2) = f[\sin(\pi/2)] = \int_0^3 \sqrt{4+t^2}\, dt. \tag{34}$$

Note: Evaluation of the integral is not required. However, a trig substitution followed by an integration by parts gives

$$\int_0^3 \sqrt{4+t^2}\, dt = \left[\frac{1}{2}t\sqrt{4+t^2} + 2\ln\left|\frac{1}{2}\left(t+\sqrt{4+t^2}\right)\right|\right]\Bigg|_0^3 \tag{35}$$

$$= \frac{3}{2}\sqrt{13} + 2\ln\frac{3+\sqrt{13}}{2}. \tag{36}$$

5 Problem 5

5.1 Part a

Inflections points for g occur at local extrema of g'. There are two such on the given graph: One at $x = 1$, and one at $x = 4$.

5.2 Part b

From the picture, we see that $g'(x) < 0$ throughout the intervals $[-3, -1)$ and $(2, 6)$. Consequently, g is decreasing on the intervals $[-3, -1]$ and $[2, 6]$, and g is increasing on the intervals $[-1, 2]$ and $[6, 7]$. [**Note:** A function that is continuous on an interval $[a, b]$ and increasing on (a, b) must necessarily be increasing on $[a, b]$.] It follows that the absolute maximum value of (x) must lie either at $x = 2$, which is the boundary between an interval where g increases to an interval where g decreases, or at one of the endpoints of the interval $[-3, 7]$.

We are given $g(2) = 5$. Making repeated use of the fact that the area of a triangle is one-half its altitude times its base, and that the Fundamental Theorem of Calculus guarantees us that $g(x) = \int_2^x g'(\xi)\, d\xi$, we find that

$$g(-3) = 5 - \left(\frac{3}{2} - 4\right) = \frac{15}{2} \text{ and} \tag{37}$$

$$g(7) = 5 - 4 + \frac{1}{2} = \frac{3}{2}. \tag{38}$$

We now see that $g(-3) = 15/2$ gives the absolute maximum value for $g(x)$ when $-3 \leq x \leq 7$.

5.3 Part c

The average rate of change of $g(x)$ on the interval $[-3, 7]$ is

$$\frac{g(7) - g(-3)}{7 - (-3)} = \frac{3/2 - 15/2}{7 + 3} = -\frac{3}{5}, \tag{39}$$

where we have used the values of $g(-3)$ and $g(7)$ that we computed in Part b, above.

5.4 Part d

The average rate of change of $g'(x)$ on the interval $[-3, 7]$ is

$$\frac{g'(7) - g'(-3)}{7 - (-3)} = \frac{1 - (-4)}{7 - (-3)} = \frac{1}{2}, \tag{40}$$

were we have read $g'(-3) = -4$ and $g'(7) = 1$ from the given graph.

The Mean Value Theorem does not apply to the function g' on the interval $[-3, 7]$, because the hypotheses of that theorem require that $g''(x)$ exist for all values of x that lie in $(-3, 7)$. However, $g''(1)$ and $g''(4)$ do not exist for this function. (This can be seen by considering the left and right derivatives of g' at the points in question.)

6 Problem 6

6.1 Part a

Using the geometric series to expand $(1 + x^2)^{-1}$ in powers of x, we find that

$$\frac{2x}{1 + x^2} = 2x \left(\frac{1}{1 + x^2} \right) \tag{41}$$

$$= 2x \left(1 - x^2 + x^4 - x^6 + \cdots + (-1)^k x^{2k} + \cdots \right), \text{ or} \tag{42}$$

$$\frac{2x}{1 + x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + \cdots = \sum_{k=0}^{\infty} 2(-1)^k x^{2k+1} \tag{43}$$

throughout the interval where $x^2 < 1$.

6.2 Part b

The series diverges when $x = 1$ because

$$\lim_{k \to 0} \left[2(-1)^k x^{2k+1} \right] \neq 0. \tag{44}$$

(In fact, the limit doesn't even exist.)

6.3 Part c

We have

$$\int_0^x \frac{2t}{1+t^2} = \ln(1+x^2). \tag{45}$$

Because

$$\frac{2t}{1+t^2} = 2t - 2t^3 + 2t^5 - 2t^7 + \cdots \tag{46}$$

when $|x| < 1$, it follows that

$$\ln(1+x^2) = \int_0^x \left(2t - 2t^3 + 2t^5 - 2t^7 + \cdots \right) \, dt \tag{47}$$

$$= \int_0^x 2t \, dt - \int_0^x 2t^3 \, dt + \int_0^x 2t^5 \, dt - \int_0^x 2t^7 \, dt + \cdots \tag{48}$$

$$= x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \cdots + \frac{(-1)^k}{k+1} x^{2k+2} + \cdots \tag{49}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{2k+2} \tag{50}$$

when $|x| < 1$.

6.4 Part d

From Part c of this problem, we know that

$$\ln \frac{5}{4} = \ln \left[1 + \left(\frac{1}{2} \right)^2 \right] \tag{51}$$

$$= \left(\frac{1}{2} \right)^2 - \frac{1}{2} \left(\frac{1}{2} \right)^4 + \frac{1}{3} \left(\frac{1}{2} \right)^6 - \frac{1}{4} \left(\frac{1}{2} \right)^8 + \cdots \tag{52}$$

The terms of this series are clearly decreasing in magnitude, and have limit zero, so the Alternating Series Test guarantees that the magnitude of the error introduced by truncating the series is at most the magnitude of the first discarded term. But

$$\frac{1}{3}\left(\frac{1}{2}\right)^6 = \frac{1}{2 \cdot 64} = \frac{1}{192} < \frac{1}{100}, \tag{53}$$

so the desired rational number is given by

$$A = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4 = \frac{1}{4} - \frac{1}{32} = \frac{7}{32}. \tag{54}$$