AP Calculus 2008 BC FRQ Solutions

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1 Problem 1

1.1 Part a

The area of the region R is

$$\int_0^2 \left[\sin \pi x - \left(x^3 - 4x\right)\right]\,dx = \left[-\frac{1}{\pi}\cos \pi x - \left(\frac{1}{4}x^4 - 2x^2\right)\right]\Bigg|_0^2 \tag{1}$$

$$= \left[\cancel{-\frac{1}{\pi}\cos 2\pi} - (4-8)\right] - \left[\cancel{-\frac{1}{\pi}\cos 0} - 0\right] = 4. \tag{2}$$

1.2 Part b

We must first find solutions, in the interval $[0, 2]$, of the equation $x^3 - 4x = -2$ to find the limits of integration. We do this numerically, and find that the solutions we need are $x_1 \sim 1.67513$ and $x_2 \sim 0.53919$.

The areas of that part of the region R which lies below the horizontal line $y = -2$ is given by the integral $\displaystyle\int_{x_2}^{x_1} \left[-2 - \left(x^3 - 4x\right)\right]\,dx$.

1.3 Part c

The area, $A(t)$, of a cross section of the solid perpendicular to the x-axis at $x = t$ is given by

$$A(t) = \left[\sin \pi t - \left(t^3 - 4t\right)\right]^2. \tag{3}$$

Thus, the volume of the solid is

$$\int_0^2 \left[\sin \pi t - \left(t^3 - 4t\right)\right]^2 \, dt \sim 9.97834, \tag{4}$$

where we have evaluated the integral numerically because, although the integral is elementary, the calculation is lengthy and requires integration by parts.

Note: The exact value of the integral is $\dfrac{1129}{105} - \dfrac{24}{\pi^3}$.

1.4 Part d

Under the conditions given, the pool is a region in three-dimensional space whose base is R and whose cross section perpendicular to the x-axis at $x = t$ has area $A(t)$ given by

$$A(t) = \left[\sin \pi t - (t^3 - 4t)\right](3 - t). \tag{5}$$

The required volume is thus

$$\int_0^2 \left[\sin \pi t - (t^3 - 4t)\right](3 - t)\, dt \sim 8.36995, \tag{6}$$

Where we have again integrated numerically to avoid a tedious calculation requiring integration by parts.

Note: The exact value of the integral is $\dfrac{116}{15} + \dfrac{2}{\pi}$.

2 Problem 2

2.1 Part a

At 5:30 pm, the rate at which the number of people standing in line was changing was approximately

$$\frac{L(7) - L(4)}{7 - 4} = \frac{150 - 126}{7 - 4} = 8 \text{ people per hour.} \tag{7}$$

2.2 Part b

The average number of people standing in line during the first four hours that tickets were on sale was

$$\frac{1}{4-0}\int_0^4 L(t)\,dt \sim \frac{1}{4}\left[\frac{120+156}{2}(1-0)+\frac{156+176}{2}(3-1)+\frac{176+126}{2}(4-3)\right] \tag{8}$$
$$\sim \frac{621}{4} = 155.25. \tag{9}$$

2.3 Part c

The function L is given twice differentiable on $[0,9]$. It is therefore continuous on $[a,b]$ and differentiable on (a,b) when $[a,b]$ is any subinterval of $[0,9]$, and we may apply the Mean Value Theorem to L on any such interval. There must be points, then, $\xi_1 \in (1,3)$ and $\xi_2 \in (3,4)$, such that

$$L'(\xi_1) = \frac{L(3)-L(1)}{3-1} = \frac{176-156}{3-1} > 0, \text{ and} \tag{10}$$
$$L'(\xi_2) = \frac{L(4)-L(3)}{4-3} = \frac{126-176}{1} < 0. \tag{11}$$

But L'' exists throughout $[0,9]$, so L' is a continuous function on $[\xi_1,\xi_2]$. By the Intermediate Value Theorem for continuous functions, there must be a number $\eta_1 \in (\xi_1,\xi_2)$ such that $L'(\eta_1) = 0$. By similar reasoning there must $\xi_3 \in (4,7)$ for which $L'(\xi_3) > 0$, and so $\eta_2 \in (\xi_2,\xi_3)$ where $L'(\eta_2) = 0$. Further, there must be $\xi_4 \in (7,8)$ for which $L'(\xi_4) < 0$, and this guarantees $\eta_3 \in (\xi_3,\xi_4)$ for which $L'(\eta_3) = 0$.

We conclude that $L'(t)$ takes on the value 0 at least three times in the interval $(0,9)$.

Note: We can make this argument even if L is given merely differentiable instead of twice differentiable, although we can no longer depend on the continuity of L'. However, derivatives necessarily have the Intermediate Value Property in spite of the fact that they may fail to be continuous[1]. To see that this is so, suppose that f is differentiable on an interval (a,b) and let $a < \alpha < \beta < b$. Suppose that $f'(\alpha) < \lambda < f'(\beta)$. We let F be the function defined on $[\alpha,\beta]$ by

$$F(x) = f(x) - \lambda x, \text{ whence} \tag{12}$$
$$F'(x) = f'(x) - \lambda. \tag{13}$$

[1] This fact is not ordinarily a part of elementary calculus, and it is to be presumed that examinees who want to use it must state it explicitly.

Now F is continuous on $[\alpha, \beta]$, and so must have an absolute minimum on that interval—which must occur at either an endpoint or a critical point. But $F(\alpha)$ can't be a minimum because $F'(\alpha) = f'(\alpha) - \lambda < 0$. Similarly, we deduce that $F(\beta)$ can't be a minimum because $F'(\beta) > 0$. It follows that there must be a critical number $x_0 \in (\alpha, \beta)$—that is, a number x_0 for which $F'(x_0) = 0$. But $F'(x_0) = 0$ is equivalent to $f'(x_0) = \lambda$.•

2.4 Part d

If $T(t)$ denotes the number of tickets that have been sold by time t, we are given that $T(0) = 0$ and $T'(t) = 550te^{-t/2}$. By the Fundamental Theorem of Calculus,

$$T(t) = T(0) + \int_0^t T'(\tau)\,d\tau = 550 \int_0^t \tau e^{-\tau/2}\,d\tau \tag{14}$$

The integral is elementary, but requires integration by parts, so we integrate numerically to learn that $T(3) \sim 972.78412$. Thus, 973 tickets have been sold by 3:00 pm.

3 Problem 3

3.1 Part a

The first-degree Taylor polynomial, $T_1(x)$, for h about $x = 2$ is

$$T_1(x) = h(2) + h'(2)(x-2) = 80 + 128(x-2). \tag{15}$$

Putting $x = 1.9$, we obtain $T_1(1.9) = 67.2$. Thus, our estimated value for $h(1.9)$ is 67.2. Because $h''(1) = 42$ and we are given that $h''(x)$ is increasing on the interval $[1, 3]$, we know that $h''(x) > 0$ on $[1, 3]$. From this, we may infer that the graph of h is concave upward throughout this interval. Consequently, the graph of the Taylor polynomial of degree one at $x = 2$, which is the graph of the tangent line at $x = 2$, lies below the curve in $[1, 3]$. Thus, our estimate of 67.2 for $h(1.9)$ is less than $h(1.9)$.

3.2 Part b

The third degree Taylor polynomial about $h = 2$ for h is

$$T_3(x) = h(2) + h'(2)(x-2) + \frac{1}{2}h''(2)(x-2)^2 + \frac{1}{6}(x-2)^3 \tag{16}$$

$$= 80 + 128(x-2) + \frac{244}{3}(x-2)^2 + \frac{224}{9}(x-2)^3. \tag{17}$$

Thus, $T_3(1.9) \sim 67.98844$.

3.3 Part c

The Lagrange error estimate for the third degree Taylor polynomial $T_3(x)$ assures us that

$$|h(1.9) - T_3(1,9)| \leq \frac{M}{24}|1.9 - 2|^4, \tag{18}$$

where M is chosen so that $\left|h^{(4)}(x)\right| \leq M$ when x lies in $[1.9, 2]$.

Now $h^{(4)}$ is increasing on the interval $[1, 3]$, so if $x \in [1.9, 2]$, then

$$h^{(4)}(x) \leq h^{(4)}(2) = \frac{584}{9}, \tag{19}$$

and we may take $M = \dfrac{584}{9}$ in the estimate (18). Thus, we find that

$$|h(1.9) - T_3(1.9)| \leq \frac{1}{24} \cdot \frac{584}{9} \cdot \frac{1}{10000} = \frac{73}{270000} \sim 0.0002704 < 3 \times 10^{-4}, \tag{20}$$

as required.

4 Problem 4

4.1 Part a

Applying the Fundamental Theorem of Calculus to what we are given we find that

$$x(t) = -2 + \int_0^t v(\tau)\, d\tau. \tag{21}$$

This means that $x(3) = -10$, $x(5) = -7$, and $x(6) = -9$. From the figure and the other information given, we have $x'(t) = v(t) < 0$ for $0 < t < 3$ and for $5 < t < 6$, while $x'(t) > 0$ for $3 < t < 5$. Thus, x is decreasing when $0 \leq t \leq 3$ and when $5 \leq t \leq 6$, while x is increasing when $3 \leq t \leq 5$. thus, the particle is farthest to the left when $t = 3$, and its position at that instant is $x = -10$.

Note: If function continuous on $[a, b]$ is increasing (respectively, decreasing) on (a, b), it is necessarily increasing (respectively, decreasing) on $[a, b]$. We should thus include the endpoints. Historically, the readers haven't taken this subtlety into account.

4.2 Part b

Because $x(0) = -2$ and $x(3) = -10$, (see Part a, above), the particle moves through $x = -8$ once (leftward bound) when $0 < t < 3$. Because $x(3) = -10$ and $x(5) = -7$ (see Part a again) it moves through $x = -8$ again (rightward bound) at some time in the interval $(3, 5)$. Because $x(5) = -7$ and $x(6) = -9$ (see Part a again) it moves through -8 still again (now leftward bound) at some time in the interval $(5, 6)$. The existence of these times is guaranteed, in each case, because the differentiable function x must be continuous on $[0, 6]$, and continuous functions have the intermediate value property. That these three instances are the only instances is guaranteed by the fact the x must be monotonic on each of the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ because velocity, the derivative of x, doesn't change sign at a point interior to any of these intervals.

4.3 Part c

Let $\sigma(t)$ denote the particle's speed at time t. Then

$$\sigma(t) = |v(t)|, \text{ so that} \tag{22}$$

$$[\sigma(t)]^2 = [v(t)]^2, \text{ and} \tag{23}$$

$$2\sigma(t)\sigma'(t) = 2v(t)v'(t), \text{ or, provided } \sigma(t) \neq 0, \tag{24}$$

$$\sigma'(t) = \frac{v(t)}{\sigma(t)}v'(t) = \frac{v(t)v'(t)}{|v(t)|}. \tag{25}$$

The numerator of this last fraction is positive ($\sigma(t)$ being non-zero), so the sign of $\sigma'(t)$ is the same as the sign of the product $v(t)v'(t)$. On the interval $(2, 3)$, we see from the graph that $v(t) < 0$, but that $v(t)$ is increasing, so that $v'(t) > 0$. It follows that $v(t)v'(t) < 0$ on $(2, 3)$, and, therefore, that speed is decreasing on $(2, 3)$.

4.4 Part d

Acceleration if $v'(t)$. Thus, acceleration is negative on intervals where $v(t)$ is decreasing. From the graph and what we have been given about it, acceleration is negative on $[0, 1)$ and on $(4, 6]$, and only on those intervals.

5 Problem 5

5.1 Part a

$f'(x) = (x-3)e^x$ is positive for $x > 3$, negative for $x < 3$, and zero when $x = 3$. Thus, f is an increasing function on $[3, \infty)$ and a decreasing function on $(-\infty, 3]$. By the First Derivative Test, this means that $f(3)$ is a local minimum for f.

5.2 Part b

$f''(x) = (x-2)e^x$, which is positive when $x > 2$, negative when $x < 2$. Thus, f'' is concave upward on $(2, \infty)$, concave downward on $(-\infty, 2)$. Combining these observations with those of Part a, above, we conclude that f is both decreasing and concave upward on the interval whose endpoints are $x = 2$ and $x = 3$.

Note: Whether to include the endpoints in intervals of concavity depends on which of several commonly used definitions of concavity the writer prefers to adopt.

5.3 Part c

By the Fundamental Theorem of Calculus,

$$f(3) - f(1) = \int_1^3 f'(t)\,dt \tag{26}$$

$$= \int_1^3 (t-3)e^t\,dt \tag{27}$$

$$= (t-4)e^t \Big|_1^3 \tag{28}$$

$$= 3e - e^3. \tag{29}$$

But it is given that $f(1) = 7$, so $f(3) = 7 + 3e - e^3$.

6 Problem 6

6.1 Part a

See Figure 1.

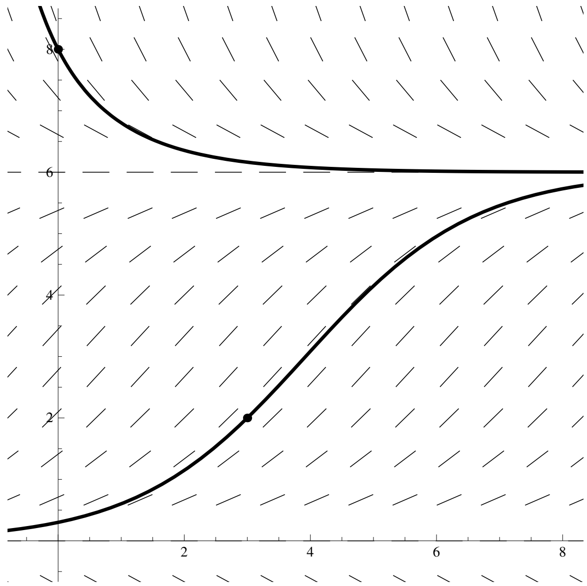

Figure 1: Problem 6, Part a

6.2 Part b

According to Euler, if $f(t)$ is a solution of the differential equation that stisfies $f(0) = 8$, then

$$f\left(\frac{1}{2}\right) \sim 8 + \frac{1}{2} \cdot \frac{8}{8} \cdot (6-8) = 7, \text{ and} \tag{30}$$

$$f(1) \sim 7 + \frac{1}{2} \cdot \frac{7}{8} \cdot (6-7) = \frac{105}{16}. \tag{31}$$

6.3 Part c

Because

$$y' = \frac{y}{8}(6-y), \tag{32}$$

we have

$$y'' = \frac{y'}{8}(6-y) - \frac{y}{8}y' \tag{33}$$

$$= \frac{1}{8} \cdot \frac{y}{8}(6-y) \cdot (6-y) - \frac{1}{8}y \cdot \frac{y}{8}(6-y) \tag{34}$$

$$= \frac{1}{32}y(3-y)(6-y). \tag{35}$$

Thus, $y'(0) = -2$ and $y''(0) = 8(6-8)(3-8)/32 = \frac{5}{2}$. So the second-degree Taylor polynomial for f about $t = 0$ is

$$T_2(t) = y(0) + y'(0)t + \frac{1}{2}y''(0)t^2 = 8 - 2t + \frac{5}{4}t^2. \tag{36}$$

Thus,

$$T_2(1) = 8 - 2 + \frac{5}{4} = \frac{29}{4}. \tag{37}$$

6.4 Part d

This differential equation is a logistic equation with an attracting equilibrium solution $y(t) \equiv 6$ and a repelling equilibrium solution $y \equiv 0$. Thus, all positive solutions of this equation decay toward the stable solution $y(t) \equiv 6$ as $t \to \infty$. In particular, solutions of the differential equation for which $y(0) > 6$ remain larger than 6 but decrease toward, and, in fact, approach, the horizontal line $y = 6$ as $t \to \infty$. So the range of the solution f, which is determined by the initial condition $f(0) = 8$, is $(6, 8]$.