AP Calculus 2009 BC (Form B) FRQ Solutions

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1 Problem 1

1.1 Part a

The area of the discarded cardboard is

$$600 + 20\int_0^{30} \sin\frac{\pi x}{30}\,dx = 600 - \frac{600}{\pi}\cos\frac{\pi x}{30}\bigg|_0^{30} = 600 - \frac{600}{\pi}\cos\pi + \frac{600}{\pi}\cos 0 \tag{1}$$

$$= 600 - \frac{1200}{\pi}\text{ square centimeters.} \tag{2}$$

1.2 Part b

The radius of the semicircle at $x = x_0$ is $10\sin(\pi x_0/30)$, so the area of that semicircle is

$$\frac{\pi}{2}r^2 = \frac{\pi}{2}\cdot 100\sin^2\frac{\pi x_0}{30}, \tag{3}$$

and the volume of the cake is

$$50\pi\int_0^{30}\sin^2\frac{\pi x}{30}\,dx = 25\pi\int_0^{30}\left(1-\cos\frac{\pi x}{15}\right)\,dx \tag{4}$$

$$= 25\pi\left[x - \frac{15}{\pi}\sin\frac{\pi x}{15}\right]\bigg|_0^{30} = 750\pi\text{ cubic centimeters.} \tag{5}$$

At $1/20$ grams of unsweetened chocolate per cubic centimeter, there must be $75\pi/2$ grams of unsweetened chocolate in the cake.

1.3 Part c

The perimeter of the base is

$$20 + \int_0^{30} \sqrt{1 + \frac{4\pi^2}{9} \cos^2 \frac{\pi x}{30}}\, dx \sim 81.80370 \text{ centimeters,} \tag{6}$$

where we have had no choice but to carry out the integration numerically.

2 Problem 2

2.1 Part a

Let $D(t)$ denote the distance, in meters, from the road to the edge of the water at time t hours after the beginning of the storm. We are given $D(0) = 35$, $D'(t) = \sqrt{t} + \cos t - 3$. By the Fundamental Theorem of Calculus,

$$\begin{aligned} D(t) &= 35 + \int_0^t D'(\tau)\, d\tau & (7)\\ &= 35 + \int_0^t \left(\sqrt{\tau} + \cos\tau - 3\right)\, d\tau & (8)\\ &= 35 + \left[\frac{2}{3}\tau^{3/2} + \sin\tau - 3\tau\right]\Bigg|_0^t & (9)\\ &= 35 - 3t + \frac{2}{3}t^{3/2} + \sin t. & (10) \end{aligned}$$

Substituting 5 for t, we obtain $D(5) \sim 26.49464$. Thus, at the end of the five-hour storm, the distance from road to water is about 26.49464 meters.

2.2 Part b

If $f'(4) = 1.007$, then $D''(4) = 1.007$, so after four hours of the storm, the rate at which distance from road to water is changing is increasing at 1.007 meters per hour per hour.

Note: This is something of a misstatement: Very few texts give any definition for the phrase "increasing at a point."

2.3 Part c

We are to find the absolute minimum of $f(t)$ on the interval $[0, 5]$. Such a minimum lies at either a critical point or an endpoint. The critical points for f are the zeros of

$$f'(t) = \frac{1}{2\sqrt{t}} - \sin t. \tag{11}$$

We solve numerically and find these critical points are $t \sim 0.66186$ and $t \sim 2.84038$. We find

$$\begin{aligned} f(0) &= -2, && (12)\\ f(0.66186) &\sim -1.39760, && (13)\\ f(2.84038) &\sim -2.26963, && (14)\\ f(5) &\sim -0.48027. && (15) \end{aligned}$$

The smallest of these is $f(2.84038)$, so the distance from water to road was decreasing most rapidly about 2.84038 hours after the storm began.

2.4 Part d

If sand is restored to the beach in such a way that the rate of change of the distance from water to road is $g(p)$ meters per day, where p is the number of days since pumping began, then, by the Fundamental Theorem of Calculus the number of days, P, of pumping required to restore the original disance between road and water, satisfies the equation

$$35 = D(5) + \int_0^P g(p)\, dp. \tag{16}$$

3 Problem 3

We don't appear to have been given quite enough information to solve this problem. We must assume that the line segment and the curved portion of the curve meet at the point $(0, 2)$. In what follows, we make this assumption.

3.1 Part a

The line segment that gives the portion of the curve that lies to the left of the y-axis has slope $2/3$, so

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \frac{2}{3}. \tag{17}$$

On the other hand, it is apparent that

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} < 0. \tag{18}$$

The left- and right-hand limits of the difference quotient for f at zero being different, $f'(0)$ cannot exist, and f is not differentiable at $x = 0$.

3.2 Part b

The average rate of change of f over the interval $[a, 6]$ is $\frac{f(6) - f(a)}{6 - a}$. This can be zero only if $f(a) = f(6) = 1$ while $a \neq 6$. The horizontal line through $(5, f(1)) = (6, 1)$ intersects the curve in just two other points, so there are just two values of a for which the average rate of change of f over $[a, 6]$ is zero.

3.3 Part c

We note that f is continuous on $[3, 6]$ and differentiable on $(3, 6)$, so the hypotheses of the Mean Value Theorem are satisfied. Hence there is a number $c \in [3, 6]$ such that

$$f'(c) = \frac{f(6) - f(3)}{6 - 3} = \frac{1}{3}. \tag{19}$$

Thus, we may take $a = 3$.

3.4 Part d

If

$$g(x) = \int_0^x f(t)\, dt, \tag{20}$$

then $g'(x) = f(x)$ and $g''(x) = f'(x)$. Thus, g is concave upward on intervals where $f'(x) > 0$ or, more generally, on intervals where f' is increasing. We conclude that g is concave upward on $(-4, 0)$ and on $(3, 6)$. Whether or not we may conclude that g is concave upward on the closures of these intervals depends upon which of several definitions of concavity we choose. In the past, the readers have paid no attention to this subtlety.

4 Problem 4

4.1 Part a

The arc that forms the lower boundary of the region S is traced out when $0 \leq \theta \leq \pi/3$. (To see why this is so, simply note that $r = 1 - 2\cos\theta < 0$ precisely when $0 < \theta < \pi/3$.) The required integral is thus

$$\frac{1}{2}\int_0^{\pi/3} (1 - 2\cos\theta)^2 \, d\theta = \frac{1}{2}\left(\pi - \frac{3\sqrt{3}}{2}\right). \tag{21}$$

Note: Evaluation of the integral is not required. However, it is an elementary—though somewhat tedious—integral, and we include its value for the sake of completeness.

4.2 Part b

We have

$$x = r\cos\theta = (1 - 2\cos\theta)\cos\theta \text{ and} \tag{22}$$

$$y = r\sin\theta = (1 - 2\cos\theta)\sin\theta. \tag{23}$$

Therefore,

$$\frac{dx}{d\theta} = -\sin\theta + 4\cos\theta\sin\theta \text{ and} \tag{24}$$

$$\frac{dy}{d\theta} = \cos\theta + 2\sin^2\theta - 2\cos^2\theta. \tag{25}$$

4.3 Part c

We have

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{\cos\theta + 2\sin^2\theta - 2\cos^2}{4\cos\theta\sin\theta - \sin\theta} \tag{26}$$

When $\theta = \pi/2$, this reduces to -2, and we have $x = 0$, $y = 1$. The required equation is therefore $y = 1 - 2(x - 0)$, or $2x + y = 1$.

5 Problem 5

5.1 Part a

If

$$g(x) = e^{f(x)}, \tag{27}$$

then

$$g'(x) = f'(x)e^{f(x)}, \tag{28}$$

so

$$g'(1) = f'(1)e^{f(1)} = -4e^2. \tag{29}$$

Hence, an equation for the line tangent to the curve $y = g(x)$ at the point corresponding to $x = 1$ is

$$y = g(1) + g'(1)(x - 1), \tag{30}$$

or

$$y = e^2 - 4e^2(x - 1). \tag{31}$$

5.2 Part b

By the First Derivative Test, g has a local maximum at any point where $g'(x)$ changes sign from positive to negative. But $g'(x) = f'(x)e^{f(x)}$, and $e^{f(x)}$ is always positive. Therefore the local maxima of g are to be found at points where $f'(x)$ changes sign from positive to negative. From the graph given, we see that there is just one such point in the interval $(-1.2, 3.2)$: $x = -1$. The function g therefore has a local maximum only at $x = 1.1$ in the interval $(-1.2, 3.2)$.

5.3 Part c

Because

$$g''(x) = e^{f(x)} \left(\left[f'(x) \right]^2 + f''(x) \right) \tag{32}$$

and $e^{f(x)} > 0$, the sign of $g''(x)$ is the same as the sign of $\left([f'(x)]^2 + f''(x)\right)$. Now, as is given, $(f'(-1))^2 = 0$, and we see from the graph that, f' being a decreasing function in a neighborhood of $x = -1$, it must be the case that $f''(-1) < 0$. So $g''(-1) < 0$.

5.4 Part d

The average rate of change of g' over the interval $[1, 3]$ is $[g'(3) - g'(1)]/[3 - 1]$. But, as we saw in Part a of this problem, above, $g'(1) = -4e^2$. We also have $g'(3) = f'(3)e^{f(3)} = 0$, $f'(30) = 0$ being given. The desired average rate of change is

$$\frac{0 - (-4e^2)}{2} = 2e^2. \tag{33}$$

6 Problem 6

6.1 Part a

The given series is a geometric series with common ratio $x+1$; it converges when $|x+1| < 1$ and diverges otherwise. The interval of convergence is therefore $-2 < x < 0$.

6.2 Part b

The sum of the geometric series $1 + r + r^2 + \cdots$ is $\dfrac{1}{1-r}$ when $|r| < 1$. Consequently,

$$1 + (1 + x) + (1 + x)^2 + \cdots = \frac{1}{1 - (1 + x)} = -\frac{1}{x}. \tag{34}$$

throughout the interval $-2 < x < 0$.

6.3 Part c

Because $f(x) = -1/x$ when $-2, x, 0$, we may write

$$g\left(-\frac{1}{2}\right) = \int_{-1}^{-1/2} f(t)\, dt \tag{35}$$

$$= -\int_{-1}^{-1/2} \frac{dt}{t} \tag{36}$$

$$= -\ln|t|\Big|_{-1}^{-1/2} \tag{37}$$

$$= -\ln\frac{1}{2} + \ln 1 = \ln 2. \tag{38}$$

6.4 Part d

When $-2 < x^2 - 1 < 0$, or, equivalently, when $-1 < x < 1$, we may replace each instance of the variable x in the given power series with $x^2 - 1$. Noting that $(x + 1)$ becomes $[(x^2 - 1) + 1] = x^2$ under this substitution, we find that

$$h(x) = f(x^2 - 1) = 1 + x^2 + x^4 + \cdots + x^{2n} + \cdots \tag{39}$$

$$= \sum_{n=0}^{\infty} x^{2n}. \tag{40}$$

Because $x = 1/2$ lies in the interval $(-1, 1)$, we may write

$$h\left(\frac{1}{2}\right) = f\left[\left(\frac{1}{2}\right)^2 - 1\right] = f\left(-\frac{3}{4}\right) = -\frac{1}{-3/4} = \frac{4}{3}. \tag{41}$$

Alternate Solution: We may write $h(x) = f(x^2 - 1) = 1/(1 - x^2)$, and, appealing once more to our knowledge of the geometric series, conclude that

$$h(x) = 1 + x^2 + x^4 + x^6 + \cdots + x^{2n} + \cdots , \tag{42}$$

as long as $-1 < x < 1$.