AP Calculus 2009 BC FRQ Solutions

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1 Problem 1

1.1 Part a

At time $t = 7.5$, the acceleration of Caren's bicycle is

$$\frac{0.3 - 0.2}{7 - 8} = -\frac{1}{10} \text{ miles per minute per minute.} \tag{1}$$

1.2 Part b

The integral $\displaystyle\int_0^{12} |v(t)|\,dt$ gives, in miles, the total distance that Caren traveled during the period $0 \le t \le 12$. The value of this integral is $\dfrac{9}{5}$.

1.3 Part c

Her turn-around time corresponds to the point on the graph where the sign of her velocity changes from positive to negative. That's $t = 2$ minutes.

1.4 Part d

Caren lives $\displaystyle\int_5^{12} v(t)\,dt = \frac{7}{5}$ miles from school because she left home at $t = 5$, arrived at school at $t = 12$, traveled in one direction only, and the distance she traveled during that time is given by the integral. Larry's distance is the integral of his velocity over the interval $[0, 12]$, or

$$\frac{\pi}{15}\int_0^{12} \sin\left(\frac{\pi t}{12}\right)\,dt = -\frac{\pi}{15}\cdot\frac{12}{\pi}\cos\left(\frac{\pi t}{12}\right)\Bigg|_0^{12} = \frac{8}{5}. \tag{2}$$

Larry lives farther from school than Caren, who lives only $\frac{7}{5}$ miles away.

2 Problem 2

2.1 Part a

At time $t = 2$, the auditorium contains

$$\int_0^2 \left(1380t^2 - 675t^3\right)\,dt = \left[460t^3 - \frac{675}{4}t^4\right]\Bigg|_0^2 = 460\cdot 8 - 675\cdot 4 = 980 \text{ people.} \tag{3}$$

2.2 Part b

We are given

$$R(t) = 1380t^2 - 675t^3, \text{ so that} \tag{4}$$

$$R'(t) == 2760t - 2025t^2 = 15(184 - 135t). \tag{5}$$

Thus, R is increasing on the interval $\left[0, \frac{184}{135}\right]$ and decreasing on the interval $\left[\frac{184}{135}, 2\right]$, because $R'(t) > 0$ on the interior of the first of those intervals and $R'(t) < 0$ on the interior of the second. It follows that the maximal rate at which people enter the auditorium occurs at $t = \frac{184}{135}$ hours.

2.3 Part c

We have $R(t) = 1380t^2 - 675t^3$, and $w'(t) = (2-t)R(t)$. By the Fundamental Theorem of Calculus, the difference $w(2) - w(1)$ is given by

$$w(2) - w(1) = \int_1^2 w'(t)\,dt = \int_1^2 (2-t)(1380t^2 - 675t^3)\,dt \tag{6}$$

$$= \int_1^2 (2760t^2 - 2730t^3 + 675t^4)\,dt = \left(920t^3 - \frac{1365}{2}t^4 + 135t^5\right)\Bigg|_1^2 \tag{7}$$

$$= 760 - \frac{745}{2} = \frac{775}{2} \text{ hours.} \tag{8}$$

2.4 Part d

From Part a of this problem, above, we know that there are 980 people in the auditorium at time $t = 2$. We also know that the total wait time for these 980 people is

$$\int_1^2 w'(t)\,dt = \left(920t^3 - \frac{1365}{2}t^4 + 135t^5\right)\Bigg|_0^2 = 760 \text{ hours.} \tag{9}$$

Consequently, average waiting time is

$$\frac{760}{980} = \frac{38}{49} \text{ hours.} \tag{10}$$

3 Problem 3

3.1 Part a

The maximum vertical distance from the water's surface to the diver's shoulders must occur when $y'(t) = 3.6 - 9.8t = 0$, or when $t = \frac{18}{49}$ seconds. But from $y'(t) = 3.6 - 9.8t$ and $y(0) - 11.4$ meters, it follows from the Fundamental Theorem of Calculus that

$$y\left(\frac{18}{49}\right) = 11.4 + \int_0^{18/49} (3.6 - 9.8t)\,dt \tag{11}$$

$$= 11.4 + (3.6t + 4.9t^2)\Bigg|_0^{18/49} \sim 12.06122. \tag{12}$$

Thus, the diver's shoulders reach a maximum distance from the water's surface of 12.06122 meters.

3.2 Part b

The diver's shoulders enter the water when $t > 0$ and $y(t) = 0$. But, as in Part a of this problem, above,

$$y(t) = 11.4 + \int_0^t (3.6 - 9.8\tau)\,d\tau = 11.4 + 3.6t - 4.9t^2. \tag{13}$$

Thus, her shoulders enter the water at time $T > 0$ where $11.4 + 3.6T - 4.9T^2 = 0$. Thus,

$$T = \frac{-3.6 - \sqrt{(3.6)^2 - 4 \cdot (-4.9) \cdot (11.4)}}{2 \cdot (-4.9)} \tag{14}$$

$$= \frac{3.6 + \sqrt{236.4}}{9.8}, \tag{15}$$

where we have chosen the minus sign in the quadratic formula because we need the quotient to be positive. This gives $T \sim 1.93626$ seconds.

3.3 Part c

Let T be as above, in Part b. The total distance the diver's shoulders travel from the time she leaves the platform until they enter the water is

$$\int_0^T \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \int_0^T \sqrt{(0.8)^2 + (3.6 - 9.8\tau)^2}\, d\tau \tag{16}$$

Numerical integration gives approximately 12.94621 meters for this distance.

3.4 Part d

The slope of her path at time t is

$$\frac{y'(t)}{x'(t)} = \frac{3.6 - 9.8t}{0.8}, \tag{17}$$

and because this slope is negative when her shoulders enter the water at time T (from Part b, above), this will be the negative of the tangent of the acute angle which her path makes with the surface of the water at the moment of entry. Thus, the required angle is

$$-\arctan \frac{3.6 - 9.8T}{0.8} \sim 1.51881 \text{ radians.} \tag{18}$$

4 Problem 4

4.1 Part a

Euler's method, with step-size h, for approximating the solution to an initial value problem

$$y' = g(x, y); \tag{19}$$
$$y(a) = b; \tag{20}$$

is given by the recursion relations

$$x_0 = a; \tag{21}$$
$$y_0 = b; \tag{22}$$
$$x_k = x_{k-1} + h. \text{ when } k > 0; \tag{23}$$
$$y_k = y_{k-1} + g(x_{k-1}, y_{k-1})h, \text{ when } k > 0. \tag{24}$$

Here, $g(x, y) = 6x^2 - x^2 y$, $a = -1$, $b = 2$, and we are to take $h = 1/2$. Thus

$$x_0 = -1; \tag{25}$$
$$y_0 = 2; \tag{26}$$
$$x_1 = x_0 + h = -1 + \frac{1}{2} = -\frac{1}{2}; \tag{27}$$
$$y_1 = y_0 + g(x_0, y_0)h = 2 + (6x_0^2 - x_0^2 y_0)h = 2 + (6-2)\frac{1}{2} = 4; \tag{28}$$
$$x_2 = -\frac{1}{2} + \frac{1}{2} = 0; \tag{29}$$
$$y_2 = 4 + \left[6 \cdot \left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)^2 \cdot 4\right] \cdot \frac{1}{2} = \frac{17}{4}. \tag{30}$$

Thus, $f(0) \sim \frac{17}{4}$.

4.2 Part b

If f is the particular solution of the differential equation $\frac{dy}{dx} = 6x^2 - x^2 y$ for which $f(-1) = 2$, then

$$f'(-1) = 6(11)^2 - (-1)^2 \cdot (2) = 4, \tag{31}$$

and we have been given that $f''(-1) = 12$. Thus, $T_2(x)$, the second degree Taylor polynomial for f about $x = -1$, is

$$T_2(x) = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+2)^2 \tag{32}$$

$$= 2 + 4(x+1) + 6(x+1)^2. \tag{33}$$

4.3 Part c

If f is the particular solution of Parts a and b, above, then

$$f'(x) = 6x^2 - x^2 f(x). \tag{34}$$

As the solution to a differential equation passing through $(-1, 2)$, f must be continuous near $x = 1$. It then folllows that the quantity $6 - f(x)$ is positive in some open interval, I, centered about $x = 1$. In particular, $6 - f(x) \neq 0$ throughout I, and for any ξ in I we may rewrite (34) as

$$\frac{f'(\xi)}{6 - f(\xi)} = \xi^2. \tag{35}$$

If x is any number in I, we have

$$\int_{-1}^{x} \frac{f'(\xi)}{6 - f(\xi)}\, d\xi = \int_{-1}^{x} \xi^2\, d\xi, \tag{36}$$

from which we see that

$$\ln\left[6 - f(\xi)\right]\Big|_{-1}^{x} = -\frac{\xi^3}{3}\Big|_{-1}^{x}, \tag{37}$$

or

$$\ln\left[6 - f(x)\right] - \ln 4 = -\left(\frac{x^3}{3} + \frac{1}{3}\right). \tag{38}$$

Solving for $f(x)$ we find that

$$f(x) = 6 - 4e^{-\frac{x^3+1}{3}}. \tag{39}$$

5 Problem 5

5.1 Part a

$$f'(4) \sim \frac{f(5) - f(3)}{5 - 3} = \frac{-2 - 4}{5 - 3} = -3. \tag{40}$$

5.2 Part b

$$\int_2^{13} \left[3 - 5f'(x)\right] dx = [3x - 5f(x)]\Big|_2^{13} \tag{41}$$
$$= [3 \cdot 13 - 5 \cdot f(13)] - [3 \cdot 2 - 5 \cdot f(2)] \tag{42}$$
$$= (39 - 30) - (6 - 5) = 8. \tag{43}$$

5.3 Part c

The desired left Riemann sum is

$$f(2) \cdot (3 - 1) + f(3) \cdot (5 - 3) + f(3) \cdot (8 - 5) + f(8) \cdot (13 - 8) = 1 + 8 - 6 + 15 \tag{44}$$
$$= 18. \tag{45}$$

5.4 Part d

An equation for the line tangent to the curve $y = f(x)$ at the point on the curve that corresponds to $x = 5$ is

$$y = f(5) + f'(5)(x - 5), \text{ or} \tag{46}$$
$$y = -2 + 3(x - 5). \tag{47}$$

Now $f''(x) < 0$ for all x in the interval $[5, 8]$, so the curve is concave downward throughout that interval; thus, the tangent line at $x = 5$ lies above the curve on $[5, 8]$. That is, when $5 \leq x \leq 8$, we have $f(x) \leq -2 + 3(x - 5)$. Consequently, $f(7) \leq -2 + 3(7 - 5) = 4$.

On the other hand, $f''(x) < 0$ on $[5, 8]$, and this implies that the curve $y = f(x)$, being concave downward there, lies above the secant line determined by the point $(5, f(5)) =$

$(5,-2)$ and the point $(8, f(8)) = (8,3)$. An equation for this secant line is

$$y = f(5) + \frac{f(8) - f(5)}{8-5}(x-5), \text{ or} \tag{48}$$

$$y = -2 + \frac{5}{3}(x-5). \tag{49}$$

Consequently, when $5 \leq x \leq 8$, we have $-2 + \frac{5}{3}(x-5) \leq f(x)$. Thus,

$$\frac{4}{3} = -2 + \frac{5}{3}(7-5) \leq f(7). \tag{50}$$

6 Problem 6

6.1 Part a

We obtain the require series by substituting $(x-1)^2$ for x in the expansion, in powers of x, that we have for e^x:

$$e^{(x-1)^2} = \sum_{k=0}^{\infty} \frac{\left[(x-1)^2\right]^k}{k!} \tag{51}$$

$$= \sum_{k=0}^{\infty} \frac{(x-1)^{2k}}{k!} \tag{52}$$

$$= 1 + (x-1)^2 + \frac{1}{2!}(x-1)^4 + \frac{1}{3!}(x-1)^6 + \cdots + \cdots + \frac{(x-1)^{2k}}{k!} + \cdots. \tag{53}$$

6.2 Part b

We have

$$f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2} \tag{54}$$

$$= \frac{\sum_{k=0}^{\infty} \frac{[(x-1)^2]^k}{k!} - 1}{(x-1)^2} \tag{55}$$

$$= \frac{\sum_{k=1}^{\infty} \frac{(x-1)^{2k}}{k!}}{(x-1)^2} \tag{56}$$

$$= \sum_{k=1}^{\infty} \frac{(x-1)^{2k-2}}{k!} \tag{57}$$

$$= 1 + \frac{1}{2!}(x-1)^2 + \frac{1}{3!}(x-1)^4 + \frac{1}{4!}(x-1)^6 + \cdots \frac{(x-1)^{2k-2}}{k!} + \cdots . \tag{58}$$

6.3 Part c

Using the ratio test, we find that

$$\lim_{k\to\infty} \left[\left| \frac{(x-1)^{2k}}{(k+1)!} \right| \cdot \left| \frac{k!}{(x-1)^{2k-2}} \right| \right] = |x-1|^2 \lim_{k\to\infty} \frac{1}{k+1} = 0. \tag{59}$$

This limit has magnitude less than one, no matter what x may be, and we conclude that our series converges for all values of x.

6.4 Part d

The series for f converges everywhere, so we can obtain a series expansion, valid everywhere, for f'' by differentiating this series, term-by-term, twice in succession. We obtain

$$f''(x) = 1 + 2(x-1)^2 + \frac{5}{4}(x-1)^4 + \frac{7}{15}(x-1)^6 + \cdots + \frac{2n(2n-1)}{(n+1)!}(x-1)^{2(n-1)} + \cdots . \tag{60}$$

But every term of this series is non-negative for all x, while the first term is, in fact, positive for all x. It follows that $f''(x) \geq 1 > 0$ for all x, and therefore that f is concave upward everywhere. f consequently has no inflection points.