AP Calculus 2010 BC (Form B) FRQ Solutions

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1 Problem 1

1.1 Part a

The area of the region R is

$$\int_0^2 [6 - 4\ln(3 - x)]\ dx \sim 6.81665, \tag{1}$$

where we have carried out the integration numerically.

Note: The integral is elementary, but is most easily accomplished using integration by parts, which isn't on the AB syllabus. We have

$$\int_0^2 [6 - 4\ln(3 - x)]\ dx = 6x\Big|_0^2 - 4(x-3)\ln(3-x)\Big|_0^2 + 4\int_0^2 dx = 20 - 12\ln 3 \tag{2}$$

$$= 4(5 - 3\ln 3).. \tag{3}$$

1.2 Part b

The volume obtained by revolving R about the line $y = 8$ is given by

$$\pi \int_0^2 \left([8 - 4\ln(3 - x)]^2 - 4\right)\ dx \sim 168.17954, \tag{4}$$

where we have once again integrated numerically.

Note: The integral is elementary, and symbolic integration is possible. However it is lengthy, and numerical integration saves a fair amount of time. Symbolic integration gives

$$\pi \int_0^2 \left([8 - 4\ln(3-x)]^2 - 4 \right) \, dx = 24\pi \left[13 - 2(6 - \ln 3)\ln 3 \right]. \tag{5}$$

1.3 Part c

The volume of this solid is

$$\int_0^2 [6 - 4\ln(3-x)]^2 \, dx \sim 26.26660. \tag{6}$$

Note: Once more, we have carried out the integration numerically, though a symbolic integration is possible. Once again, the numerical integration saves time. For the curious,

$$\int_0^2 [6 - 4\ln(3-x)]^2 \, dx = 16 \left[20 - 3(6 - \ln 3)\ln 3 \right]. \tag{7}$$

2 Problem 2

2.1 Part a

The slope $m(t)$ of the tangent line to the path is given by

$$m(t) = \frac{y'(t)}{x'(t)}, \tag{8}$$

at least when this fraction is meaningful. Thus, we can find vertical tangent lies by solving the equation $x'(t) = 0$ and being sure that $y'(t)$ is not simultaneously zero. (To see that we must include the latter condition, consider the curve $x = t^3, y = t^2$ at the origin.)

Working numerically, we find that the solutions $\theta_1 < t_2$ we seek for the equation

$$14 \cos\left(t^2\right) \sin\left(e^t\right) = 0 \tag{9}$$

are $t_1 \sim 1.14473$ and $t_2 \sim 2.93258$. It is easily checked that neither of these values in a zero of $y'(t)$.

2.2 Part b

By the Fundamental Theorem of Calculus,

$$x(1) = x(0) + \int_0^t x'(\tau)\, d\tau \sim 9.31470 \tag{10}$$

and

$$y(1) = y(0) + \int_0^t y'(\tau)\, d\tau \sim 4.62054, \tag{11}$$

where we have carried out the integrations numerically. We also have $m(1) \sim 0.86345$, so an equation for the line tangent to the path at the point corresponding to $t = 1$ is (approximately)

$$y = 4.62054 + 0.86345(x - 9.31470). \tag{12}$$

2.3 Part c

Speed $\sigma(t)$ at time t is given by

$$\sigma(t) = \sqrt{[x'(t)]^2 + [y'(t)]^2}. \tag{13}$$

Thus,

$$\sigma(1) = \sqrt{[x'(1)]^2 + [y'(1)]^2} \sim 4.10526. \tag{14}$$

2.4 Part d

The acceleration vector $\mathbf{a}(t)$ at time t is given by

$$\mathbf{a}(t) = \langle x''(t), y''(t) \rangle \tag{15}$$
$$= \langle 14e^t \cos e^t \cos t^2 - 28t \sin e^t \sin t^2, 4t \cos t^2 \rangle. \tag{16}$$

Thus,

$$\mathbf{a}(1) \sim \langle -28.42531, 2.16121 \rangle. \tag{17}$$

3 Problem 3

3.1 Part a

The three-equal-interval midpoint Riemann sum that approximates the amount of water pumped into the pool during the time interval $0 \leq t \leq 12$ is

$$f(2) \cdot 4 + f(6) \cdot 4 + f(10) \cdot 4 = 660. \tag{18}$$

Consequently, about 660 cubic feet of water was pumped into the pool in the given twelve-hour interval.

3.2 Part b

If water leaked at the rate $R(t) = 25e^{-0.05t}$ cubic feet per hour, then, during the interval $0 \leq t \leq 12$, the amount of water lost was, in cubic feet,

$$25 \int_0^{12} e^{-t/20}\, dt = -500e^{-t/20}\Big|_0^{12} = 500(1 - e^{-3/5}). \tag{19}$$

3.3 Part c

The total amount of water, in cubic feet, in the pool at the end of the twelve-hour period is thus about

$$1000 + 660 - 500(1 - e^{-3/5}) \sim 1434.40582. \tag{20}$$

To the nearest cubic foot, at time $t = 12$ the pool contains 1434 cubic feet of water.

3.4 Part d

The rate at which the volume of water in the pool is increasing is $P(t) - R(t)$ cubic feet per hour. When $t = 8$ this rate is $60 - 25e^{-2/5} \sim 43.24200$ cubic feet per hour.

The relationship between the height, h, of water in the tank and the volume, V, of water in the tank is $V = \pi r^2 h = 144\pi h$. Thus,

$$h = \frac{V}{144\pi}, \text{ and} \tag{21}$$

$$\frac{dh}{dt} = \frac{1}{144\pi}\frac{dV}{dt}. \tag{22}$$

Setting $t = 8$ in this latter equation gives (from what we have seen above)

$$\left.\frac{dh}{dt}\right|_{t=8} = \frac{1}{144\pi}(60 - 25e^{-2/5}) \sim 0.09558 \text{ feet per hour.} \quad (23)$$

4 Problem 4

4.1 Part a

The squirrel's direction changes when its velocity changes sign. That happens only at those values of t where the graph of velocity crosses the t-axis. There are two such places: $t = 9$ and $t = 15$.

4.2 Part b

The squirrel's distance from Building A at time T is the integral of its velocity from 0 to T. This integral is the algebraic sum of the signed areas associated with the appropriate regions between the velocity curve and the t-axis, taking the regions above the axis to have positive area and assigning negative areas to the regions below the t-axis. The area under the velocity curve on the interval $[0, 9]$ is clearly larger than the area below the axis on the interval $[9, 15]$, and this latter area is clearly smaller than the area above the axis on the interval $[15, 18]$. Thus, the squirrel is farthest from Building A when $t = 9$, and this distance is the area enclosed by the trapezoid whose vertices are $(0, 0)$, $(2, 20)$, $(7, 20)$, and $(9, 0)$. The area of this trapezoid is 140, so at time $t = 9$, the squirrel is 140 feet from Building A and is closer at all other times.

4.3 Part c

Summing the magnitudes of the signed areas, we find that the squirrel has traveled a total distance of $140 + 50 + 25 = 215$ feet.

4.4 Part d

On the interval $7 < t < 10$, the squirrel's velocity is given by a line, of slope

$$\frac{20 - (-10)}{7 - 10} = -10, \quad (24)$$

which passes through the point $(7, 20)$, and therefore has equation

$$v(t) = 20 - 10(t - 7) = 90 - 10t \text{ feet per second.} \tag{25}$$

For the wquirrel's acceleration during this interval, we have

$$a(t) = v'(t) = -10 \text{ feet per second per second.} \tag{26}$$

For distance, $x(t)$, from Building A, we have, when $7 \leq t \leq 10$,

$$x(t) = x(7) + \int_7^t v(\tau)\, d\tau \tag{27}$$

$$= 120 + \int_7^t (90 - 10\tau)\, d\tau \tag{28}$$

$$= -5t^2 + 90t - 265 \text{ feet.} \tag{29}$$

5 Problem 5

5.1 Part a

From

$$g(x) = \frac{4x}{1 + x^2}, \text{ we find} \tag{30}$$

$$g'(x) = \frac{4(1 + x^2) - 4x(2x)}{(1 + x^2)^2} = \frac{4(1 - x^2)}{(1 + x^2)^2}. \tag{31}$$

Thus, $g'(x)$ exists for all real values of x. The only point in the interval $(0, \infty)$ where $g'(x)$ vanishes is at $x = 1$, so if $g(x)$ takes on an absolute maximum in this interval, it must do so at $x = 1$. Now $g(x) > 0$ on $(0, 1)$ so g is increasing on that interval. On the other hand $g'(x) < 0$ on $(1, \infty)$, which means that g is decreasing on that interval. We conclude that g does have an absolute maximum on $(0, \infty)$, that the absolute maximum occurs at $x = 1$, and that the value of that absolute maximum is $g(1) = 4/5$.

If g is to have an absolute minimum in $(0, \infty)$, it must lie at a critical point for g. However, g has only one critical point, and we have seen that g has an absolute maximum at that point. The function g is not a constant function, so it can't have a minimum value that's equal to its maximum value, and we conclude that g does not take on an absolute mininimum value on $(0, \infty)$.

5.2 Part b

The required area is given by the improper integral

$$\int_1^\infty \left[\frac{1}{x} - \frac{4x}{1+4x^2}\right] dx = \int_1^\infty \frac{dx}{x(1+4x^2)}. \tag{32}$$

But we are concerned only with positive values of x, so

$$\int \frac{dx}{x(1+4x^2)} = \int \left[\frac{1}{x} - \frac{4x}{1+4x^2}\right] dx \tag{33}$$

$$= \ln x - \frac{1}{2}\ln(1+4x^2) = \ln \frac{x}{\sqrt{1+4x^2}}. \tag{34}$$

Thus,

$$\int_1^\infty \frac{dx}{x(1+4x^2)} = \lim_{T\to\infty} \int_1^T \frac{dx}{x(1+4x^2)} \tag{35}$$

$$= \lim_{T\to\infty} \ln \frac{x}{\sqrt{1+4x^2}}\bigg|_1^T \tag{36}$$

$$= \lim_{T\to\infty} \left[\ln \frac{T}{\sqrt{1+4T^2}} - \ln \frac{1}{\sqrt{5}}\right] \tag{37}$$

$$= \lim_{T\to\infty} \ln \frac{1}{\sqrt{T^{-2}+4}} - \ln \frac{1}{\sqrt{5}} \tag{38}$$

$$= \ln \frac{1}{2} - \ln \frac{1}{\sqrt{5}} = \ln \frac{\sqrt{5}}{2}. \tag{39}$$

6 Problem 6

6.1 Part a

We apply the Ratio Test to determine the radius of convergence:

$$\lim_{n\to\infty} \left[\left(\frac{|2x|^{n+1}}{n}\right)\left(\frac{n-1}{|2x|^n}\right)\right] = |2x| \lim_{n\to\infty} \frac{n-1}{n} = 2|x|, \tag{40}$$

and this is less than one when $|x| < \frac{1}{2}$ but greater than one when $|x| > \frac{1}{2}$. Thus the series converges when $-\frac{1}{2} < x < \frac{1}{2}$ but diverges when $x < -\frac{1}{2}$ and when $x > \frac{1}{2}$. When $x = \frac{1}{2}$, the series becomes the alternating harmonic series, which converges. When $x = -\frac{1}{2}$, the series becomes the harmonic series, which diverges. The interval of convergence is therefore $\left(-\frac{1}{2}, \frac{1}{2}\right]$, and the radius of convergence is $\frac{1}{2}$.

6.2 Part b

If

$$y = f(x) = \sum_{n=2}^{\infty} \left[(-1)^n \frac{(2x)^n}{n-1} \right] \text{ when } |x| < \frac{1}{2}, \tag{41}$$

then, differentiating term-by-term, we obtain

$$y' = f'(x) = \sum_{n=2}^{\infty} \left[(-1)^n \frac{2n}{n-1} (2x)^{n-1} \right], \tag{42}$$

and we may write

$$xy' - y = \sum_{n=2}^{\infty} \left[(-1)^n \frac{n}{n-1} (2x)^n \right] - \sum_{n=2}^{\infty} \left[(-1)^n \frac{(2x)^n}{n-1} \right] \tag{43}$$

$$= \sum_{n=2}^{\infty} \left[(-1)^n \left(\frac{n}{n-1} - \frac{1}{n-1} \right) (2x)^n \right], \tag{44}$$

at least for $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, because both of the series being combined converge absolutely on that interval. Thus

$$xy' - y = \sum_{n=2}^{\infty} (-2x)^n, \tag{45}$$

and this is a geometric series with common ratio $(-2x)$. It follows from this last observation that, on the interval $\left((-\frac{1}{2}, \frac{1}{2}\right)$,

$$xy' - y = \frac{4x^2}{1+2x}, \tag{46}$$

which shows that $y = f(x)$, as given by the series above, is a solution of the differential equation (45).

Note: If we multiply equation (45) through by $\frac{1}{x^2}$, the equation becomes

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{4}{1+2x}. \tag{47}$$

We can easily obtain the initial condition $y(0) = 0$ from the original series. Solving this initial value problem gives $f(x) = \sum_{n=2}^{\infty} \left[(-1)^n \frac{(2x)^n}{n-1} \right] = 2x \ln(1+2x)$ on the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.