# AP Calculus 2010 BC FRQ Solutions

Louis A. Talman, Ph.D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

June 7, 2017

## 1 Problem 1

### 1.1 Part a

At 6 AM, $7\int_0^6 t\,e^{\cos t}\,dt$ cubic feet of snow have accumulated. Integrating numerically, we find that $142.17469$ cubic feet have accumulated by $6$ AM.

### 1.2 Part b

At $8$ AM, snow is falling at the rate of $56e^{\cos 8}$ cubic feet per hour, but Janet is removing it at the rate of $108$ cubic feet per hour. So at $8$ AM, the rate of change of the volume of snow on the driveway is $56e^{\cos 8} - 108 \sim -59.58297$ cubic feet per hour.

### 1.3 Part c

We have

$$h(t) = \begin{cases} 0 & \text{when } 0 \le t \le 6. \\ 125(t-6) & \text{when } 6 \le t < 7. \\ 125 + 108(t-7) & \text{when } 7 \le t \le 9. \end{cases} \tag{1}$$

### 1.4 Part d

$7\int_0^9 t\,e^{\cos t}\,dt - 341$ is the total amount, in cubic feet, of snow on the driveway at $t = 9$. Numeric integration gives this as $26.33461$ cubic feet of snow at $9$ AM.

# 2 Problem 2

## 2.1 Part a

At $t = 6$, the approximate rate at which entries were being made is given by the fraction

$$\frac{E(7) - E(5)}{7 - 5} = \frac{21 - 13}{2} = 4 \text{ hundred entries per hour.} \tag{2}$$

## 2.2 Part b

The trapezoidal approximation is

$$\frac{1}{8}\int_0^8 E(t)\,dt \sim \frac{1}{16}[(4+0)\cdot 2 + (13+4)\cdot 3 + (21+13)\cdot 2 + (23+21)\cdot 1 = \frac{171}{16}. \tag{3}$$

This means that the average rate of deposits at any time during the 8-hour period was about $171/16$ hundreds of entries per hour.

## 2.3 Part c

The number $U(t)$, of entries not processed at a given time $t$, $8 \leq t \leq 12$, is given by

$$U(t) = 2300 - 100\int_8^t P(\tau)\,d\tau, \tag{4}$$

where $P(t) = t^3 - 30t^2 + 298t - 976$. Thus,

$$U(t) = 2300 - 100\left(\frac{1}{4}\tau^4 - 10\tau^3 + 149\tau^2 - 976\tau\right)\Big|_8^t \tag{5}$$

$$= -2tt^2 + 1000t^3 - 14900t^2 + 97600t - 234500., \tag{6}$$

and

$$U(12) = 700. \tag{7}$$

According to this model, 700 entries remain unprocessed at midnight.

### 2.4 Part d

We want the maximal rate at which entries were being processed in the time interval $[8, 12]$. Such a maximum must lie at a critical point or at an end-point. The critical points are the solutions of the equation $P'(t) = 3t^2 - 60t + 298 = 0$ or

$$t = \frac{30 \pm \sqrt{6}}{3}. \tag{8}$$

Thus, the critical points are $t_1 \sim 9.18350$ and $t_2 \sim 10.81650$. We find that

$$P(8) = 0; \tag{9}$$
$$P(t_1) \sim 5.08866; \tag{10}$$
$$P(t_2) \sim 2.91134; \text{ and} \tag{11}$$
$$P(12) = 8. \tag{12}$$

The largest of these numbers must be the maximum, so the entries were being processed most quickly at midnight.

## 3 Problem 3

### 3.1 Part a

Speed at time $t$ is

$$\sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{(2t-4)^2 + (te^{t-3} - 1)^2}. \tag{13}$$

Consequently, when $t = 3$, speed is $\sqrt{4+4} = 2\sqrt{2}$ meters per second.

### 3.2 Part b

To obtain distance traveled, we integrate speed over the time interval. This particular integral must be done numerically; we get

$$\int_0^4 \sqrt{(2t-4)^2 + (te^{t-3} - 1)^2}\, dt \sim 11.58767. \tag{14}$$

The total distance traveled during the time interval from $t = 0$ to $t = 4$ is about $11.58767$ meters.

### 3.3 Part c

The tangent line is horizontal at points where

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right) = 0. \tag{15}$$

This is possible only where $\frac{dy}{dt} = 0$, which is to say, when $te^{t-3} - 1 = 0$. Because $d^2y/dt^2 = (1 + t)e^{t-3}$ is negative when $t < -1$ and positive when $t > -1$, we see that $y'(t)$ is a decreasing function on $(-\infty, -1)$ and an increasing function on $(-1, \infty)$. Moreover,

$$\lim_{t\to-\infty}(te^{t-3} - 1) = \left(\lim_{t\to-\infty}\frac{t}{e^{3-t}}\right) - 1, \tag{16}$$

and, because both the limit in the numerator and the limit in the denominator are infinite, we may attempt l'Hôpital's Rule. Application of that rule gives

$$\lim_{t\to-\infty}(te^{t-3} - 1) = \left(\lim_{t\to-\infty}\frac{1}{-e^{3-t}}\right) - 1 = -1. \tag{17}$$

From the value of this limit, the fact that $y'(-1) \sim -1.018$, and the fact that $y'$ is decreasing on $(-\infty, -1]$ we conclude that $y'(t) \neq 0$ for any value of $t < -1$. On the other hand, $y'$ is an increasing function on $[-1, \infty)$ whose limit at $\infty$ is $\infty$, so there is exactly one value of $t$ for which $y'(t) = 0$. Numerical solution gives $t \sim 2.20794$ as the only value of $t$ for which this curve's tangent line is horizontal.

### 3.4 Part d

When $x = 5$, we have

$$t^2 - 4t + 8 = 5; \tag{18}$$

$$t^2 - 4t + 3 = 0; \tag{19}$$

$$(t - 1)(t - 3) = 0; \tag{20}$$

so that $t = 1$ or $t = 3$.

The slope of the tangent line when $t = 1$ is

$$\frac{y'(1)}{x'(1)} = \frac{1 \cdot e^{-2} - 1}{2 - 4} = \frac{1}{2}\left(1 - e^{-2}\right), \tag{21}$$

while the slope of the tangent line at $t = 3$ is

$$\frac{y'(3)}{x'(3)} = \frac{3 \cdot e^0 - 1}{6 - 4} = 1. \tag{22}$$

If $y(2) = 3 + e^{-1}$, then, by the Fundamental Theorem of Calculus and an integration by parts,

$$y(t) = 3 + e^{-1} + \int_2^t \left(\tau e^{\tau - 3} - 1\right) \, d\tau \tag{23}$$
$$= e^{t-3}(t-1) - t + 5. \tag{24}$$

Thus, $y(1) = y(3) = 4$.

# 4 Problem 4

## 4.1 Part a

The area of $R$ is

$$\int_0^9 \left[6 - 2\sqrt{x}\right] \, dx = \left[6x - \frac{4}{3}x^{3/2}\right]\Bigg|_0^9 = 18. \tag{25}$$

## 4.2 Part b

The volume of the solid generated by revolving $R$ about the line $y = 7$ is given by

$$\pi \int_0^9 \left[\left(7 - \sqrt{x}\right)^2 - 1\right] \, dx = \pi \int_0^9 \left(48 - 14\sqrt{x} + x\right) \, dx. \tag{26}$$

**Note:** Evaluation of the integral is not required. However,

$$\pi \int_0^9 \left(48 - 14\sqrt{x} + x\right) \, dx = \pi \left(48x - 14 \cdot \frac{2}{3}x^{3/2} + \frac{x^2}{2}\right)\Bigg|_0^9 \tag{27}$$
$$= \pi \left(48 \cdot 9 - 14 \cdot \frac{2}{3} \cdot 27 + \frac{81}{2}\right) = \frac{441}{2}\pi. \tag{28}$$

### 4.3 Part c

The volume of this solid is $\frac{3}{16}\int_0^6 y^4\,dy$.

**Note:** Evaluation of this integral isn't required either. However

$$\frac{3}{16}\int_0^6 y^4\,dy = \frac{3}{16}\cdot\frac{y^5}{5}\bigg|_0^6 = \frac{1458}{5} \tag{29}$$

# 5 Problem 5

### 5.1 Part a

In order to approximate the solution of the initial value problem $y' = f(x,y) = 1-y$, $y(1) = 0$, on the interval $[0,1]$, using Euler's method with step-size $-1/2$, we write

$$x_0 = 2; \tag{30}$$
$$y_0 = 0; \tag{31}$$

and for $k = 1, 2, \ldots$

$$x_{k+1} = x_k - \frac{1}{2}; \tag{32}$$
$$y_{k+1} = y_k - \frac{1}{2}f(x_k, y_k). \tag{33}$$

Thus,

$$x_1 = \frac{1}{2}; \tag{34}$$
$$y_1 = 0 - \frac{1}{2}(1-0) = -\frac{1}{2}; \tag{35}$$
$$\tag{36}$$
$$x_2 = 0; \tag{37}$$
$$y_2 = -\frac{1}{2} - \frac{1}{2}\left(1+\frac{1}{2}\right) = -\frac{5}{4}. \tag{38}$$

We conclude that $y(0) \sim -5/4$.

## 5.2 Part b

If $y = f(x)$ solves the initial value problem of Part a, then $y$ must be a differentiable function near $x = 1$, and so must be continuous near $x = 1$. It follows then from $y(1) = 0$ that $\lim_{x\to 1} f(x) = 0$. Because, also, $x^3 - 1 \to 0$ as $x \to 1$, we may attempt to evaluate the limit of the quotient $f(x)/(x^3 - 1)$ by l'Hôpital's rule. We get

$$\lim_{x\to 1} \frac{f(x)}{x^3 - 1} = \lim_{x\to 1} \frac{f'(x)}{3x^2} \tag{39}$$

$$= \lim_{x\to 1} \frac{1 - f(x)}{3x^2} = \frac{1}{3}, \tag{40}$$

where we have used the relation $f'(x) = 1 - f(x)$ together with the fact, mentioned above, that $f(x) \to 0$ as $x \to 1$.

## 5.3 Part c

If $f'(x) = 1 - f(x)$, then, at least as long as $f(x) \neq 1$—which, as we see from the continuity of $f$ we deduced in Part b, above, must be correct for all $x$ in some interval centered at $x = 0$—we have

$$\frac{f'(x)}{1 - f(x)} = 1. \tag{41}$$

Thus, provided that $|1 - x|$ isn't too large and $t$ lies in the interval whose endpoints are $1$ and $x$, we know that $1 - f(t) > 0$. For such values of $x$, we may write

$$\int_1^x \frac{f'(t)}{1 - f(t)}\, dt = \int_1^x dt, \tag{42}$$

$$-\ln\left[1 - f(t)\right]\Big|_1^x = t\Big|_1^x, \tag{43}$$

$$1 - f(x) = e^{1-x}. \tag{44}$$

Thus, the solution is given by $f(x) = 1 - e^{1-x}$,

# 6 Problem 6

## 6.1 Part a

The Taylor expansion about $x = 0$ for $\cos x$, which converges to $\cos x$ for all values of $x$, is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}. \tag{45}$$

Consequently, the Taylor expansion for the function $f$ given in the problem is

$$f(x) = \frac{1}{x^2}\left[\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\right) - 1\right] = \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n-2}}{(2n)!} \tag{46}$$

$$= -\frac{1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!} + \cdots + (-1)^n \frac{x^{2n-2}}{(2n)!} + \cdots. \tag{47}$$

## 6.2 Part b

The general term of the Taylor series about $x = 0$ for a function $F(x)$ is $\frac{F^{(k)}(0)}{k!}x^k$ and from this and the expansion we have given above, we see that $f'(0) = 1$ and $f''(0) = 1/24 > 0$. Thus, $x = 0$ gives a critical point for $f$ where $f''$ is positive. By the Second Derivative Test, $f$ has a local minimum at $x = 0$.

## 6.3 Part c

If

$$g(x) = 1 + \int_0^x f(t), dt, \tag{48}$$

then we can obtain the Taylor expansion for $g$ by integrating the expansion for $f$ term by term. This leads to

$$g(x) = 1 + \left[\sum_{n=1}^{\infty}(-1)^n \int_0^x \frac{t^{2n-2}}{(2n)!}\, dt\right] \tag{49}$$

$$= 1 + \left[\sum_{n=1}^{\infty}(-1)^n \frac{x^{2n-1}}{(2n-1)(2n)!}\right] \tag{50}$$

$$= 1 - \frac{x}{2!} + \frac{x^3}{3 \cdot 4!} - \frac{x^5}{3 \cdot 6!} + \cdots. \tag{51}$$

The required Taylor polynomial, $P_5(x)$ is thus

$$P_5(x) = 1 - \frac{x}{2!} + \frac{x^3}{3 \cdot 4!} - \frac{x^5}{5 \cdot 6!} \tag{52}$$

### 6.4 Part d

The approximate value for $g(1)$ that results from replacing $g$ with its third degree Taylor polynomial at $x = 0$ is

$$g(1) \sim 1 - \frac{1}{2!} + \frac{1^3}{3 \cdot 4!} = 1 - \frac{1}{2} + \frac{1}{72} = \frac{37}{72}. \tag{53}$$

When $x = 1$ this series for $g$ has alternating signs and the magnitudes of the terms clearly decrease to zero. Thus, by the Alternating Series Test, the error in the estimate above is, in magnitude, at most the magnitude of the fourth term of the series, which is

$$\frac{1}{5 \cdot 6!} < \frac{1}{6!}, \tag{54}$$

and this guarantees that the error in our estimate for $g(1)$ differs from the actual value of $g(1)$ by less than $1/6!$.