AP Calculus 2011 BC, Form B, FRQ Solutions

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1 Problem 1

1.1 Part a

According to the model, the height of the water in the can at the end of the 60-day period is

$$\int_0^{60} [2\sin(0.03t) + 1.5]\,dt = \left[-\frac{2}{0.03}\cos(0.03t) + 1.5t\right]\Bigg|_0^{60} \tag{1}$$

$$= \left(-\frac{200}{3}\cos(9/5) + 90\right) + \frac{200}{3} = \left[\frac{470}{3} - \frac{200}{3}\cos\left(\frac{9}{5}\right)\right]\ \text{mm}. \tag{2}$$

1.2 Part b

The average rate of change in the height of water in the can over the 60-day period is

$$\frac{1}{60}\int_0^{60} S'(t)\,dt = \frac{1}{60}\left[\frac{470}{3} - \frac{200}{3}\cos\left(\frac{9}{5}\right)\right] = \left[\frac{47}{18} - \frac{10}{9}\cos\left(\frac{9}{5}\right)\right]\ \text{mm/day}, \tag{3}$$

where we have inserted the value of the integral from equation (2).

1.3 Part c

The volume $V(t)$ of water in the can at time t is given by

$$V(t) = 100\pi S(t), \text{ so} \tag{4}$$
$$V'(t) = 100\pi S'(t). \tag{5}$$

Consequently,

$$V'(7) = 100\pi S'(7) = 150\pi + 200\pi \sin\left(\frac{21}{100}\right) \text{ cubic mm/sec.} \tag{6}$$

1.4 Part d

We have $M'(t) = \frac{1}{400}(9t^2 - 60t + 330)$. Using $S'(t)$ as given, we find that

$$M'(0) - S'(0) = \frac{33}{40} - \frac{3}{2} = -\frac{27}{40} < 0, \text{ while} \tag{7}$$
$$D(60) = M'(60) - S'(60) = \frac{2853}{40} - 2\sin\left(\frac{9}{5}\right) > \frac{2853}{40} - 2 > 69 > 0. \tag{8}$$

Because D is a continuous function on $[0, 60]$, it follows from the Intermediate Value Theorem that there is a time $t_0 \in (0, 60)$ such that $D(t_0) = 0$, which is to say that $M'(t_0) = S'(t_0)$, or the two rates are the same.

2 Problem 2

2.1 Part a

The area of the polar curve $r = r(\theta)$ corresponding to the interval $\alpha \leq \theta \leq \beta$ is given by $\frac{1}{2}\int_\alpha^\beta [r(\theta)]^2\, d\theta$, so we calculate

$$\frac{1}{2}\int_{\pi/2}^{\pi} [r(\theta)]^2 \; d\theta = \frac{1}{2}\int_{\pi/2}^{\pi} [3\theta + \sin\theta]^2 \; d\theta \tag{9}$$

Numeric integration gives 47.51322 for the required area. The symbolic integration is elementary, but moderately tedious (involving as it does, an integration by parts and a use of the half-angle formula for the sine function), and there are better ways to spend time on the exam. For the sake of completeness,

$$\begin{aligned}
\frac{1}{2}\int_{\pi/2}^{\pi}\left[3\theta+\sin\theta\right]^2\,d\theta &= \frac{1}{2}\int_{\pi/2}^{\pi}\left[9\theta^2+6\theta\sin\theta+\sin^2\theta\right]\,d\theta && (10)\\
&= \frac{1}{2}\int_{\pi/2}^{\pi}\left[9\theta^2+6\theta\sin\theta+\frac{1}{2}\left(1-\cos 2\theta\right)\right]\,d\theta && (11)\\
&= \frac{3}{2}\theta^3\Big|_{\pi/2}^{\pi}-3\theta\cos\theta\Big|_{\pi/2}^{\pi}+3\int_{\pi/2}^{\pi}\cos\theta\,d\theta+\frac{1}{4}\theta\Big|_{\pi/2}^{\pi}-\frac{1}{8}\sin 2\theta\Big|_{\pi/2}^{\pi} && (12)\\
&= \frac{21}{16}\pi^3+3\pi+3\sin\theta\Big|_{\pi/2}^{\pi}+\frac{\pi}{8} && (13)\\
&= \frac{21}{16}\pi^3+\frac{25}{8}\pi-3. && (14)
\end{aligned}$$

2.2 Part b

We are to solve the equation $r(\theta)\cos\theta = -3$, with $r(\theta)$ as above. We call the solution θ_0 for future reference. Thus, the equation to be solved is

$$(3\theta+\sin\theta)\cos\theta = -3. \tag{15}$$

Numerical solution gives $\theta_0 \sim 2.01692$.

We have

$$y(\theta_0) = r(\theta_0)\sin\theta_0 \sim 6.27238. \tag{16}$$

2.3 Part c

We have $y(\theta) = r(\theta)\sin\theta$, so that

$$y'(\theta) = r'(\theta)\sin\theta + r(\theta)\cos\theta. \tag{17}$$

This gives

$$\begin{aligned}
y'\left(\frac{2\pi}{3}\right) &= \frac{5\sqrt{3}}{4}+\frac{1}{2}\left(-\frac{\sqrt{3}}{2}-2\pi\right) && (18)\\
&= \sqrt{3}-\pi. && (19)
\end{aligned}$$

But $\frac{dy}{dt} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dt}$, and we are given $\frac{d\theta}{dt} = 2$. Thus $\frac{dy}{dt} = 2(\sqrt{3} - 2)$when $\theta = 2\pi/3$. This is the y-component of the velocity of the particle at the instant in question; it is negative, so the particle's y-component is decreasing at that instant.

3 Problem 3

3.1 Part a

The area of the pictured region R is

$$\int_0^4 \sqrt{x}\, dx + \int_4^6 (6 - x)\, dx = \frac{2}{3} x^{3/2} \bigg|_0^4 + \left(6x - \frac{x^2}{2}\right) \bigg|_4^6 = \frac{16}{3} + 2 = \frac{22}{3}. \qquad (20)$$

3.2 Part b

A cross section of this solid perpendicular to the y-axis at $y = t$ is a rectangle whose height is $2t$ and whose base extends from the curve $x = y^2$ to the curve $x = 6 - y$. The area of such a cross section is therefore $2t\left[(6 - t) - t^2\right]$, so the required integral is $2 \displaystyle\int_0^2 \left[6t - t^2 - t^3\right]\, dt$.

Note: Evaluation of this integral is not required. For the curious,

$$2 \int_0^2 \left[6t - t^2 - t^3\right]\, dt = 2 \left[3t^2 - \frac{1}{3}t^3 - \frac{1}{4}t^4\right] \bigg|_0^2 \qquad (21)$$

$$= 2 \left[12 - \frac{8}{3} - 4\right] = \frac{32}{3}. \qquad (22)$$

3.3 Part c

The slope of the line $y = 6 - x$ is -1, so we seek a point on the curve $y = \sqrt{x}$ where $y' = 1$. But $y' = \frac{1}{2}x^{-1/2}$ = 1 when $x^{-1/2} = 2$, or, equivalently, when $x = \frac{1}{4}$. The point P therefore has coordinates $\left(\frac{1}{4}, \frac{1}{2}\right)$.

4 Problem 4

In effect, it is given that

$$f'(3) = 0, \tag{23}$$

$$f'(8) = 0, \tag{24}$$

$$\int_0^5 f(x)\,dx = -10, \tag{25}$$

$$\int_5^{10} f(x)\,dx = 27, \tag{26}$$

$$\int_0^5 \sqrt{1+[f'(x)]^2}\,dx = 11, \text{ and} \tag{27}$$

$$\int_5^{10} \sqrt{1+[f'(x)]^2}\,dx = 18. \tag{28}$$

4.1 Part a

The average value of f over the interval $[0,5]$ is

$$\frac{1}{5}\int_0^5 f(x)\,dx = \frac{1}{5}\cdot(-10) = -2. \tag{29}$$

4.2 Part b

$$\int_0^{10} (3f(x)+2)\;dx = 3\int_0^{10} f(x)\,dx + 2\int_0^{10} dx \tag{30}$$

$$= 3\int_0^5 f(x)\,dx + 3\int_5^{10} f(x)\,dx + 2x\Big|_0^{10} \tag{31}$$

$$= 3\cdot(-10) + 3\cdot 27 + 2\cdot 10 - 2\cdot 0 = 71. \tag{32}$$

4.3 Part c

If $g(x) = \int_5^x f(t)\,dt$, then, by the Fundamental Theorem of Calculus, $g'(x) = f(x)$. But $f(x) < 0$ for on $x \in (0,5)$, and $f(x) > 0$ on $(5,10)$. Thus, the continuous function g is decreasing on $[0,5]$. The graph of g is concave upward on intervals where $g' = f$ is increasing. But, from the graph and the critical points given for f, we see that f is increasing on $[3, 8]$. It follows that g is both concave upward and decreasing on the interval $(3, 5)$. Whether or not we include the endpoints in this interval depends upon which definition (there are several) we have adopted for the term *concave upward*. The readers haven't worried about this subtlety in the past.

4.4 Part d

The required arc-length is given by $\int_0^{20} \sqrt{1 + \left[f'\left(\frac{t}{2}\right)\right]^2}\, dt$. Taking $t = 2x$, we see that $dt = 2\,dx$, that $x = 0$ when $t = 0$, and that $x = 10$ when $t = 20$. Consequently,

$$\int_0^{20} \sqrt{1 + \left[f'\left(\frac{t}{2}\right)\right]^2}\, dt = 2\int_0^{10} \sqrt{1 + [f'(x)]^2}\, dx \tag{33}$$

$$= 2\left[\int_0^{5} \sqrt{1 + [f'(x)]^2}\, dx + \int_5^{10} \sqrt{1 + [f'(x)]^2}\, dx\right] \tag{34}$$

$$= 2 \cdot 11 + 2 \cdot 18 = 54. \tag{35}$$

5 Problem 5

5.1 Part a

Ben's acceleration at time $t = 5$ is approximately

$$\frac{v(10) - v(0)}{10 - 0} = \frac{2.3 - 2.0}{10} = 0.03 \text{ meters per second per second.} \tag{36}$$

5.2 Part b

The integral $\int_0^{60} |v(t)|\, dt$ is the integral of Ben's speed. It measures the total distance Ben has traveled over the interval $0 \leq t \leq 60$. We have

$$\int_0^{60} |v(t)|\, dt \sim 2.0 \cdot (10 - 0) + 2.3 \cdot (40 - 10) + 2.5 \cdot (60 - 4) = 139, \tag{37}$$

so the total distance Ben traveled during this minute is about 139 meters.

5.3 Part c

We have

$$\frac{B(60) - B(40)}{60 - 40} = \frac{49 - 9}{60 - 40} - \frac{40}{20} = 2. \tag{38}$$

We may apply the Mean Value Theorem here, because we are given that B is a twice differentiable function, and this latter fact guarantees that B is continuous on $[40, 60]$ and differentiable on $(40, 60)$—which are the hypotheses of the Mean Value Theorem. Thus, there must be a time $t_0 \in (40, 60)$ when $v(t_0) = B'(t_0) = 2$.

Note: We are cheating a bit, but this has to be what the examiners expected. We haven't been told just *where* B is twice-differentiable or what the domain of B is, and it's not really clear what it would mean for $B''(60)$ to exist if the domain of B is $[0, 60]$. We adopt the convention that the problem takes differentiability at an end-point to be the appropriate one-sided differentiability there; if we don't do so, our conclusion that B is continuous at $t = 60$ is unsupportable.

5.4 Part d

From $L^2 = 144 + B^2$, we find that $2LL' = 2BB' = 2Bv$. Thus, when $t = 40$ we have

$$2LL' = 2Bv = \cancel{2} \cdot 9 \cdot \frac{5}{\cancel{2}} = 45. \tag{39}$$

However, when $t = 40$, we also have $L^2 = 144 + 81 = 225$, so that $L = 15$. Thus, at $t = 40$, $45 = 2LL' = 2 \cdot 15 \cdot L'$, and $L' = \frac{45}{30} = \frac{3}{2}$ meters per second.

6 Problem 6

6.1 Part a

We may substitute x^3 for x in the Maclaurin series for $\ln(1 + x)$ to obtain that for $f(x) = \ln(1 + x^3)$, and the resulting series is

$$x^3 - \frac{1}{2}x^6 + \frac{1}{3}x^9 - \frac{1}{4}x^{12} + \cdots + \frac{(-1)^{n+1}}{n}x^{3n} + \cdots .$$

6.2 Part b

The series for f must converge in the open interval $(-1, 1)$, because $x = 0$ is the center of the expansion, and the radius of convergence is given to be 1.

The only real issue is whether the series converges at endpoints. When $x = 1$ the series becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \cdots$, which is the convergent alternating harmonic series. When $x = -1$, the series becomes $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots$, which is the negative of the divergent harmonic series. The interval of convergence for the Maclaurin series for f is therefore $(-1, 1]$.

6.3 Part c

We obtain the Maclaurin series for f' by term-by-term differentiation of the Maclaurin series for f, and we obtain

$$3x^2 - 3x^5 + 3x^8 - 3x^{11} + \cdots + (-1)^{n+1} x^{3n-1} + \cdots .$$

Consequently, the first four terms of the Maclaurin series for $f'(t^2)$ are

$$3t^4 - 3t^{10} + 3t^{16} - 3t^{22}.$$

Replacing $f'(t^2)$ with the first two terms of this series in $\int_0^1 f'(t^2)\, dt$ gives

$$\int_0^1 \left(3t^4 - 3t^{10}\right)\, dt = \left[\frac{3}{5}t^5 - \frac{3}{11}t^{11}\right]\Big|_0^1 = \frac{3}{5} - \frac{3}{11} = \frac{18}{55} \sim 0.32727. \tag{40}$$

6.4 Part d

The Maclaurin series for g begins with the terms

$$\frac{3}{5}x^5 - \frac{3}{11}x^{11} + \frac{3}{17}x^{17},$$

and we have been given that the series meets the hypotheses of the Alternating Series Test. In Part c, above, we used the first two terms of this series to approximate $g(1)$. Hence, the error in our approximation is bounded by the magnitude of the third term, which is

$$\left|\frac{3}{17} \cdot 1^{17}\right| = \frac{3}{17} < \frac{3}{15} = \frac{1}{5} \tag{41}$$

when $x = 1$.

Note: In fact, it can be shown that

$$\int_0^1 f'(t^2)\,dt = \frac{1}{4}\left[2\pi + \sqrt{3}\ln\left((7 - 4\sqrt{3}\right)\right] \sim 0.43028, \tag{42}$$

so the approximation of Part c is a pretty miserable one.