AP Calculus 2011 BC FRQ Solutions

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1 Problem 1

1.1 Part a

Speed is the magnitude of the velocity vector, which we are given as

$$\mathbf{v}(t) = \langle x'(t), y'(t) \rangle = \langle 4t + 1, \sin t^2 \rangle. \tag{1}$$

Thus, speed is $\sqrt{(4t+1)^2 + \sin^2 t^2}$. When $t = 3$, this is $\sqrt{169 + \sin^2 9}$. The acceleration vector $\mathbf{a}(t)$ is given by $\mathbf{a}(t) = \mathbf{v}'(t) = \langle 4, 2t \cos t^2 \rangle$. At time $t = 3$, $\mathbf{a}(3) = \langle 4, 6 \cos 9 \rangle$.

1.2 Part b

The slope of the line tangent to the path at $t = 3$ is

$$\frac{y'(3)}{x'(3)} = \frac{\sin 9}{13} \sim 0.03170. \tag{2}$$

1.3 Part c

The position, $\mathbf{r}(t)$, of the particle at time t is

$$\mathbf{r}(t) = \langle 0, 4 \rangle + \int_0^t \langle 4\tau + 1, \sin \tau^2 \rangle \, d\tau, \text{ so} \tag{3}$$

$$\mathbf{r}(3) = \langle 0, 4 \rangle + \int_0^3 \langle 4\tau + 1, \sin \tau^2 \rangle \, d\tau \tag{4}$$

Numerical integration gives $\mathbf{r}(3) \sim \langle 21.00000, 3.22644 \rangle$.

1.4 Part d

Total distance traveled during $0 \leq t \leq 3$ is the definite integral of speed from 0 to 3. We calculated speed in Part a, above. By numerical integration, we have

$$\int_0^3 \sqrt{(4\tau + 1)^2 + \sin^2 \tau^2}\, d\tau \sim 21.09119 \tag{5}$$

2 Problem 2

2.1 Part a

The rate at which the temperature of the tea is changing at time $t = 3.5$ is given, approximately, by the difference quotient

$$\frac{H(3.5 + 1.5) - H(3.5 - 1.5)}{(3.5 + 1.5) - (3.5 - 15)} = \frac{52 - 60}{3} = -\frac{8}{3} \text{ degrees per minute.} \tag{6}$$

2.2 Part b

The average value $\bar{T}$ of the temperature of the tea, in degrees Celsius, is

$$\bar{T} = \frac{1}{10} \int_0^{10} H(t)\, dt. \tag{7}$$

The trapezoidal approximation for this integral is

$$\frac{1}{10} \cdot \frac{1}{2} \sum_{k=1}^{4} \left[H(t_{k-1}) + H(t_k)\right](t_k - t_{k-1}) \tag{8}$$

$$= \frac{1}{20}\left[(66 + 60)(2 - 0) + (60 + 52)(5 - 2) + (52 + 44)(9 - 5) + (44 + 43)(10 - 9)\right] \tag{9}$$

$$= \frac{1059}{20}. \tag{10}$$

2.3 Part c

By the Fundamental Theorem of Calculus, $\int_0^{10} H'(t)\, dt = H(10) - H(0) = -23$. Thus, -23° C is, again by the Fundamental Theorem of Calculus, the amount by which the temperature had changed over the interval $0 \leq t \leq 10$.

2.4 Part d

$B(t)$ is given by

$$B(t) = 100 - 13.84 \int_0^t e^{-0.173\tau}\, d\tau. \tag{11}$$

Therefore

$$B(10) = 100 - 13.84 \int_0^{10} e^{-0.173\tau}\, d\tau \tag{12}$$

$$= 100 - 13.84 \left(-\frac{1}{0.173} e^{-0.173\tau} \right) \Bigg|_0^{10} \sim 34.18275. \tag{13}$$

We seek $H(10) - B(10) = 43 - 34.18275 = 8.81725$. So the biscuits are about $8.81725°$ C. cooler than the tea at time $t = 10$.

3 Problem 3

3.1 Part a

The perimeter, P, of the region shown consists of three line segments and the piece of the curve $y = e^{2x}$ corresponding to $0 \leq x \leq k$. This is given by

$$P = 1 + k + e^{2k} + \int_0^k \sqrt{1 + 4e^{4x}}\, dx, \tag{14}$$

where we have used the arc-length integral to find the arc-length of the portion of the perimeter that is not a straight line.

3.2 Part b

The area of a cross section of the volume perpendicular to the x-axis at $x = t$ is $\pi(e^{2t})^2 = \pi e^{4t}$, so the volume of the solid is

$$\pi \int_0^k e^{4t}\, dt = \frac{\pi}{4} e^{4t} \Big|_0^k = \frac{\pi}{4} \left(e^{4k} - 1 \right). \tag{15}$$

3.3 Part c

From Part b, above, we have $V(k) = \frac{\pi}{4}(e^{4k} - 1)$. Thus

$$\frac{dV}{dt} = \frac{dV}{dk} \cdot \frac{dk}{dt} = \pi e^{4k} \cdot \frac{1}{3}. \tag{16}$$

When $k = \frac{1}{2}$, this is $\frac{\pi}{3}e^2$.

4 Problem 4

4.1 Part a

$$g(-3) = -6 + \int_0^{-3} f(t)\,dt = -6 - \frac{1}{4}\pi \cdot 3^2 = -6 - \frac{9}{4}\pi; \tag{17}$$

$$g'(x) = \frac{d}{dx}\left[2x + \int_0^x f(t)\,dt\right] = 2 + f(x). \tag{18}$$

$$G'(3) = 2 + f(-3) = 2. \tag{19}$$

4.2 Part b

The absolute maximum of g must occur at an endpoint of the interval $[-4, 3]$ or at a critical point interior to that interval. But $g'(x) = 2 + f(x)$, and this is simply the curve $y = f(x)$ shifted 2 units upward. Note that all of the shifted curve that lies to the left of the y-axis lies above the x-axis, so that $g'(x) > 0$ when x lies to the left of the y-axis—and for a substantial interval just to the right of the y-axis. For $0 \le x \le 3$, we then have $g'(x) = 5 - 2x$, so that $g'(x) = 0$ when $x = \frac{5}{2}$. Thus, $g'(x) > 0$ for $-4 \le x < \frac{5}{2}$, negative for $\frac{5}{2} < x \le 3$, and zero when $x = \frac{5}{2}$. The latter value is the only critical value for g. It is clear, on geometric ground, that the area under g' on the interval $[-4, \frac{5}{2}]$ is positive and exceeds, in magnitude, the area between the g' curve and the x-axis on the interval $[\frac{5}{2}, 2]$. Consequently, $0 = f(-4) < g(\frac{5}{2})$ and $g(3) < g(\frac{5}{2})$. The absolute maximum therefore occurs at $x = \frac{5}{2}$.

4.3 Part c

The function g' [see Part b, above, for an explicit description of g'] is increasing on $[-4, 0]$ and decreasing on $[0, 3]$. Inflection points are to be found where the monotonicity of the derivative changes, so $x = 0$ is the location of the only inflectiion point for this curve.

4.4 Part d

We have $f(-4) = -1$ and $f(3) = -3$. The average rate of change of f on the interval $[-4, 3]$ is therefore

$$\frac{f(3) - f(-4)}{4 - (-3)} = \frac{(-3) - (-1)}{7} = -\frac{2}{7}. \tag{20}$$

That $f'(c) = -\frac{2}{7}$ fails for all c in $(-4, 3)$ doesn't contradict the Mean Value Theorem because $f'(0)$ doesn't exist. The hypotheses of the Mean Value Theorem require, among other things, that a function f be differentiable on $(-4, 3)$ before we may apply the theorem to that function on the interval $[-4, 3]$. This is not so for this f, so there is no contradiction.

5 Problem 5

5.1 Part a

We are given

$$W'(t) = \frac{1}{25}[W(t) - 300], \tag{21}$$

so $W'(0) = \frac{1400-300}{25} = 44$, and the equation for the line tangent to the solution curve for the initial value problem, in (t, w) coordinates, at $t = 0$ is $w = W(0) + W'(0)(t - 9) = 1400 + 44t$. When $t = \frac{1}{4}$, this gives $w = 1400 + 11 = 1411$, so the approximate amount of solid waste at the end of the first three months of 2010 is 1411 tons.

5.2 Part b

Differentiating both sides of (21), we see that

$$\frac{d^2W}{dt^2} = \frac{1}{25} \cdot \frac{d}{dt}\left[W(t) - 300\right] \tag{22}$$

$$= \frac{1}{25} W'(t), \text{ which, again by (21), is} \tag{23}$$

$$\frac{d^2W}{dt^2} = \frac{1}{625}\left[W(t) - 300\right]. \tag{24}$$

Thus, $W''(0) = \frac{44}{25} > 0$, and, $W''(t)$ being continuous, the solution curve must be concave upward near $t = 0$. This means that the tangent line to the curve at $t = 0$ lies below the curve, so the estimate given in Part a is an underestimate.

From equation (21), we see that either $W(t) \equiv 300$ or

$$\frac{W'(t)}{W(t) - 300} = \frac{1}{25}, \tag{25}$$

which means that

$$\int_0^t \frac{W'(\tau)}{W(\tau) - 300}\, d\tau = \int_0^t \frac{1}{25}\, d\tau. \tag{26}$$

We discard the constant solution because it doesn't satisfy the initial conditon, and we carry out the integration. Thus

$$\ln|W(\tau) - 300|\Big|_0^t = \frac{\tau}{25}\Big|_0^t \tag{27}$$

Now $W(0) = 1400$, so $W(0) - 300 > 0$ and we may write

$$\ln(W(t) - 300) - \ln(1400 - 300) = \frac{t}{25}, \text{ or} \tag{28}$$

$$\ln\left[\frac{W(t) - 300}{1100}\right] = \frac{t}{25}. \tag{29}$$

This leads to

$$W(t) = 300 + 1100e^{t/25}. \tag{30}$$

6 Problem 6

6.1 Part a

The first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$ are

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

It follows that the first four non-zero terms of the Taylor series for $\sin x^2$ are

$$x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!}.$$

6.2 Part b

The first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$ are

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}.$$

Thus, the first four nonzero terms of the Taylor series for $\sin x^2 + \cos x$ about $x = 0$ are

$$\left(x^2 - \frac{x^6}{3!}\right) + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right) \tag{31}$$

$$= 1 + \left(1 - \frac{1}{2!}\right)x^2 + \frac{1}{4!}x^4 - \left(\frac{1}{3!} + \frac{1}{6!}\right)x^6 \tag{32}$$

$$= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{121}{720}x^6. \tag{33}$$

6.3 Part c

The coefficient of x^n in the Taylor series about $x = 0$ for a function g is $\dfrac{g^{(n)}(0)}{n!}$. From Part b, above, we see that

$$\frac{f^{(6)}(0)}{6!} = -\frac{121}{720}, \tag{34}$$

and, 6! being 720, it follows that $f^{(6)}(0) = -121$.

6.4 Part d

By Taylor's Theorem with Lagrange Remainder, $P_4(x)$, the Taylor polynomial of degree 4 in powers of x for $f(x)$, approximates $f(x)$ to within $\dfrac{M}{5!}|x|^5$, provided that M is chosen so that $|f^{(5)}(t)| \le M$ on $[0, x]$. From the graph, we see that $|f^{(5)}(t)| \le 40$ on any interval of the form $[0, x]$, where, say, $\frac{1}{4} \le x \le \frac{11}{40}$. Consequently,

$$\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| \le \frac{40}{120} \cdot \left(\frac{1}{4}\right)^5 = \frac{1}{3} \cdot \frac{1}{1024} < \frac{1}{3} \cdot \frac{1}{1000} = \frac{1}{3000}. \tag{35}$$