AP Calculus 2015 BC FRQ Solutions

Louis A. Talman, Ph.D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

May 25, 2017

1 Problem 1

1.1 Part a

The rate of change, in cubic feet per hour, of the volume of water in the pipe at time t is $R(t) - D(t)$, where

$$R(t) = 20\sin\left(\frac{t^2}{35}\right), \tag{1}$$

and

$$D(t) = -0.04t^3 + 0.4t^2 + 0.96t. \tag{2}$$

Thus,

$$v(t) = R(t) - D(t) \tag{3}$$

$$= 0.04t^3 - 0.4t^2 - 0.96t + 20\sin\left(\frac{t^2}{35}\right). \tag{4}$$

Integrating numerically, we find that the amount of water, in cubic feet, that flows into the tank in the eight-hour time interval $0 \le t \le 8$ is

$$\int_0^8 R(t)\,dt = \int_0^8 20\sin\left(\frac{t^2}{35}\right)\,dt \sim 76.57035. \tag{5}$$

1.2 Part b

The rate of change of volume of water in the pipe at time $t = 3$ is

$$v(3) = -0.31363 \text{ cubic feet per hour.} \tag{6}$$

This is negative, and $v'(t)$ is continuous, so the volume of water in the tank is decreasing when t is near 3.

Note: We have phrased our answer this way because very few authors give a definition for the phrase "increasing at $x = a$ Instead, the usual definition is for "increasing on an interval".

1.3 Part c

From a plot, we see that $v(t)$ is zero at a value $t = t_0$ near $t = 3$, negative immediately to the left of this zero, and positive to the right. It follows from the First Derivative Test that this zero of $v(t)$ gives a relative minimum for the amount of water in the pipe. Neither endpoint can be a global minimum, because $v(t)$ is negative to the right of $t = 0$ and $v(t)$ is positive to the left of $t = 8$. Solving numerically, we find that $t_0 \sim 3.27155$ hours. Hence, volume is minimal at about $t = 3.27155$ hours.

1.4 Part d

There are intially 30 cubic feet of water in the pipe, and the pipe can hold 50 cubic feet of water before overflowing, so, using what we have seen in Part a, above, we can determine the time of overflow by solving the equation

$$30 + \int_0^t v(\tau)\, d\tau = 50. \tag{7}$$

for t.

Note: Solution of this equation is not required. However, numerical methods give $t \sim 8.23202$, which lies outside the domain we were given. We conclude that the pipe doesn't overflow during the specified interval.

2 Problem 2

2.1 Part a

If $\mathbf{v}(t) = \langle x'(t), y'(t)\rangle = \langle \cos t^2, e^{0.5t}\rangle$ and the particle is at $(3, 5)$ when $t = 1$, then the particle's position vector $\mathbf{r}(t)$ at time t is given by

$$\mathbf{r}(t) = \langle x(t), y(t)\rangle = \langle 3, 5\rangle + \int_1^t \mathbf{v}(\tau)\, d\tau \tag{8}$$

Thus, $x(2) = 3 + \displaystyle\int_1^2 \cos \tau^2\, d\tau$. Integrating numerically, we obtain $x(2) \sim 2.55694$.

2.2 Part b

The slope of the particle's curve at time t is

$$\frac{y'(t)}{x'(t)} = \frac{e^{0.5t}}{\cos t^2}. \tag{9}$$

We set this quotient equal to 2 and solve numerically. We obtain $t \sim 0.84016$.

2.3 Part c

The particle's speed is $\sqrt{[x'(t)]^2 + [y'(t)]^2}$, and this is 3 when $\cos^2 t^2 + e^t = 9$. Solving numericallly yields $t \sim 2.19590$.

2.4 Part d

The total distance traveled by the particle from time $t = 0$ to time $t = 1$ is the integral of speed over that time interval: $\int_0^1 \sqrt{\cos^2 \tau^2 + e^\tau}\, d\tau$. Numerical integration gives distance traveled at time $t = 1$ as approximately 1.59461.

3 Problem 3

3.1 Part a

Using data from the table, we find that the approximate value of the derivative $v'(16)$ is

$$v'(1.8) \sim \frac{v(20) - v(12)}{20 - 12} = \frac{240 - 200}{20 - 12} = 5 \text{ meters/min.} \tag{10}$$

3.2 Part b

The definite integral $\int_0^{40} |v(t)|\, dt$ gives the actual distance, in meters, that Johanna traveled in the time interval $0 \leq t \leq 40$. The right Riemann sum approximation of this distance is $12 \cdot |200| + 8 \cdot |240| + 4 \cdot |-220| + 16 \cdot |150| = 7600$ m.

3.3 Part c

If Bob's velocity at time t is $B(t) = t^3 - 6t^2 + 300$ meters per minute, then his acceleration at time t is $B'(t) = 3t^2 - 12t$. Thus his acceleration at time $t = 5$ is $v'(5) = 15$ meters per minute per minute.

3.4 Part d

Bob's average velocity over the interval $[0, 10]$ is

$$\frac{1}{10}\int_0^{10} B(\tau)\,d\tau = \frac{1}{10}\int_0^{10} \left(\tau^3 - 6\tau^2 + 300\right)\,d\tau \tag{11}$$

$$= \frac{1}{10}\left[\frac{\tau^4}{4} - 2\tau^3 + 300\tau\right]\Bigg|_0^{10} = 350 \text{ meters per minute.} \tag{12}$$

4 Problem 4

4.1 Part a

See Figure 1.

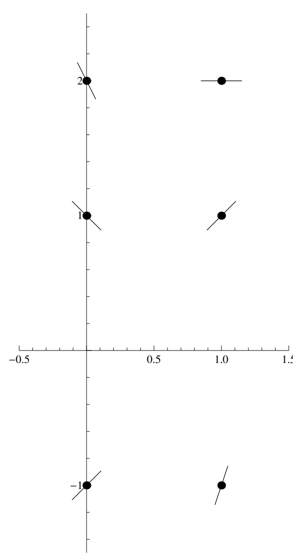

Figure 1: Problem 4, Part a: Slope Field

4.2 Part b

If $y' = 2x - y$, then $y'' = 2 - y' = 2 - 2x + y$. In the second quadrant, $x < 0$ and $y > 0$, so $y'' > 0$ throughout that quadrant. It follows that any solution curve that passes through any part of the second quadrant must be concave upward there.

4.3 Part c

If f is a solution of the differential equation $y' = 2x - y$ for which $f(2) = 3$, then $f'(2) = 2 \cdot 2 - 3 = 1$, so f doesn't have a critical point at $x = 2$. Thus, f has neither a maximum nor a minimum at $x = 2$.

4.4 Part d

If $y = mx + b$ is a solution of the differential equation $y' = 2x - y$, then, on the one hand, direct differentiation of the solution shows that we must have $y' = m$. On the other hand, the differential equation implies that we must also have $y' = 2x - (mx + b) = (2 - m)x - b$. Consequently, $m = (2 - m)x - b$, which we can rewrite and $(2 - m)x - (m + b) \equiv 0$. It follows that $m = 2$ and $b = -2$.

4.5 Remark

The differential equation $y' = 2x - y$ can be rewritten as $y' + y = 2x$, which has the form $y' + p(x)y = q(x)$. Such a differential equation can always be solved by choosing a function $P(x)$ such that $P'(x) = p(x)$ and multiplying the differential equation through by $e^{P(x)}$. After doing so, we find that the new equation can always be put in the form

$$\frac{d}{dx}\left[ye^{P(x)}\right] = q(x)e^{P(x)}. \tag{13}$$

All that then remains is to carry out the integration on the right side.

Applying this technique to the current equation, we obtain

$$ky'e^x + ye^x = 2xe^x, \text{ or} \tag{14}$$

$$\frac{d}{dx}\left[ye^x\right] = 2xe^x \tag{15}$$

An integration by parts on the right side now gives

$$ye^x = 2xe^x - 2e^x + C, \text{ or} \tag{16}$$

$$y = 2x - 2 + Ce^{-x}. \tag{17}$$

This is the general solution to the differential equation.

The initial value problem of Part c can now be solved by substituting 3 for y and 2 for x, to learn that we must take $C = e^2$. This gives the solution $y = 2x - 2 + e^{2-x}$. We can use this solution to confirm the conclusions obtained above for Part c of the problem.

We can also obtain the linear solution required in Part d by taking $C = 0$ in the general solution.

5 Problem 5

5.1 Part a

If $k = 3$,

$$f(x) = \frac{1}{x^2 - kx}, \text{ and} \tag{18}$$

$$f'(x) = \frac{k - 2x}{(x^2 - kx)^2}, \tag{19}$$

then

$$f(4) = \frac{1}{16 - 12} = 4 \text{ and} \tag{20}$$

$$f'(4) = \frac{3 - 8}{(16 - 12)^2} = -\frac{5}{16}. \tag{21}$$

An equation for the desired tangent line is therefore

$$y = \frac{1}{4} - \frac{5}{16}(x - 4). \tag{22}$$

5.2 Part b

Putting $k = 4$, we have

$$f'(x) = \frac{4 - 2x}{(x^2 - 4x)^2}, \tag{23}$$

so $f'(2) = 0$. The denominator of the derivative is never negative, so the sign of the derivative is the same as the sign of its numerator (except of course, when $x = 4$). Thus $f'(x) > 0$ when x is in the region immediately to the left of $x = 2$, and $f'(x) < 0$ when x is in the region immediately to the right of $x = 2$. By the First Derivative Test, f has a relative maximum at $x = 2$. (Note: It is also possible to draw this conclusion by way of the Second Derivative Test, but doing so requires more computation than the First Derivative Test requires.)

5.3 Part c

If $f'(-5) = 0$ then

$$\frac{k - 2 \cdot (-5)}{[(-5)^2 - k \cdot (-5)]^2} = 0, \tag{24}$$

Thus, $k = -10$.

5.4 Part d

If

$$\frac{A}{x} + \frac{B}{x-6} = \frac{A(x-6) + Bx}{x^2 - 6x} = \frac{1}{x^2 - 6x}, \tag{25}$$

then equating coefficients of like powers of x in the numerators gives $A + B = 0$ and $-6A = 1$. Thus, $A = -1/6$ and $B = 1/6$. It follows that

$$\frac{1}{x^2 - 6x} = \frac{1}{6}\left[\frac{1}{x-6} - \frac{1}{x}\right], \text{ so that} \tag{26}$$

$$\int \frac{dx}{x^2 - 6x} = \frac{1}{6}\int \left[\frac{1}{x-6} - \frac{1}{x}\right] dx \tag{27}$$

$$= \frac{1}{6} \ln \left|\frac{x-6}{x}\right| + C \tag{28}$$

for some constant C.

6 Problem 6

6.1 Part a

We have

$$\lim_{n\to\infty} \left[\frac{3^n |x|^{n+1}}{n+1} \cdot \frac{n}{3^{n-1}|x|^n}\right] = 3|x| \lim_{n\to\infty} \frac{1}{1 + (1/n)} = 3|x|. \tag{29}$$

By the Ratio Test, the series converges when this limit is less than one; it diverges when this limit is greater than one. Hence, the radius of convergence is $1/3$.

6.2 Part b

We can obtain the first four nonzero terms of the Maclaurin series for f' by differentiating the original series term by term. The radius of convergence of the derived power series is the same as that of the original power series—which is $1/3$ in this case. The derivative of the n-th term of the given series here is $(-3)^{n-1}x^{n-1}$, so the first four nonzero terms of the derived series are

$$1 - 3x + 9x^2 - 27x^3.$$

This is the geometric series with common ratio $-3x$, so $f'(x) = \dfrac{1}{1+3x}$ when $|x| < 1/3$.

6.3 Part c

We have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \text{ Higher Order Terms.} \tag{30}$$

We obtain the Maclaurin polyomial of degree three for $e^x f(x)$ by multiplying the series for e^x and the series for $f(x)$ and retaining the low order terms. This gives

$$e^x f(x) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \text{ H. O. T.}\right)\left(x - \frac{3x^2}{2} + 3x^3 + \text{ H. O. T.}\right) \tag{31}$$

$$= x - \frac{x^2}{2} + 2x^3 - \frac{13x^4}{3} + \text{ H. O. T.} \tag{32}$$

So the required polynomial is

$$P(x) = x - \frac{x^2}{2} + 2x^3. \tag{33}$$