AP Calculus 2016 BC FRQ Solutions

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1 Problem 1

1.1 Part a

We estimate $R'(2)$ as

$$R'(2) \sim \frac{R(3) - R(1)}{3 - 1} = \frac{950 - 1190}{2} = -120 \text{ liters/hour}^2. \tag{1}$$

1.2 Part b

To estimate the total amount of water removed from the tank during the time interval $[0, 8]$ with a left Riemann sum having four sub-intervals, we may write

$$R(0) \cdot [1 - 0] + R(1) \cdot [3 - 1] + R(3) \cdot [6 - 3] + R(6) \cdot [8 - 6] = \tag{2}$$

$$1340 \cdot 1 + 1190 \cdot (3 - 1) + 950 \cdot (6 - 3) + 740 \cdot (8 - 6) = 8050. \tag{3}$$

The function R is decreasing, so the left-hand endpoint of each subinterval gives the maximum value of R on that subinterval. Thus, a left-hand Riemann sum gives an overestimate of the integral.

1.3 Part c

The total amount of water in the tank at time t is

$$50000 + \int_0^t [W(\tau) - R(\tau)]\, d\tau = 50000 + 2000 \int_0^t e^{-\tau^2/20}\, d\tau - \int_0^t R(\tau)\, d\tau, \tag{4}$$

or, when $t = 8$,

$$\sim 50000 + 2000 \int_0^8 e^{-\tau^2/20}\, d\tau - 8050. \tag{5}$$

Thus, after carrying out the remaining integration numerically, we find that the amount of water in the tank when $t = 8$ is approximately 49786.19532 liters. To the nearest liter, this is 49786 liters.

1.4 Part d

We consider the function $F(t) = W(t) - R(t)$. The functions W and R are both continuous on the interval $[0, 8]$, so the function F is also continuous on that interval. We have $F(0) = 660$, while $F(8) \sim -618.5$ to the nearest tenth. Thus, $F(0) > 0$ while $F(8) < 0$, and, by the Intermediate Value Propery of continouus functions, there is a point ξ somewhere in the interval $(0, 8)$ for which $F(\xi) = 0$. For this ξ we have $W(\xi) - R(\xi)$, so the answer to the question is "Yes."

2 Problem 2

2.1 Part a

The graph gives $y(3) = -1/2$. We obtain $x(3)$ from

$$x(t) = x(0) + \int_0^t x'(\tau)\, d\tau, \text{ whence} \tag{6}$$

$$x(3) = 5 + \int_0^t \left[\tau^2 + \sin 3\tau^2\right] d\tau \sim 14.37704, \tag{7}$$

via numerical integration. Thus, the position of the particle at time $t = 3$ is approximately $(14.37704, -0.5)$.

2.2 Part b

The slope of the line tangent to the curve $\big(x(t), y(t)\big)$ at the point where $t = 3$ is

$$\left.\frac{y'(t)}{x'(t)}\right|_{t=3} = \left.\frac{y'(t)}{t^2 + \sin 3t^2}\right|_{t=3} \tag{8}$$

$$= \frac{1/2}{9 + \sin 27} \sim 0.05022, \tag{9}$$

where we have read $y'(3)$ from the given graph.

2.3 Part c

The speed $\sigma(t)$, of the particle at time t is

$$\sigma(t) = \sqrt{[x'(t)]^2 + [y'(t)]^2}, \tag{10}$$

so speed at time $t = 3$ is

$$\sigma(3) = \sqrt{[9 + \sin 27]^2 + (1/4)} \sim 9.96892. \tag{11}$$

2.4 Part d

The total distance, s traveled over the time interval $[0, 2]$ is

$$s = \int_0^2 \sigma(\tau)\, d\tau = \int_0^2 \sqrt{[x'(\tau)]^2 + [y'(\tau]^2}\, d\tau \tag{12}$$

$$= \int_0^1 \sqrt{[\tau^2 + \sin 3\tau^2]^2 + (-2)^2}\, d\tau + \int_1^2 [\tau^2 + \sin 3\tau^2]\, d\tau \tag{13}$$

$$\sim 2.23787 + 2.11200 = 4.34987. \tag{14}$$

Note: It must be noted, in the course of these numerical integrations, that the second integral is over the interval $[1, 2]$, where the graph gives $y'(\tau) \equiv 0$. For $1 \leq t \leq 2$, we then have

$$|\sin 3t^2| \leq 1 \leq t^2, \tag{15}$$

so that $t^2 + \sin 3t^2 \geq 0$ when $1 \leq t \leq 2$. This means that, on the interval $[1, 2]$, we may replace $\sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{[t^2 + \sin 3t^2]^2 + 0}$ with $t^2 + \sin 3t^2$—as we have done. In my opinion, a solution that fails to make this observation explicitly is incomplete.

3 Problem 3

For a graph of g, see Figure 1.

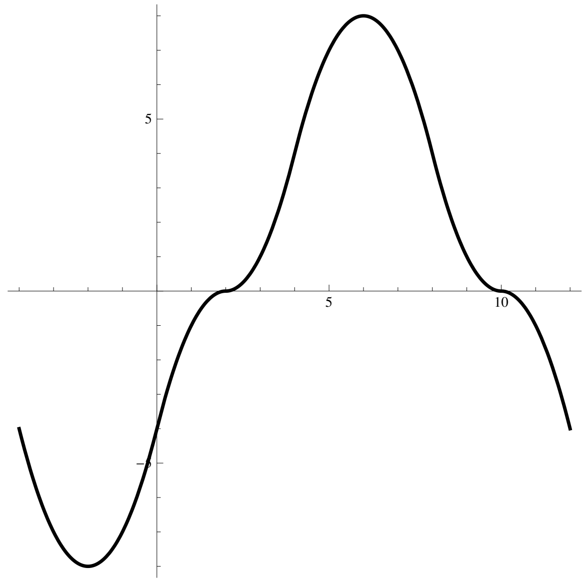

Figure 1: Problem 3, Graph of g

3.1 Part a

If $g(x) = \int_2^x f(t)\,dt$ then, by the Fundamental Theorem of Calculus, $g'(x) = f(x)$. While $g'(10) = -$, we see that $g'(x)$ is negative for all values of x in some punctured neighborhood of $x = 10$. Thus, by the First Derivative Test, g has neither a relative minimum nor a relative maximum at $x = 10$.

3.2 Part b

Arguing again from the given graph, which is that of g', we see that g' is increasing on an interval just to the left of $x = 4$ but decreasing on an interval just to the right of $x = 4$. Thus, g has an inflection point where $x = 4$. (In fact, g is concave upward immediately to the left of $x = 4$ and concave downward immediately to the right of $x = 4$.)

3.3 Part c

The absolute minimum value must occur either at an endpoint of the interval or at a point where $g'(x)$ undergoes a sign change from negative to positive as x increases. The only points that qualify are $x = -4$, $x = -2$, and $x = 12$. Summing the areas of the appropriate triangles (with appropriate signs), we see that $g(-4) = -4$, $g(-2) = -9$, and $g(12) = -4$. Thus, g has its absolute minimum at $x = -8$.

Similar reasoning shows that the absolute maximum of $g(x)$ can only be at $x = -4$, $x = 6$, or $x = 12$. But this makes $g(6) = 8$ the absolute maximum. (We evaluated the other two possibilities in the preceding paragraph.)

3.4 Part d

On any interval of the form $[x, 2]$, with $-4 \leq x < 2$, the area between the curve $y = f(x)$ and the x-axis, and lying above the x-axis, exceeds that below the x-axis. Thus guarantees that, for such x, $g(x) < 0$.

On the other hand, on any interval of the form $[2, x]$, with $x > 2$, the area of the region bounded by f and below the x-axis doesn't exceed that of the region above the x-axis unless $x > 10$. This means that $g(x) \geq 0$ for $x \leq x \leq 10$, and $g(x) < 0$ when $10 < x$.

The desired intervals are $[-4, 2]$ and $[10, 12]$.

4 Problem 4

4.1 Solution 1

4.1.1 Part a

From the equation $\dfrac{dy}{dx} = x^2 - \dfrac{1}{2}y$, we have

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) && (16)\\
&= \frac{d}{dx}\left(x^2 - \frac{1}{2}y\right) && (17)\\
&= 2x - \frac{1}{2}\frac{dy}{dx} && (18)\\
&= 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right). && (19)
\end{aligned}$$

4.1.2 Part b

At the point $(-2, 8)$, we have

$$\left.\frac{dy}{dx}\right|_{(-2,8)} = (-2)^2 - \frac{1}{2} \cdot 8 = 0, \text{ and} \tag{20}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(-2,8)} = 2(-2) - \frac{1}{2} \cdot 0 = -4 < 0. \tag{21}$$

By the Second Derivative Test, this curve has a local maximum at $(-2, 8)$.

4.1.3 Part c

By the continuity of the solution of a differential equation, we have

$$\lim_{x \to -1} [g(x) - 2] = g(-1) - 2 = 0. \tag{22}$$

Also,

$$\lim_{x \to -1} 3(x+1)^2 = 0. \tag{23}$$

We may therefore attempt to evaluate the limit by l'Hôpital's Rule. This gives

$$\lim_{x \to -1} \frac{g(x) - 2}{3(x+1)^2} = \lim_{x \to -1} \frac{g'(x)}{6(x+1)}, \tag{24}$$

provided the latter limit exists.

But

$$\lim_{x \to -1} \frac{g'(x)}{6(x+1)} = \lim_{x \to -1} \frac{x^2 - g(x)/2}{6(x+1)} \tag{25}$$

$$= \lim_{x \to -1} \frac{2x^2 - g(x)}{12(x+1)}. \tag{26}$$

Here, $2x^2 - g(x) = [2x^2 - 2] + [2 - g(x)] \to 0$ and $12(x+1) \to 0$ as $x \to -1$, so we may attempt l'Hôpital's Rule again. This gives

$$\lim_{x \to -1} \frac{2x^2 - g(x)}{12(x+1)} = \lim_{x \to -1} \frac{4x - g'(x)}{12} \tag{27}$$

$$= \lim_{x \to -1} \frac{4x - [x^2 - g(x)/2]}{12} - \frac{-4 - [1 - g(-1)/2]}{12} = -\frac{1}{3}. \tag{28}$$

We conclude that the required limit is $-1/3$

4.1.4 Part d

The Euler's method recursion, with step-size $1/2$, to the solution $y = h(x)$ of the initial-value problem

$$y' = x^2 - \frac{y}{2}; \tag{29}$$

$$y(0) = 2; \tag{30}$$

is

$$x_0 = 0; \tag{31}$$

$$y_0 = 2; \tag{32}$$

$$x_k = x_{k-1} + \frac{1}{2}; \tag{33}$$

$$y_k = y_{k-1} + \frac{1}{2}\left(x_{k-1}^2 - \frac{1}{2}y_{k-1}\right). \tag{34}$$

Therefore,

$$x_1 = x_0 + \frac{1}{2} = \frac{1}{2}, \tag{35}$$

$$y_1 = y_0 + \frac{1}{2}\left(x_0^2 - \frac{1}{2}y_0\right) = 2 + \frac{1}{2}\left(0^2 - \frac{1}{2}\cdot 2\right) = \frac{3}{2}; \tag{36}$$

$$x_2 = x_1 + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1, \tag{37}$$

$$y_2 = y_1 + \frac{1}{2}\left(x_1^2 - \frac{1}{2}y_1\right) = \frac{3}{2} + \frac{1}{2}\left(\frac{1}{4} - \frac{3}{4}\right) = \frac{5}{4}; \tag{38}$$

The required Euler's method approximation to $h(1)$ is thus $h(1) \sim \frac{5}{4}$.

4.2 Solution 2

4.2.1 Part a

We have

$$\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right), \tag{39}$$

as in 4.1.1, above. Note that we can continue this procedure, seeing thus that y possesses derivatives of all orders throughout its domain.

4.2.2 Part b

There is a local maximum at $(-2, 8)$, as in 4.1.2 above.

4.2.3 Part c

We now know that

$$g(-1) = 2; \tag{40}$$

$$g'(-1) = (-1)^2 - \frac{1}{2}g(-1) = 0; \tag{41}$$

$$g''(-1) = 2(-1) - \frac{1}{2}g'(-1) = -2. \tag{42}$$

Thus, by Taylor's Theorem and our earlier observation regarding higher order derivatives, there is a function r, defined on some open interval centered at $x = -1$, continuous at $x = -1$, and such that

$$g(x) = g(-1) + g'(-1)(x+1) + \frac{1}{2}g''(-1)(x+1)^2 + r(x)(x+1)^3 \tag{43}$$

$$= 2 - (x+1)^2 + r(x)(x+1)^3. \tag{44}$$

So

$$\lim_{x \to -1} \frac{g(x) - 2}{3(x+1)^2} = \lim_{x \to -1} \frac{[2 - (x+1)^2 + r(x)(x+1)^3] - 2}{(x+1)^2} \tag{45}$$

$$= \lim_{x \to -1} \frac{[r(x)(x+1) - 1]\cancel{(x+1)^2}}{3\cancel{(x+1)^2}} \tag{46}$$

$$= \frac{1}{3} \lim_{x \to -1} [r(x)(x+1) - 1] = -\frac{1}{3}. \tag{47}$$

4.2.4 Part d

$h(1) \sim 5/4$, as in 4.1.4, above.

4.3 Solution 3

4.3.1 Part a

We have

$$\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right), \tag{48}$$

as in 4.1.1, above. Note that we can continue this procedure, seeing thus that y possesses derivatives of all orders throughout its domain.

4.3.2 Part b

If $y = f(x)$ gives a solution to the initial value problem

$$y' = x^2 - \frac{y}{2}; \tag{49}$$

$$y(-2) = 8, \tag{50}$$

then

$$f'(x) + \frac{1}{2}f(x) = x^2; \tag{51}$$

$$e^{x/2}f'(x) + \frac{1}{2}e^{x/2}f(x) = x^2e^{x/2}; \tag{52}$$

$$\frac{d}{dx}\left[e^{x/2}f(x)\right] = x^2e^{x/2}.. \tag{53}$$

so that

$$\int_{-2}^{x}\frac{d}{d\xi}\left[e^{\xi/2}f(\xi)\right]\,d\xi = \int_{-2}^{x}\xi^2e^{\xi/2}\,d\xi. \tag{54}$$

Integrating by parts twice in succession we find that

$$\int x^2e^{x/2}\,dx = 2x^2e^{x/2} - 8xe^{x/2} + 16e^{x/2}. \tag{55}$$

From (53) and (55). it now follows that we can rewrite the equation

$$\int_{-2}^{x}\frac{d}{d\xi}\left[e^{\xi/2}f(\xi)\right]\,d\xi = \int_{-2}^{x}\xi^2e^{\xi/2}\,d\xi \tag{56}$$

as

$$e^{\xi/2}f(\xi)\Big|_{-2}^{x} = \left(2\xi^2e^{\xi/2} - 8\xi e^{\xi/2} + 16e^{\xi/2}\right)\Big|_{-2}^{x}, \tag{57}$$

whence

$$e^{x/2}f(x) - e^{-1}f(-2) = \left(2x^2e^{x/2} - 8xe^{x/2} + 16e^{x/2}\right) - 40e^{-1}. \tag{58}$$

But $f(-2) = 8$, so solving the latter equation for $f(x)$ yields

$$f(x) = 2x^2 - 8x + 16 - 32e^{-1-x/2}. \tag{59}$$

If follows, now, that

$$f'(x) = 4x - 8 + 16e^{-1-x/2}, \text{ and} \tag{60}$$

$$f''(x) = 4 - 8e^{-1-x/2}. \tag{61}$$

Thus, $f'(-2) = 0$ and $f''(-2) = -4$. By the Second Derivative Test, f has a local maximum at $x = -2$.

4.3.3 Part c

Repeating the solution of the initial value problem above with the initial value $g(-1) = 2$, we find that

$$g(x) = 2x^2 - 8x + 16 - 24e^{-1-x/2}. \tag{62}$$

Thus,

$$\lim_{x\to-1} \frac{g(x) - 1}{3(x+1)^2} = \lim_{x\to-1} \frac{2x^2 - 8x + 14 - 24e^{-1-x/2}}{3(x+1)^2}. \tag{63}$$

It is easily checked that l'Hôpital's rule applies twice in succession, giving

$$\lim_{x\to-1} \frac{2x^2 - 8x + 14 - 24e^{-1-x/2}}{3(x+1)^2} = \lim_{x\to-1} \frac{4x - 8 + 12e^{-1-x/2}}{6(x+1)} \tag{64}$$

$$= \lim_{x\to-1} \frac{4 - 6e^{-1-x/2}}{6} = -\frac{1}{3}. \tag{65}$$

4.3.4 Part d

$h(1) \sim 5/4$, as in 4.1.4, above.

5 Problem 5

5.1 Part a

The average value of the funnel's radius is

$$\frac{1}{10-0}\int_0^{10}\frac{3+h^2}{20}\,dh = \frac{3}{200}\int_0^{10}dh + \frac{1}{200}\int_0^{10}h^2\,dh \tag{66}$$

$$= \frac{3}{200}\cdot 10 + \frac{1}{200}\cdot\frac{1000}{3} \tag{67}$$

$$= \frac{3}{20} + \frac{5}{3} = \frac{109}{60}. \tag{68}$$

The average value of the radius is $\frac{109}{60}$ inches.

5.2 Part b

The volume, V, of the funnel is

$$V = \pi\int_0^{10}[r(h)]^2\;dh \tag{69}$$

$$= \frac{\pi}{400}\int_0^{10}(3+h^2)^2\,dh \tag{70}$$

$$= \frac{\pi}{400}\int_0^{10}\left(9+6h^2+h^4\right)\;dh \tag{71}$$

$$= \frac{\pi}{400}\left(9h+2h^3+\frac{1}{5}h^5\right)\Big|_0^{10} \tag{72}$$

$$= \frac{\pi}{400}(90+2000+20000) = \frac{2209}{40}\pi\text{ in}^3. \tag{73}$$

5.3 Part c

The radius $r(t)$ and the height $y(t)$ are related by the equation

$$r(t) = \frac{1}{20}\left(3+[y(t)]^2\right), \tag{74}$$

so that

$$r'(t) = \frac{1}{10}y(t)y'(t), \tag{75}$$

or

$$y'(t) = 10\frac{r'(t)}{y(t)}. \tag{76}$$

Thus, at the instant when $r'(t_0) = -1/5$ in/sec and $y(t_0) = 3$ in, the height is changing at the rate

$$y'(t_0) = \frac{10}{3} \cdot \left(-\frac{1}{5}\right) = -\frac{2}{2} \text{ in/sec}. \tag{77}$$

6 Problem 6

6.1 Part a

We have

$$\frac{f(1)}{0!} = 1, \tag{78}$$

$$\frac{f'(1)}{1!} = -\frac{1}{2}, \tag{79}$$

$$\frac{f''(1)}{2!} = (-1)^2 \cdot \frac{1!}{2 \cdot 2^3} = \frac{1}{8}, \tag{80}$$

$$\frac{f^{(3)}(-1)}{3!} = (-1)^3 \frac{2}{6 \cdot 2^3} = -\frac{1}{24}, \tag{81}$$

$$\vdots \tag{82}$$

$$\frac{f^{(n)}(-1)}{n!} = (-1)^n \frac{(n-1)!}{n! \cdot 2^n} = \frac{(-1)^n}{n2^n}, \tag{83}$$

whence

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n}(x-1)^n \tag{84}$$

$$= 1 - \frac{1}{2}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{24}(x-1)^3 + \cdots + \frac{(-1)^n}{n2^n}(x-1)^n + \cdots . \tag{85}$$

6.2 Part b

If the radius of convergence for this series is 2, then, being centered at $x = 1$, it converges for all values of x in the interval $(1 - 2, 1 + 2) = (-1, 3)$, and it remains to check the endpoints of this interval for convergence.

If $x = 3$, the series becomes

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n} 2^n = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n}. \tag{86}$$

Except for the first term, this is the negative of the alternating harmonic series, which converges.

If $x = -1$, the series becomes

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n} (-2)^n = 1 + \sum_{n=1}^{\infty} \frac{1}{n}, \tag{87}$$

and (again, except for the first term) this is the harmonic series—which diverges.

The interval of convergence is thus the interval $(-1, 3]$.

6.3 Part c

We have

$$f(1.2) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n} (0.2)^n \tag{88}$$

$$= 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.04) - \frac{1}{24}(0.008) + \cdots \tag{89}$$

Taking the first three terms of this series gives

$$f(1.2) \sim 1 - 0.1 + 0.005 = 0.905. \tag{90}$$

6.4 Part d

The magnitude of the error that results from truncating an alternating series is bounded by the the magnitude of the first truncated term, and in this case, that is

$$\frac{0.008}{24} = \frac{1}{3000} < \frac{1}{1000}. \tag{91}$$

Note: The series is easily summed. From (84), we have $f(1) = 1$ and, for x inside the interval of convergence,

$$f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}(x-1)^{n-1} \tag{92}$$

$$= -\frac{1}{2}\sum_{n=1}^{\infty} \left(-\frac{x-1}{2}\right)^{n-1}. \tag{93}$$

This is a geometric series with common ratio $-\frac{x-1}{2}$. It converges when $-1 < x < 3$, and gives in that interval

$$f'(x) = -\frac{1}{2}\left(\frac{1}{1+\frac{(x-1)}{2}}\right) \tag{94}$$

$$= -\frac{1}{x+1} \tag{95}$$

Thus, $f(x) = C - \ln(1+x)$ for some constant C. But $f(1) = 1$ so $C = 1 + \ln 2$ and

$$f(x) = 1 + \ln 2 - \ln(1+x). \tag{96}$$

From this, we see easily in Part d that $f(1.2) \sim 0.90469$, leading to another verification that the error in the approximation of Part c must be less than 0.001.