AP Calculus 2017 BC FRQ Solutions

Louis A. Talman, Ph.D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

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1 Problem 1

1.1 Part a

The approximation with a left-hand sum using the intervals given is

$$50.3 \cdot 2 + 14.4 \cdot 3 + 6.5 \cdot 5 = 176.3. \qquad (1)$$

1.2 Part b

We are given that the area of cross-sections decreases as h increases, so the cross-section at the left-hand endpoint of each interval has maximal area for the cross-sections associated with that interval. Thus, the left-hand sum overestimates the volume of the tank. The required approximate volume is 176.3 cubic feet.

1.3 Part c

The volume, in cubic feet, of the tank is $\displaystyle\int_0^{10} \frac{50.3}{e^{0.2h} + h}\,dh$. Numerical integration gives an approximate volume of 101.325 cubic feet for the tank.

1.4 Part d

Let $H(t)$ denote the height of water in the tank at time t. Then the volume, in cubic feet, of water in the tank at time t is

$$V(t) = \int_0^{H(t)} \frac{50.3}{e^{0.2h} + h}\, dh \tag{2}$$

Thus, by the Fundamental Theorem of Calculus and the Chain Rule,

$$V'(t) = \frac{50.3}{e^{0.2H(t)} + H(t)} H'(t), \text{ so that} \tag{3}$$

$$V'(t_0) = 50.3 + \frac{50.3}{e^{0.2H(t_0)} + H(t_0)} H'(t_0), \tag{4}$$

where t_0 is the instant when the depth of water in the tank is five feet. Thus,

$$V'(t_0) = \frac{50.3}{e+5} \cdot 0.26 \sim 1.694 \text{ cubic feet per minute.} \tag{5}$$

2 Problem 2

2.1 Part a

The required area, A_R is given by

$$A_R = \frac{1}{2}\int_0^{\pi/2} [f(\theta)]^2\, d\theta \tag{6}$$

$$= \frac{1}{2}\int_0^{\pi/2} [1 + \sin\theta\cos 2\theta]^2\, d\theta \tag{7}$$

Although the necessary anti-derivative can be found by elementary techniques, the calculation requires a significant amount of time; and this is a calculator-active problem. Numerical integration gives $A_R \sim 0.648$.

Note: For the curious:

$$\int [1 + \sin\theta\cos 2\theta]^2\, d\theta = \int [1 + 2\sin\theta\cos 2\theta + \sin^2\theta\cos^2 2\theta]\, d\theta. \tag{8}$$

We break this into pieces. The first is easy. The second is

$$\int \sin\theta\cos 2\theta\, d\theta = \int \sin\theta\left(2\cos^2\theta - 1\right)\, d\theta \tag{9}$$

$$= 2\int \cos^2\theta\sin\theta\, d\theta - \int \sin\theta\, d\theta = -\frac{2}{3}\cos^3\theta + \cos\theta \tag{10}$$

The third is

$$\int \sin^2\theta\cos^2 2\theta\,d\theta = \frac{1}{2}\int (1-\cos 2\theta)\cos^2 2\theta\,d\theta \tag{11}$$

$$= \frac{1}{2}\int \left(\cos^2 2\theta - \cos^3 2\theta\right)\,d\theta \tag{12}$$

$$= \frac{1}{4}\int (1+\cos 4\theta)\,d\theta - \frac{1}{2}\int \left(1-\sin^2 2\theta\right)\cos 2\theta\,d\theta \tag{13}$$

$$= \frac{1}{4}\theta + \frac{1}{16}\sin 4\theta - \frac{1}{4}\sin 2\theta + \frac{1}{12}\sin^3 2\theta \tag{14}$$

2.2 Part b

The ray $\theta = k$ divides the region S into two regions of equal area when

$$\int_0^k \cos^2\theta\,d\theta = \int_k^{\pi/2} \cos^2\theta\,d\theta. \tag{15}$$

In fact, there are other possibilities. Either of the integrals of (15) may be equated to the integral $\frac{1}{2}\int_0^{\pi/2}\cos^2\theta\,d\theta$.

(For the curious: Numerical solution of equation (15), which is not required, yields $k \sim 0.4158556$.)

2.3 Part c

The distance $w(\theta)$ from $(f(\theta),\theta)$ and $(g(\theta),\theta)$ is measured along the ray θ, and, for all θ in the interval $[0,\pi/2]$, we have $g(\theta) \geq f(\theta)$. Therefore,

$$w(\theta) = g(\theta) - f(\theta) \tag{16}$$

$$= 2\cos\theta - 1 - \sin\theta\cos 2\theta. \tag{17}$$

The average value, w_A, of $w(\theta)$ over $[0,\pi/2]$ is thus given by

$$w_A = \frac{2}{\pi}\int_0^{\pi/2} \left[g(\theta) - f(\theta)\right]\,d\theta \tag{18}$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \left[2\cos\theta - 1 - \sin\theta\cos 2\theta\right]\,d\theta. \tag{19}$$

Integrating numerically again, we obtain $w_A \sim 0.485446$.

$$w_A \sim 0.485. \tag{20}$$

(For the curious: The antiderivative is again accessible by elementary techniques:

$$\int [2\cos\theta - 1 - \sin\theta\cos 2\theta]\ d\theta = \frac{1}{6}\cos 3\theta + 2\sin\theta - \frac{1}{2}\cos\theta - \theta. \tag{21}$$

This yields

$$w_A = \frac{14}{3\pi} - 1, \tag{22}$$

but, again, at the expense of time.)

2.4 Part d

We must solve the equation

$$w_A = w(\theta_A), \tag{23}$$

for θ_A, or

$$0.485446 = 2\cos\theta_A - 1 - \sin\theta_A\cos 2\theta_A. \tag{24}$$

Numerical solution of this equation gives $\theta_A \sim 0.581859$, or $\theta_A \sim 0.582$. We have

$$w'(\theta) = -2\sin\theta - \cos\theta\cos 2\theta + 2\sin\theta\sin 2\theta, \tag{25}$$

whence

$$w'(\theta_A) \sim -0.581859 < 0. \tag{26}$$

We conclude that $w(\theta)$ is decreasing near $\theta = \theta_A$ because w' is continuous there and $w'(\theta_A) < 0$.

Remark: The notions of "increasing [decreasing] at a point" are not defined in most elementary calculus texts. Strictly speaking, the observation that w' is continuous at $\theta = \theta_A$ is necessary to justify the conclusion that w is decreasing on some interval centered at $\theta = \theta_A$. It is, in fact, possible to give examples of functions φ which have the property that for some value x_0, $\varphi(x_0) < 0$ but φ is not a decreasing function on any interval, however small, centered at x_0. However, the readers generally disregard this technicality.)

3 Problem 3

3.1 Part a

By the Fundamental Theorem of Calculus, $\int_{-6}^{-2} f'(x)\,dx = f(-2) - f(-6) = 7 - f(-6)$. But the value of this integral is the area of a triangle whose base is four and whose altitude is two, so $7 - f(-6) = 4$, and $f(-6) = 3$. Similarly, $\int_{-2}^{5} f'(x)\,dx = f(5) - 7$, while the value of this integral is the area of a triangle of base three, altitude two, less the area of a half disk of radius two. Hence, $f(5) = 7 + 3 - 2\pi = 10 - 2\pi$.

3.2 Part b

The function f is increasing on the closures of those intervals where f' is positive, or on $[-6, -2]$ and on $[2, 5]$.

3.3 Part c

The absolute minimum for f on $[-6, 5]$ must occur either at an endpoint or at a critical point where the derivative changes sign from negative to positive. Thus, the only possibilities are $x = -6$, $x = 2$, and $x = 5$. We already (see Part a, above) have $f(-6) = 3$ and $f(5) = 10 - 2\pi$, which latter is about 3.717, so we need only calculate $f(2)$. But $f(2)$ is less than $f(5)$ by the area of a triangle whose base is three and whose altitude it two, so $f(2) = 7 - 2\pi \sim 0.717$. Now $7 - 2\pi < 3 < 10 - 2\pi$, so the absolute minimum we seek is $f(2) = 7 - 2\pi$.

3.4 Part d

In the vicinity of $x = -5$, the graph of f' is a line whose slope is $-1/2$, so $f''(-5) = -1/2$. Immediately to the left of $x = 3$, the graph of f' is given by a straight line of slope 2, so the left-hand derivative, $f''_{-}(3)$ of f' at $x = 3$ must be 2. Immediately to the right of $x = 3$, the graph of f' is given by a line of slope -1, so the right-hand derivative, $f''_{+}(3)$, of f' at $x = 3$ must be given by $f''_{+}(3) = -1$. The one-sided derivatives of f' at $x = 3$ are different, so $f''(3)$ doesn't exist.

4 Problem 4

4.1 Part a

We have $4H'(t) = 27 - H(t)$; $H(0) = 91$. Thus,

$$H'(0) = \frac{27 - 91}{4} = -16, \tag{27}$$

and an equation for the tangent line at $(0, H(0))$ is $H = 91 - 16t$. Setting $t = 3$ in this equation for the tangent line, we obtain the approximation $H = 43$ for the value of $H(3)$.

4.2 Part b

Differentiating the original equation, we obtain $H''(t) = -H'(t)/4$. Substituting for $H'(t)$ then gives

$$H''(t) = -\frac{1}{4} \cdot \frac{27 - H}{4} = \frac{H - 27}{16}. \tag{28}$$

Thus, $H''(0) = (91 - 27)/16 = 4 > 0$. This means that the solution curve lies above its tangent line near $t = 0$, so we expect our estimate to be an underestimate.

4.3 Part c

If $G'(t) = -[G(t) - 27]^{2/3}$ with $G(0) = 91$, then

$$\frac{G'(t)}{[G(t) - 27]^{3/2}} = -1, \tag{29}$$

and $(G(t) - 27) > 0$ when t is near $t = 0$ because G—being a solution to a differential equation—must be a continuous function. For such values of t, we may therefore write

$$\int_0^t \frac{G'(\tau)}{[G(\tau) - 27]^{2/3}}\, d\tau = -\int_0^t d\tau, \tag{30}$$

so that

$$3[G(t) - 27]^{1/3} - 3[G(0) - 27]^{1/3} = -t, \tag{31}$$

or

$$3[G(t) - 27]^{1/3} = 12 - t. \tag{32}$$

From this it follows that

$$G(t) = \frac{1}{27}(2457 - 432t + 36t^2 - t^3) \tag{33}$$

Thus, $G(3) = 54$, and the internal temperature of the potato at $t = 3$ is, according to this model, 54 degrees Celsius.

5 Problem 5

5.1 Part a

From

$$f(x) = \frac{3}{2x^2 - 7x + 5}, \tag{34}$$

we obtain

$$f'(x) = -\frac{3(4x - 7)}{(2x^2 - 7x + 5)^2}. \tag{35}$$

The slope of the tangent line at $x = 3$ is

$$f'(3) = -\frac{3(4 \cdot 3 - 7)}{(2 \cdot 3^2 - 7 \cdot 3 + 5)^2} = -\frac{15}{4}. \tag{36}$$

5.2 Part b

The only critical point for f in $(1, 5/2)$ is at $x = 7/4$, because that is the only point in the interval where $f'(x) = 0$ or $f'(x)$ is undefined. The denominator of f' has zeros only at the points $x = 1$ and $x = 5/2$, by the Quadratic Formula. Its denominator, being the square of a non-zero quantity, is positive on $(1, 5/2)$, the sign of $f'(x)$, because of the minus sign in front of the fraction, is the opposite of the sign of its numerator. Thus, $f'(x) > 0$ on $(1, 7/4)$, while $f'(x) < 0$ on $(7/4, 5/2) \cup (5/2, \infty)$. It follows by the First Derivative Test that f has a relative maximum at $x = 7/4$.

5.3 Part c

We have

$$\int_5^\infty f(x)\,dx = \lim_{T\to\infty} \int_5^T f(x)\,dx, \tag{37}$$

provided that the limit exists. Thus, we write

$$\int_5^\infty f(x)\,dx = \lim_{T\to\infty} \int_5^T \left[\frac{2}{2x-5} - \frac{1}{x-1}\right] dx \tag{38}$$

$$= \lim_{T\to\infty} \left[\ln(2x-5) - \ln(x-1)\right] \Big|_5^T \tag{39}$$

$$= \lim_{T\to\infty} \ln\left(\frac{2x-5}{x-1}\right) \Big|_5^T \tag{40}$$

$$= \lim_{T\to\infty} \left[\ln\frac{2-5/T}{1-1/T} - \ln\frac{5}{4}\right] = \ln\frac{8}{5}, \tag{41}$$

and the improper integral converges to $\ln(5/8)$.

5.4 Part d

The function $f(x) = 3/[(2x-5)(x-1)]$ is positive on $(5,\infty)$ because both of the factors in its denominator are positive there. We have seen in Part b, above, that $f'(x) < 0$ on $(5/2,\infty)$, so f is a decreasing function on $[5,\infty)$. We have seen in Part c, above, that $\int_5^\infty f(x)\,dx$ is a convergent improper integral. The Integral Test assures us that if there is a positive integer M such that

1. f is a continuous function on $[M,\infty)$,
2. f is a decreasing function on $[M,\infty)$, and
3. the improper integral $\int_M^\infty f(x)\,dx$ converges,

then the series $\sum_M^\infty f(n)$ converges. We conclude that

$$\sum_{n=5}^\infty \frac{3}{2n^2 - 7n + 5}$$

converges.

Note: The convergence of this series can also be shown by using the convergent series $\sum n^{-2}$ and the Comparison-Limit Test, or by using the convergent series $\sum 3/(2n^2)$ and the Comparison Test.

6 Problem 6

6.1 Part a

From what is given, we find that

$$f(0) = 0; \tag{42}$$
$$f'(0) = 1; \tag{43}$$
$$f''(0) = f^{(1+1)}(0) = (-1) \cdot f'(0) = -1; \tag{44}$$
$$f^{(3)}(0) = f^{(2+1)}(0) = (-2) \cdot f^{(2)}(0) = 2; \tag{45}$$
$$f^{(4)}(0) = f^{(3+1)}(0) = (-3) \cdot f^{(3)}(0) = -6, \tag{46}$$

and, by an easy induction,

$$f^{(n)}(0) = (-1)^{n-1}(n-1)! \text{ when } n \geq 1. \tag{47}$$

The coefficient a_n of the Maclaurin series for f is given by

$$a_n = \frac{f^{(n)}(0)}{n!}, \text{ so} \tag{48}$$
$$a_0 = 0; \tag{49}$$
$$a_1 = 1; \tag{50}$$
$$a_2 = \frac{-1}{2!} = -\frac{1}{2}; \tag{51}$$
$$a_3 = \frac{2}{3!} = \frac{1}{3}; \tag{52}$$
$$a_4 = \frac{-6}{4!} = -\frac{1}{4}, \tag{53}$$

and, continuing inductively,

$$a_n = (-1)^{n-1}\frac{1}{n} \tag{54}$$

The desired first four non-zero terms of the Maclaurin series for f are therefore

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4},$$

and the general term is $(-1)^{n-1}\frac{x^n}{n}$, $n = 1, 2, \ldots$.

6.2 Part b

When $x = 1$, we have

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\,, \tag{55}$$

which is the alternating harmonic series—a convergent series.

However, when $x = 1$,

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{x^n}{n}\right| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\,, \tag{56}$$

and this is the divergent harmonic series. It follows that $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ is conditionally convergent when $x = 1$.

6.3 Part c

If $g(x) = \int_0^x f(t)\,dt$, where

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots\,, \tag{57}$$

then, integrating term-by-term, we have

$$g(x) = \int_0^x \left[t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + (-1)^{n-1}\frac{t^n}{n} + \cdots\right] dt \tag{58}$$

$$= \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + (-1)^{n-1}\frac{x^{n+1}}{n(n+1)} + \cdots\,, \tag{59}$$

which gives the Maclaurin series for g. The convergence of the series for f on $(-1, 1)$ guarantees the convergence of the new series for g on $(-1, 1)$.

6.4 Part d

We note first that, from Part c, above,

$$P_4(x) = \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12}. \tag{60}$$

Thus,

$$P_4\left(\frac{1}{2}\right) = \frac{1}{8} - \frac{1}{48} + \frac{1}{192} = \frac{7}{64}. \tag{61}$$

The alternating series bound is the magnitude of the first unused term from the series, or, in this case,

$$\left|(-1)^3 \frac{(1/2)^5}{4 \cdot 5}\right| = \frac{1}{640}. \tag{62}$$

It follows that

$$\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| = \left|\frac{7}{64} - g\left(\frac{1}{2}\right)\right| \leq \frac{1}{640} < \frac{1}{500}. \tag{63}$$

6.5 Addendum

The series given in this problem is easily summed. If

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \tag{64}$$

for $-1 < x < 1$, then

$$f'(x) = \sum_{n=1}^{\infty}(-1)^{n-1}x^{n-1} \tag{65}$$

$$= \frac{1}{1+x}, \tag{66}$$

also on $-1 < x < 1$, because the series of (65) is geometric, with common ratio $-x$. Integrating from 0 to x and using the observation that $f(0) = 0$, we find that

$$f(x) = \int_0^x \frac{dt}{1+t} = \ln(1+x). \tag{67}$$

Now an easy integration by parts shows that

$$g(x) = (1+x)\ln(1+x) - x. \tag{68}$$

We have shown that $P_4(1/2) = 7/64 = 0.109375$. From (68), we see that $g(x) \sim 0.1081977$. The magnitude of the difference between these two numbers is about 0.00118.