AP Calculus 2018 BC FRQ Solutions

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1 Problem 1

1.1 Part a

During the time interval $0 \le t \le 300$,

$$\int_0^{300} r(t)\,dt = 44\int_0^{300} \left(\frac{t}{100}\right)^3 \left(1 - \frac{t}{300}\right)^7 dt = 270 \qquad (1)$$

people enter the line for the escalator. (Fortunately, the problem is calculator active; finding this antiderivative doesn't bear thinking about under examination conditions.)

1.2 Part b

People exit the line at the rate of 0.7 person per second, so $300 \cdot 0.7 = 210$ people leave the line between $t = 0$ and $t = 300$. Because there were 20 people in the line at $t = 0$ and 270 people entered the line between $t = 0$ and $t = 300$, there must be $20 + 270 - 210 = 80$ people in the line at $t = 300$.

1.3 Part c

In order to solve this problem, we must assume that people exit the line at the same rate (0.7 people per second) after $t = 300$; the problem statement is ambiguous about this. Under this assumption, the line will be empty for the first time when $t = 300 + 80/0.7 = 414.286$.

1.4 Part d

People arrive at the rate $r(t)$ and leave at the rate 0.7. Because there are 20 people in the line at $t = 0$, the number $n(t)$ of people in the line at time t is given, when $0 \leq t \leq 300$, by

$$n(t) = 20 + 44\int_0^t \left(\frac{\tau}{100}\right)^3 \left(1 - \frac{\tau}{300}\right)^7 d\tau - 0.7t. \tag{2}$$

This can be minimal only when $t = 0$, $t = 300$, or $n'(t) = 0$. The latter condition is met when

$$n'(t) = 44\left(\frac{t}{100}\right)^3 \left(1 - \frac{t}{300}\right)^7 - 0.7 = 0, \tag{3}$$

or, solving numerically, when

$$t \sim 33.0133, \tag{4}$$

and when

$$t \sim 166.5747. \tag{5}$$

Evaluating n at these four values, we find that

$$n(0) = 20, \tag{6}$$

$$n(33.0133) = 3.8034, \tag{7}$$

$$n(166.5747) = 158.0701, \tag{8}$$

$$n(300) = 80. \tag{9}$$

The minimum value of this function thus occurs at about time $t = 33.0133$. To the nearest whole number, the minimum number of people is 4.

2 Problem 2

2.1 Part a

If $p(h) = 0.2h^2 e^{-0.0025h^2}$ for $0 \leq h \leq 20$, then, because

$$p'(h) = 0.4he^{-0.0025h^2} - 0.001h^3 e^{-0.0025h^2}, \tag{10}$$

$$p'(25) \sim -1.179. \tag{11}$$

The quantity $p'(25)$ gives the rate, in millions of cells per cubic meter, at which the density of plankton cells increases as h increases. (The fact that $p'(25) < 0$ means that, at a depth of 25 meters, density decreases as depth increases.)

2.2 Part b

According to this model, the number of millions of plankton cells in a vertical column of water of constant cross-sectional area 3 square meters and extending over $0 \le h \le 30$ is

$$3\int_0^{30} p(h)\,dh = 0.6\int_0^{30} h^2 e^{-0.0025h^2}\,dh \sim 1675.415. \tag{12}$$

Thus, to the nearest million, there are 1675 million plankton cells in the column of water.

2.3 Part c

We suppose that K denotes the depth of the bottom of this 30-foot column. (The problem statement is unclear about what it means for the column to be "K meters deep.") There are two cases: either $K < 60$ or $K \ge 60$. In the second case, the entire column is at a depth of at least 30 feet, and the number of plankton in the column is

$$3\int_{K-30}^{K} f(h)\,dh, \tag{13}$$

and we have (because $K - 30 \ge 30$)

$$3\int_{K-30}^{K} f(h)\,dh \le 3\int_{K-30}^{K} u(h)\,dh \le 3\int_{30}^{\infty} u(h)\,dh < 3\cdot 105 = 315 < 2000. \tag{14}$$

If, on the other hand, we have $K < 60$, then, because also $30 < K$, the number of plankton in the column is

$$3\int_{K-30}^{30} p(h)\,dh + 3\int_{30}^{K} f(h)\,dh. \tag{15}$$

Here, we may write

$$3\int_{K-30}^{30} p(h)\,dh + \int_{30}^{K} f(h)\,dh \le 3\int_{0}^{30} p(h)\,dh + 3\int_{30}^{\infty} u(h)\,dh \tag{16}$$

$$\le 1676 + 106 = 1782 < 2000. \tag{17}$$

2.4 Part d

The total distance traveled by the boat over the time interval $0 \leq t \leq 1$ is

$$\int_0^1 \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2}\, d\tau = \int_0^1 \sqrt{438244 \sin^2 5\tau + 774400 \cos^2 6\tau}\, d\tau. \tag{18}$$

The integral is not elementary, so we integrate numerically to find that

$$\int_0^1 \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2}\, d\tau \sim 757.456 \text{ meters}. \tag{19}$$

3 Problem 3

3.1 Part a

By the Fundamental Theorem of Calculus,

$$f(x) = 3 + \int_1^x g(t)\, dt, \tag{20}$$

so $g(-5)$ is 3 added to, reading from the graph, the sum of the area of a 3×3 square, the area of a triangle of base 1 and height 3 and the negative of a triangle of base 1 and height 2. That's

$$3 + 3^2 + \frac{1}{2} \cdot 1 \cdot 3 - \frac{1}{2} \cdot 1 \cdot 2 = \frac{25}{2}. \tag{21}$$

3.2 Part b

We have

$$\int_1^6 g(t)\, dt = \int_1^3 2\, dt + 2 \int_3^6 (t-4)^2\, dt \tag{22}$$

$$= 2 \cdot 2 + \frac{2}{3}(t-4)^3 \Big|_3^6 \tag{23}$$

$$= 4 + \frac{2}{3}[8 - (-1)] = 10. \tag{24}$$

3.3 Parts c & d

There are thorny issues with these two questions—not because they involve difficult mathematics, but because there is no general agreement about a formal definition for the term “concave upward” (or for its sibling term “concave downward”). And so, of course, there can’t be general agreement about what an “inflection point” is, either. To make matters worse, even those who agree on their definitions for the two flavors of concavity disagree about what inflections points are. (The most obvious difference is that some insist that there must be a line tangent to the original curve at a point if it is to qualify as an inflection point; others omit this requirement. There are other different ways of seeing inflection points, but this is not the place to explore them.)

Some people take a region of upward concavity to be a region (which may, or may not, be required to be open, depending on whom we’re reading) where the derivative is increasing, others a region where the tangent line lies below the curve near the point of tangency. Some take positivity of the second derivative to be (to *be*—not to *imply*) upward concavity. Still others define a function F to be concave upward on an interval I provided that for any pair $x_1 < x_2$ of points in I and any number α between 0 and 1 it is true that $F[\alpha x_1 + (1-\alpha)x_2] \leq \alpha F(x_1) + (1-\alpha)F(x_2)$—that is, no matter what points x_1 and x_2 we choose in I, the curve $y = F(x)$ never rises above the line segment connecting the points $\big(x_1, F(x_1)\big)$ and $\big(x_2, F(x_2)\big)$. (Of course, these are not the only ways of defining “concave upward.”)

Figures 1 and 2 show the graphs of f and $f'' = g'$ respectively. It is clear from the graph that if we adopt the last definition we gave for upward concavity—the one in terms of line segments lying above the curve—then f is concave upward on $[-5, 3]$ and on $[4, 6]$.

If, on the other hand, we think that a function is concave upward just in those regions where its second derivative is positive, we must conclude that f is concave upward on the intervals $(-2, -1)$, $(0, 1)$, and $(4, 6)$.

That folks use the term “increasing” in different ways adds a bit of spice. (Some folks require that $f(\alpha) < f(\beta)$ when $\alpha < \beta$ and both are in I, some only that $f(\alpha) \leq f(\beta)$, for f to be increasing on .

It should be clear now that someone can reasonably assert that f is increasing and concave upward on the intervals $(-1, 3)$ and $(4, 6)$ while someone else can assert—just as reasonably—that f is both increasing and concave upward just on the intervals $(0, 1)$ and $(4, 6)$. It is only when we know precisely what these people are using for their definitions that we can decide whether they’re right or wrong.

Different (correct) decisions about different meanings for upward and downward concavity can clearly lead to different (correct) decisions about inflection points.

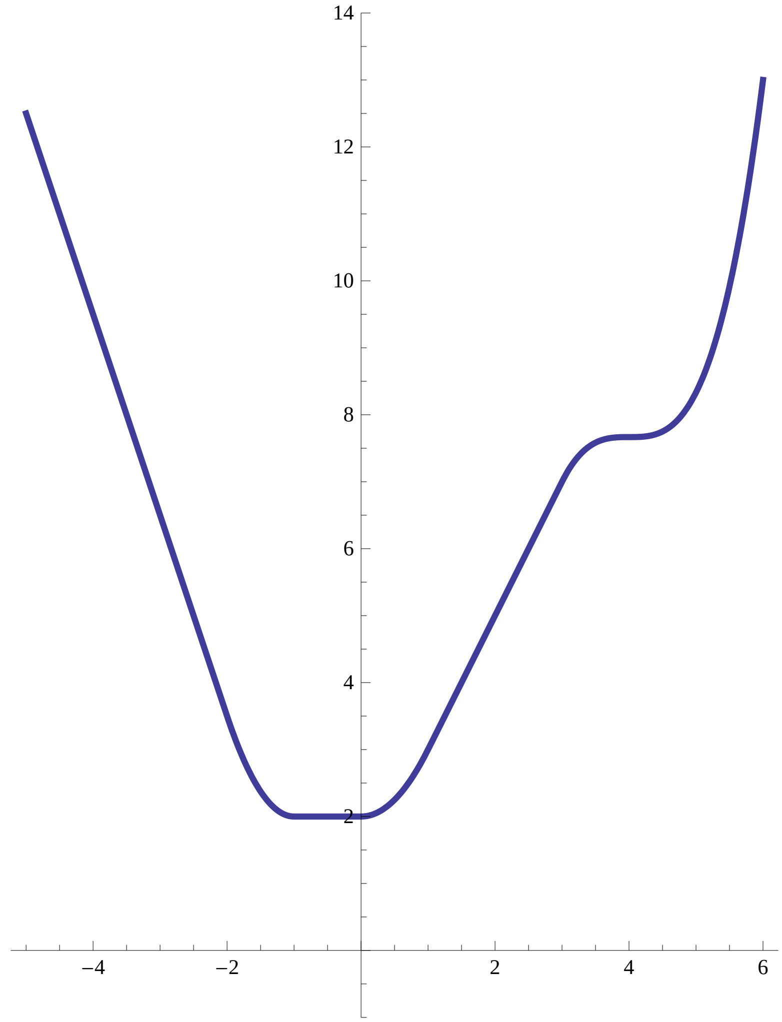

Figure 1: $y = f(x) = 3 + \int_1^x g(t)\,dt$

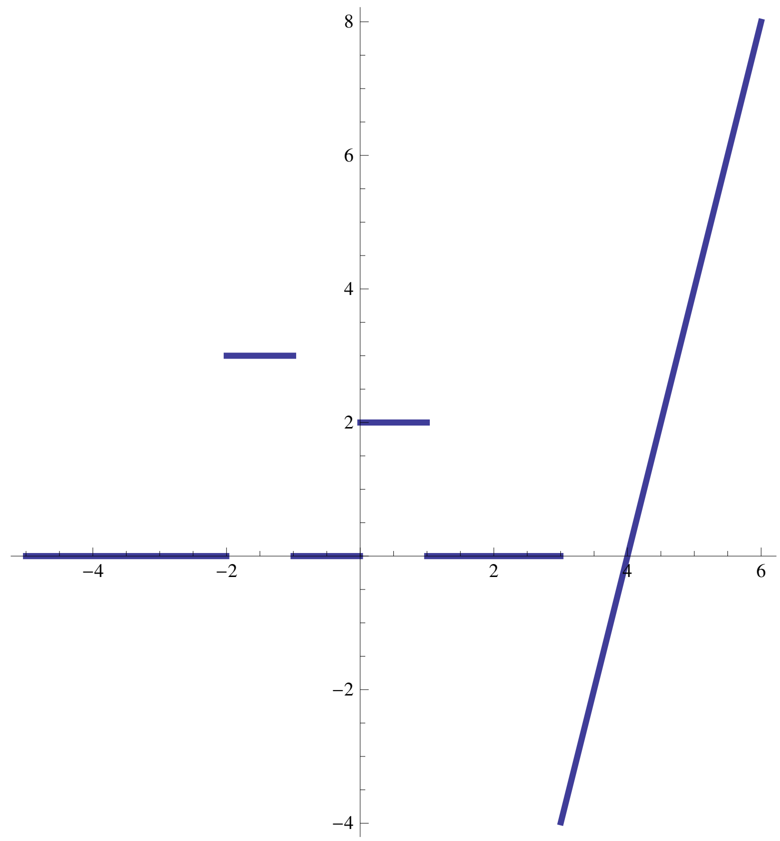

Figure 2: The graph of $f''(x) = g'(x)$

Students in elementary calculus will almost certainly have seen just one definition for each of their concepts; they won't even know that different people might use substantially different definitions. So it's very unlikely that these students will refer to their definitions—let alone state them in full—in the answers they give to these questions.

All definitions of concavity that I've seen yield the theorem that tells us that a function whose second derivative is positive on an interval is necessarily concave upward on that interval. My guess is that the development committee looked (and the readers will look) at $f''(x)$, observed (will observe) that it changes sign at just one point—at $x = 4$—and concluded (will conclude) that there is just one inflection point, which is at $x = 4$.

It's very unusual for the development committee to pose such an ambiguous problem. What the readers will do with the mess is anybody's guess. I'm glad I don't have to decide.

4 Problem 4

4.1 Part a

From the table, we estimate

$$H'(6) \sim \frac{H(7) - H(5)}{7 - 5} = \frac{11 - 6}{7 - 5} = \frac{5}{2}. \tag{25}$$

The number $H'(6)$ gives, in meters per year, the rate at which the tree is growing at time $t = 6$.

4.2 Part b

The function H is given twice differentiable, presumably (though the problem statement is vague about this) at least on the interval $(2, 10)$, so it is continuous on the interval $[3, 5]$ and differentiable on the interval $(3, 5)$. The hypotheses of the mean value theorem being satisfied, there must be at least one point ξ in the interval $(3, 5)$, and therefore in the interval $(2, 10)$, such that

$$H'(\xi) = \frac{H(5) - H(3)}{5 - 3} = \frac{6 - 2}{5 - 3} = \frac{4}{2} = 2. \tag{26}$$

4.3 Part c

The required trapezoidal sum is $\dfrac{1}{10-2} = \dfrac{1}{8}$ times

$$\begin{aligned}
&\frac{1}{2}[H(2)+H(3)](3-2) + \frac{1}{2}[H(3)+H(5)](5-3) \\
&\qquad + \frac{1}{2}[H(5)+H(7)](5-7) + \frac{1}{2}[H(10)+H(7)](10-7) \\
&= \frac{1}{2}[(1.5+2)(3-2) + (2+6)(5-3) + (6+11)(7-5) + (11+15)(10-7)] \qquad (27) \\
&= \frac{1}{2}(3.5\cdot 1 + 8\cdot 2 + 17\cdot 2 + 26\cdot 3) = 65.75. \qquad (28)
\end{aligned}$$

The average height during the period $2 \leq t \leq 10$ is therefore 8.219 meters, to three decimal places.

4.4 Part d

We are given

$$G(x) = \frac{100x}{1+x}, \tag{29}$$

where x is the diameter, in meters, of the base of the tree and $h = G(x)$ is the height, also in meters, of the tree. We have

$$\frac{dh}{dt} = \frac{dh}{dx} \cdot \frac{dx}{dt}. \tag{30}$$

Now

$$\frac{dh}{dx} = \frac{d}{dx}\left(\frac{100x}{1+x}\right) = \frac{100}{(1+x)^2}, \tag{31}$$

and $G(x) = h = 50$ implies that $x = 1$, so that we have been given $\left.\dfrac{dx}{dt}\right|_{h=50} = 0.03$. Putting this all together, we find that

$$\left.\frac{dh}{dt}\right|_{h=50} = \left(\left.\frac{dh}{dx}\right|_{h=50}\right)\left(\cdot\left.\frac{dx}{dt}\right|_{h=50}\right) = \frac{100}{(1+1)^2}\cdot 0.03 = 0.75 \text{ meters/year}. \tag{32}$$

5 Problem 5

5.1 Part a

The area, A, outside the polar curve $r = 3 + 2\cos\theta$ but inside the curve $r = 4$ is

$$\int_{\pi/3}^{\pi} \left[16 - (3 + 2\cos\theta)^2\right] d\theta = \int_{\pi/3}^{\pi} \left(7 - 12\cos\theta - 4\cos^2\theta\right) d\theta \tag{33}$$

$$= \frac{1}{6}\left(39\sqrt{3} + 20\pi\right). \tag{34}$$

(Evaluation of the integral is not required; we give it for the sake of completeness.)

5.2 Part b

On the curve $r = 3 + 2\cos\theta$, we have

$$x(\theta) = r(\theta)\cos\theta = 3\cos\theta + 2\cos^2\theta, \tag{35}$$

and

$$y(\theta) = r(\theta)\sin\theta = 3\sin\theta + 2\sin\theta\cos\theta \tag{36}$$

$$= 3\sin\theta + \sin 2\theta. \tag{37}$$

Therefore,

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{3\cos\theta + 2\cos 2\theta}{-3\sin\theta - 4\sin\theta\cos\theta} \tag{38}$$

$$= \frac{y'(\theta)}{x'(\theta)} = \frac{3\cos\theta + 2\cos 2\theta}{-3\sin\theta - 2\sin 2\theta} \tag{39}$$

It follows that

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{3\cos(\pi/2) + 2\cos 2(\pi/2)}{-3\sin(\pi/2) - 2\sin 2(\pi/2)} = \frac{0 - 2}{-3 - 0} = \frac{2}{3}, \tag{40}$$

so the slope of the line tangent to the curve $r = 3 + 2\cos\theta$ at the point where $\theta = \pi/2$ is $2/3$.

5.3 Part c

The distance from the particle to the origin is $r = 3 + 2\cos\theta$, and we are given $\dfrac{dr}{dt} \equiv 3$. Moreover, if $r = 3 + 2\cos\theta$, then

$$\cos\theta = \frac{1}{2}(r - 3), \tag{41}$$

so that

$$-\sin\theta \cdot \frac{d\theta}{dr} = \frac{1}{2}, \tag{42}$$

and

$$\frac{d\theta}{dr} = -\frac{1}{2\sin\theta}. \tag{43}$$

For the rate at which the angle changes with respect to time, we have

$$\left.\frac{d\theta}{dt}\right|_{\theta=\pi/3} = \left(\left.\frac{d\theta}{dr}\right|_{\theta=\pi/3}\right) \cdot \left(\left.\frac{dr}{dt}\right|_{\theta=\pi/3}\right) \tag{44}$$

$$= -\frac{1}{2\sin\pi/3} \cdot 3 = -\frac{3}{\sqrt{3}} = -\sqrt{3}. \tag{45}$$

6 Problem 6

6.1 Part a

The first four nonzero terms, and the general term, of the Maclaurin series for

$$f(x) = x\ln(1 + x/3) = 3\left(\frac{x}{3}\right)\ln\left[1 + \frac{x}{3}\right] \tag{46}$$

are

$$x\left(\frac{x}{3} - \frac{x^2}{2\cdot 3^2} + \frac{x^3}{3\cdot 3^3} - \frac{x^4}{4\cdot 3^4} + \cdots + (-1)^{n+1}\frac{x^n}{n\cdot 3^n} + \cdots\right), \tag{47}$$

or

$$\frac{x^2}{3} - \frac{x^3}{2\cdot 3^2} + \frac{x^4}{3\cdot 3^3} - \frac{x^5}{4\cdot 3^4} + \cdots + (-1)^{n+1}\frac{x^{n+1}}{n\cdot 3^n} + \cdots \tag{48}$$

6.2 Part b

We can obtain the Maclaurin series for $f(x) = \ln\left(1 + \frac{x}{3}\right)$ by observing that

$$
\begin{aligned}
\ln\left(1+\frac{x}{3}\right) &= \left[\ln\left(1+\frac{x}{3}\right) - \ln(1+0)\right] && (49)\\
&= \ln(3+x) - \ln 3 && (50)\\
&= \ln(3+t)\Big|_{t=0}^{t=x} && (51)\\
&= \int_0^x \left(\frac{d}{dt}\left[\ln(3+t)\right]\right)\, dt && (52)\\
&= \int_0^x \frac{dt}{3+t} && (53)\\
&= \frac{1}{3}\int_0^x \frac{dt}{1-(-t/3)} && (54)\\
&= \frac{1}{3}\int_0^x \left[\sum_{k=0}^{\infty}(-1)^k \frac{t^k}{3^k}\right] dt && (55)\\
&= \frac{1}{3}\sum_{k=0}^{\infty}\left(\int_0^x \left[(-1)^k \frac{t^k}{3^k}\right] dt\right) && (56)\\
&= \frac{1}{3}\sum_{k=0}^{\infty}(-1)^k \frac{x^{k+1}}{(k+1)3^k}. && (57)
\end{aligned}
$$

Interchanging the sum and the integral are valid within the interval of convergence of the series we see in (55), and the interiors of the intervals of convergence of the two series are identical. But the series of (55) is a geometric series, and the interior of its interval of convergence is $(-3, 3)$—which means that the same is true of the series in (57).

Distributing the initial factor of $1/3$ over the series of (57) doesn't change the interior of the interval of convergence.

Further multiplication of every term in the series by x yields the series of (48)—which therefore also has an interval of convergence whose interior is $(-3, 3)$.

All that remains is to check behavior at the endpoints.

When $x = 3$,

$$\sum_{k=0}^{\infty}(-1)^k \frac{x^{k+2}}{(k+1)3^{k+1}} = 3 - \frac{3}{2} + \frac{3}{3} - \frac{3}{4} + \cdots . \qquad (58)$$

This is the alternating harmonic series (multiplied by a factor of 3, which doesn't change its convergence properties), which we know to be convergent.

When $x = -3$,

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{k+2}}{(k+1)3^{k+1}} = \sum_{k=0}^{\infty} \frac{(-1)^{2k+2} 3}{k+1} = 3 + \frac{3}{2} + \frac{3}{3} + \frac{3}{4} + \cdots , \tag{59}$$

and this is, except for the common factor 3, the harmonic series, which we know to be divergent.

We conclude that the series of (48) converges on $(-3, 3]$ and diverges outside that interval.

6.3 Part c

In our case, the fourth degree Maclaurin polynomial for $f(x) = x \ln \left(1 + \frac{x}{3}\right)$ is

$$T_4(x) = \frac{x^2}{3} - \frac{x^3}{18} + \frac{x^4}{81} \tag{60}$$

$$= \frac{x^2}{3} \left[1 - \frac{x}{6} + \frac{x^2}{27}\right]. \tag{61}$$

The magnitude of first term of the series that we haven't used is

$$\left| -\frac{x^5}{324} \right| \Bigg|_{x=2} = \frac{8}{81}. \tag{62}$$

This is the desired error bound.