Calc I
Exams

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To the Reader

This is a collection of some of the exams I gave in my first semester calculus courses over the twenty-five-year period beginning in 1991 and extending into 2015. I offer it here as a resource for current calculus teachers—and for students, too.

I'm not concerned about students gaining access to these exams, nor should you be—there are enough problems of enough different kinds that any student who can handle a sizable fraction of them will necessarily have a good grasp of the concepts of first-semester calculus (which is roughly congruent with the Advanced Placement program's AB Calculus course). As you will see, I wasn't afraid to recycle problems, or entire exams, either. When I found something that I thought tested what I wanted students to know, I saw no reason to abandon it after a single use. And, of course, every teacher knows that there isn't always the time one would like to have to develop new things.

I taught all levels of calculus (Calc I, Calc II, Calc III, Advanced Calc I, Advanced Calc II), from 1969, when I was a callow graduate student, until my retirement in 2015. Of course I didn't teach a given calculus course every semester, and this means that I haven't included exams for every semester, or even every year, in this collection. Some exams never existed, and, inevitably, I lost some. In some cases, I lost portions (especially graphics—which, almost always had to be stored in separate files in different formats, depending on the capabilities of the software of the day) of exams.

As a rule, I gave three in-class exams and a final exam each semester until the early part of this century, when my department changed from a schedule of four one-hour meetings during each of fifteen weeks to one of two two-hour meetings over the same fifteen weeks. I didn't want to devote six hours of classroom time (which is always in short supply—even more in the college/university environment than in the high-school setting) to examinations—and I knew better to try to lecture immediately after (or worse, immediately before) an exam. So I switched to two in-class exams and a final exam from then on. A side-effect of this change was that students had only one hour to complete the earlier exams, but they had two to work on the later exams. (The final exams were all two-hour exams.)

It wasn't until 1991 that I began hanging onto the old exams, at least in a format that my current computer can read. During the first two or three decades of my career, we either wrote out our exam questions on the blackboard at the front of the classroom or wrote them out by hand on "ditto masters" for reproduction by spirit duplication—known to most of us then as "dittoing". (We also walked ten miles to and from school in sub-zero weather, uphill both ways, and always with a forty-plus-mile-per-hour wind in our faces!) It never occurred to me to hang onto those exams, neither format being amenable to long-term storage. (Every now and then, I find an old copy of something that had been produced by spirit duplication; the characteristic purple ink has now faded pretty much into illegibility.)

In the mid-Eighties, I started using computers and more modern copying machines, but, again, it either didn't occur to me to hang onto things or, if it did, I had stored them in formats my computer will no longer read. For example, I produced some of the early exams in this collection in the version of Microsoft Word which was then current. (It was terrible for producing printed mathematics, and the current version seems no better.) If I hadn't saved them in PDF, I wouldn't be able to read them today.

The Web didn't come along until 1992, and several more years passed before my institution provided me with my own website. At about the same time, I gained access to LaTeX, which makes it possible for anyone with a computer and a decent printer to produce mathematics in typeset form.

It was a few years after that when I began typesetting the solutions to exam problems and posting them on my website where students could find them immediately (or nearly so) after they'd left the examination room. This freed up precious classroom time, it no longer being necessary to give a thorough treatment of each problem in class after I had read and returned students' work. Because LaTeX source files are stored in ASCII, today's computers will read them—and compile them. Neither LaTeX nor ASCII is going way in the foreseeable future. Thus, in order to complete this collection, I had simply (Hah!) to find all of those exams somewhere on multi-terabytes of internal and external hard drive. (Fortunately, I'm something of a packrat, and I don't throw many things away; unfortunately, I store things with the same kind of "organization" that packrats use.) But some of them were in earlier formats, so I also had to re-typeset many of them, and I had to typeset some of the solutions for inclusion here.

Most of the exams included in this collection appear in two forms: one without solutions—just as I handed it to the class that sat for it—and one that includes the solutions that I hoped the best of my students would produce. Whether you are teaching elementary calculus or learning it, I hope you find this collection helpful.

Louis A. Talman
Denver, CO
February 3, 2019

MTH141, Calculus I, Final Examination May 8, 1991

Instructions: Write the exam on your own paper. Give your reasoning, and provide enough detail to support your reasoning. Correct work lacking supporting detail is worth at most half credit. Incorrect work without supporting detail is worth no credit. Your paper is due at 3:55pm.

1. (a) Find
$$\lim_{x\to -2}\frac{x^3-2x^2-5x+6}{x^2+5x+6}.$$
   (b) Find
$$\lim_{x\to 0}\frac{x\sin x}{1-\cos x}.$$
   [Hint: Multiply top and bottom by $(1+\cos x)$.]

2. Use the definition of the derivative to find $f'(x)$ when $f(x)=x^2+x$.

3. Find the equation of the line tangent to the curve
$$x^2-9y^2+4x^2y^2=11$$
at the point $(2,-1)$.

4. Find
   (a) $\displaystyle\int \frac{1+x}{(2x+x^2)^3}dx$
   (b) $\displaystyle\int \frac{\sin\left(x^{-1}\right)}{x^2}dx$

5. Find all of the critical points for the function $f(x)=\sin^3 x$ in the interval $\left(-\frac{\pi}{4},\frac{7\pi}{4}\right)$. Describe the behavior of f at each of the critical points you have found.

6. A radar station is tracking an airplane as the airplane travels southward at 240 mile per hour. At noon, the airplane is 60 miles due east of the radar station. How quickly is the bearing of the airplane from the radar station changing then? (Alternative but equivalent way to phrase the question: How rapidly must the radar antenna be turning in order to track the airplane?)

7. Find the area of the bounded region between the curves $y=x+1$ and $y=x^2-4x+5$.

8. Find the volume generated when the region bounded by $x=0$, $x=4$, the x-axis, and the curve $y=4x-x^2$ is revolved about the x-axis.

9. Find the length of the curve $y=\dfrac{2}{3}x^{3/2}$ on the interval $1\le x\le 4$.

10. How many subdivisions will be required to obtain accuracy to three digits to the right of the decimal when we use the trapezoidal rule to approximate the value of $\displaystyle\int_0^1\sqrt{4-x^2}\,dx$?

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1. (a) Find
$$\lim_{x\to -2} \frac{x^3 - 2x^2 - 5x + 6}{x^2 + 5x + 6}.$$

   (b) Find
$$\lim_{x\to 0} \frac{x \sin x}{1 - \cos x}.$$

   [Hint: Multiply top and bottom by $(1 + \cos x)$.]

   Solution:

   (a)
$$\lim_{x\to -2} \frac{x^3 - 2x^2 - 5x + 6}{x^2 + 5x + 6} = \lim_{x\to -2} \frac{\cancel{(x+2)}(x^2 - 4x + 3)}{\cancel{(x+2)}(x+3)} = 15. \tag{1}$$

   (b)
$$\lim_{x\to 0} \frac{x \sin x}{1 - \cos x} = \frac{x \cancelto{1}{\sin x}(1 + \cos x)}{\cancelto{\sin x}{1 - \cos^2 x}} = \lim_{x\to 0} \left[\left(\frac{x}{\sin x} \right) \cdot (1 + \cos x) \right] = 1 \cdot 2 = 2. \tag{2}$$

2. Use the definition of the derivative to find $f'(x)$ when $f(x) = x^2 + x$.

   Solution:
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{3}$$
$$= \lim_{h\to 0} \frac{[(x+h)^2 + (x+h)] - [x^2 + x]}{h} \tag{4}$$
$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{x} + h - \cancel{x^2} - \cancel{x}}{h} \tag{5}$$
$$= \lim_{h\to 0} \frac{(2x + h + 1)\cancel{h}}{\cancel{h}} = 2x + 1. \tag{6}$$

3. Find the equation of the line tangent to the curve
$$x^2 - 9y^2 + 4x^2y^2 = 11$$
   at the point $(2, -1)$.

   Solution: We treat y as a function of x, and we compute
$$\frac{d}{dx}\left(x^2 - 9y^2 + 4x^2y^2\right) = \frac{d}{dx}11; \tag{7}$$
$$2x - 18yy' + 8xy^2 + 8x^2yy' = 0. \tag{8}$$

So when $x = 2$ and $y = -1$,

$$4 + 18y' + 16 - 32y' = 0, \tag{9}$$

We conclude that $y' = \dfrac{10}{7}$ when $x = 2$ and $y = -1$. We know that an equation for the line of slope m through the point (x_0, y_0) is $y = y_0 + m(x - x_0)$. Thus, an equation for the line tangent to the curve $x^2 - 9y^2 + 4x^2y^2 = 11$ at the point $(2, -1)$ is

$$y = -1 + \frac{10}{7}(x + 2). \tag{10}$$

4. Find

   (a) $\displaystyle\int \frac{1+x}{(2x+x^2)^3}dx$

   (b) $\displaystyle\int \frac{\sin\left(x^{-1}\right)}{x^2}dx$

   Solution:

   (a) Let $u = 2x + x^2$. Then $du = (2 + 2x)\,dx = 2(1 + x)\,dx$, or $(1 + x)\,dx = \frac{1}{2}du$. Thus

$$\int \frac{1+x}{(2x+x^2)^3}dx = \int (2x+x^2)^{-3}(1+x)\,dx \tag{11}$$

$$= \frac{1}{2}\int u^{-3}\,du = -\frac{1}{4}u^{-2} + c \tag{12}$$

$$= -\frac{1}{4(2x+x^2)^2} + c. \tag{13}$$

   (b) Let $u = x^{-1}$. Then $du = -x^{-2}\,dx$, or $\dfrac{dx}{x^2} = -du$. So

$$\int \frac{\sin\left(x^{-1}\right)}{x^2}dx = -\int \sin u\,du \tag{14}$$

$$= \cos u + c = \cos x^{-1} + c. \tag{15}$$

5. Find all of the critical points for the function $f(x) = \sin^3 x$ in the interval $\left(-\frac{\pi}{4}, \frac{7\pi}{4}\right)$. Describe the behavior of f at each of the critical points you have found.

   Solution: The function f is everywhere differentiable, so the critical points of f in the interval $I = \left(-\frac{\pi}{4}, \frac{7\pi}{4}\right)$ are the points where $f(x) = 3\sin^2 x\cos x = 0$. Now in the interval I, we know that $\sin x$ vanishes at $x = 0$ and at $x = \pi$, and the factor $\sin^2 x$ is never negative. The factor $\cos x$ vanishes when $x = \dfrac{\pi}{2}$ and when $x = \dfrac{3\pi}{2}$; moreover, it changes sign from positive to negative at the first of these, from negative to positive at the second. Therefore, $f'(x)$ is positive on the interval $\left(-\frac{\pi}{4}, 0\right)$, positive on the interval $\left(0, \frac{\pi}{2}\right)$, negative on the interval $\left(-\frac{\pi}{2}, \pi\right)$, negative on $\left(\pi, \frac{3\pi}{2}\right)$, and positive on the interval $\left(\frac{3\pi}{2}, \frac{7\pi}{4}\right)$, with critical points at $x = 0, \dfrac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$. By the First Derivative Test, we conclude that f has a local maximum at $x = \frac{\pi}{2}$, a local minimum at $x = \frac{3\pi}{2}$, and neither at the other two critical points.

6. A radar station is tracking an airplane as the airplane travels southward at 240 mile per hour. At noon, the airplane is 60 miles due east of the radar station. How quickly is the bearing of the airplane from the radar station changing then? (Alternative but equivalent way to phrase the question: How rapidly must the radar antenna be turning in order to track the airplane?)

Solution: Place an xy-coordinate system with its origin at the radar station and its positive x-axis pointing directly east. Let y be the vertical coordinate of the airplane at a given time, and let θ be the corresponding angle, measured counterclockwise, that the line from $(0,0)$ to the location of the airplane at $(60,y)$ makes with the x-axis. Then

$$\theta = \arctan\frac{y}{60}, \text{ so} \tag{16}$$

$$\frac{d\theta}{dt} = \frac{1}{60}\frac{1}{1+\left(\frac{y}{60}\right)^2}\frac{dy}{dt} = \frac{60}{3600+y^2}\frac{dy}{dt}. \tag{17}$$

We are given that $\frac{dy}{dt} = -240$, so when $y = 0$,

$$\frac{d\theta}{dt} = \frac{60}{3600+0^2}\cdot(-240) = -\frac{6\not{0}\cdot 24\not{0}}{36\not{0}\not{0}} = -4. \tag{18}$$

As the airplane crosses the x-axis, the radar antenna is rotating clockwise at the rate of 4 radians per hour, which is $1/900$ radians per second or $\frac{1}{5\pi}$ degrees per second.

7. Find the area of the bounded region between the curves $y = x+1$ and $y = x^2 - 4x + 5$.

Solution: We begin by determining the points where the curves intersect. We accomplish this by solving the system

$$y = x + 1; \tag{19}$$

$$y = x^2 - 4x + 5 \tag{20}$$

simultaneously. Substituting the right side of equation (19) for y in (20), we find that $x^2-5x+4 = 0$ or $(x-1)(x-4) = 0$. Thus, the curves intersect when $x = 1$ and when $x = 4$. But $y = x+1$ at the intersection points, so the intersections points are $(1,2)$ and $(4,5)$. Equation (20) is the equation of a parabola opening upward, while (19) gives a straight line of slope one. We conclude that the parabola lies below the line between the intersection points. Therefore, the desired area is

$$\int_1^4 [(x+1)-(x^2-4x+5)]\,dx = -\int_1^4 (x^2-5x+4)\,dx \tag{21}$$

$$= -\left[\frac{x^3}{3} - \frac{5}{2}x^2 + 4x\right]\Bigg|_1^4 \tag{22}$$

$$= -\left[\frac{64}{3} - \frac{80}{2} + 16\right] + \left[\frac{1}{3} - \frac{5}{2} + 4\right] = \frac{9}{2}. \tag{23}$$

8. Find the volume generated when the region bounded by $x = 0$, $x = 4$, the x-axis, and the curve $y = 4x - x^2$ is revolved about the x-axis.

Solution: The required volume is

$$\pi\int_0^4 (4x-x^2)^2\,dx = \pi\int_0^4 (16x^2 - 8x^3 + x^4)\,dx \tag{24}$$

$$= \pi\left[\frac{16}{3}x^3 - 2x^4 + \frac{1}{5}x^5\right]\Bigg|_0^4 \tag{25}$$

$$= \pi\left[\frac{1024}{3} - 512 + \frac{1024}{5}\right] = \frac{512\pi}{15}. \tag{26}$$

9. Find the length of the curve $y = \frac{2}{3}x^{3/2}$ on the interval $1 \le x \le 4$.

Solution: $y' = \sqrt{x}$, so the required length is

$$\int_1^4 \sqrt{1 + (y')^2}\, dx = \int_1^4 \sqrt{1+x}\, dx \tag{27}$$

$$= \frac{2}{3}(1+x)^{3/2}\Big|_1^4 \tag{28}$$

$$= \frac{10}{3}\sqrt{5} - \frac{4}{3}\sqrt{2}. \tag{29}$$

10. How many subdivisions will be required to obtain accuracy to three digits to the right of the decimal when we use the trapezoidal rule to approximate the value of $\int_0^1 \sqrt{4 - x^2}\, dx$?

Solution: The error in an n-subdivision trapezoid rule approximation to $\int_a^b f(t)\, dt$ is at most $M\frac{(b-a)^3}{12n^2}$, where M is any number for which $|f''(x)| \le M$ for all x in $[a, b]$. Now if $f(x) = \sqrt{4 - x^2}$, then

$$f'(x) = -\frac{x}{\sqrt{4-x^2}}, \text{ and} \tag{30}$$

$$f''(x) = -\frac{4}{(4-x^2)^{3/2}}, \tag{31}$$

so that $f''(x) < 0$ on $[0, 1]$. But the quantity $(4 - x^2)$, and therefore the quantity $(4 - x^2)^{3/2}$ as well, decreases as x increases from 0 to 1. Consequently, the negative-valued function $f''(x)$, which is $-4(4 - x^2)^{-3/2}$, decreases in that interval. This means that $|f''(x)| = 4(4 - x^2)^{-3/2}$ increases in $[0, 1]$ and assumes its maximum value in that interval at $x = 1$, where $|f''(x)|$ is very slightly smaller than 0.77. We may therefore take $M = 77/100$. Accuracy to three digits to the right of the decimal requires that error be no more than $5/10000 = 1/2000$, so we must solve for n in the inequality

$$M\frac{(b-a)^3}{12n^2} \le \frac{1}{2000}. \tag{32}$$

Here, $a = 0$ and $b = 1$, so we need to take n large enough to guarantee that

$$\frac{77}{100} \cdot \frac{1}{12n^2} \le \frac{1}{2000}, \text{or so that} \tag{33}$$

$$n^2 \ge \frac{77 \cdot 20\cancel{00}}{\cancel{100} \cdot 12} = \frac{385}{3}. \tag{34}$$

Thus, we need $n > \sqrt{\frac{385}{3}}$ which latter is about 11.33. But n must be a whole number. Thus, to guarantee a trapezoidal rule approximation accurate to three digits to the right of the decimal for the integral $\int_0^1 \sqrt{4 - x^2}\, dx$, we will need at least 12 subdivisions.

MTH 141, Calculus I, Exam I **June 20, 1991**

Instructions: Work the following problems on your own paper. The way in which you present your thinking is at least as important as the answers you get. You must hand in your paper by 9:00pm.

1. Evaluate:
   (a) $\lim_{x\to 2} \dfrac{x^3 - 3x + 2}{2x^2 + x + 1}$
   (b) $\lim_{x\to 3} \dfrac{x - 3}{2(x^2 - 9)}$
   (c) $\lim_{x\text{ø}1} \dfrac{x^3 - 2x^2 - 5x + 6}{2x^3 - 7x + 5}$
   (d) $\lim_{x\to 0} \dfrac{x + \tan x}{\sin x}$
2. Find a value for c for which the function given below is continuous everywhere:
$$f(x) = \begin{cases} 3x - 2; & \text{if } x \leq 1 \\ cx + 3; & \text{if } 1 < x. \end{cases}$$
   Be sure to explain the reasoning, based on the definition of continuity, that leads to the solution.
3. *Use the definition of the derivative as a limit* to find $f'(2)$ when f is given by:
$$f(x) = 3x^2 + x - 2.$$
4. *Use the definition of the derivative* to find $f'(x)$ when f is given by:
$$f(x) = \frac{1}{3\sqrt{x}}.$$
5. Use whatever method seems to you to be wisest and simplest to find $f'(x)$ given that:
   (a) $f(x) = 3x^4 - 2x^3 + 7x^2 + x - 8.$
   (b) $f(x) = (x^3 - x^2 + x)(3x^2 + 2x - 9).$
   (c) $f(x) = \dfrac{x^2 - 3x + 2}{2x^2 + x + 1}.$
   (d) $f(x) = (3x^2 - 4x + 3)^{12}.$
6. Find the equation of the line tangent to the curve
$$y = \frac{x - 1}{x + 1}$$
   at the point corresponding to $x = 1$.
7. An object shot upward has height $x = -5t^2 + 30t$ meters after t seconds of flight. Compute its velocity after $\frac{3}{2}$ seconds, its maximum height, and the speed with which it strikes the ground.
8. For a certain function f.
$$f(0) = 5;$$
$$f'(0) = -2.$$
   If $g(x) = [f(x)]^2$, and $h(x) = [f(x)]^3$, find $g'(0)$ and $h'(0)$.

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1. Evaluate:

   (a) $\lim_{x\to 2} \frac{x^3 - 3x + 2}{2x^2 + x + 1}$

   (b) $\lim_{x\to 3} \frac{x-3}{2(x^2-9)}$

   (c) $\lim_{x\to 1} \frac{x^3 - 2x^2 - 5x + 6}{2x^3 - 7x + 5}$

   (d) $\lim_{x\to 0} \frac{x + \tan x}{\sin x}$

   Solution:

   (a)

$$\lim_{x\to 2} \frac{x^3 - 3x + 2}{2x^2 + x + 1} = \frac{8 - 6 + 2}{8 + 2 + 1} = \frac{4}{11}. \tag{1}$$

   (b)

$$\lim_{x\to 3} \frac{x-3}{2(x^2-9)} = \lim_{x\to 3} \left[\frac{1}{2} \cdot \frac{\cancel{x-3}}{\cancel{(x-3)}(x+3)}\right] = \frac{1}{12}. \tag{2}$$

   (c)

$$\lim_{x\to 1} \frac{x^3 - 2x^2 - 5x + 6}{2x^3 - 7x + 5} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2 - x - 6)}{\cancel{(x-1)}(2x^2 + 2x - 5)} = \frac{1 - 1 - 6}{2 + 2 - 5} = 6. \tag{3}$$

   (d)

$$\lim_{x\to 0} \frac{x + \tan x}{\sin x} = \lim_{x\to 0} \left[\frac{x}{\sin x} + \frac{\tan x}{\sin x}\right] \tag{4}$$

$$= \lim_{x\to 0} \frac{x}{\sin x} + \lim_{x\to 0} \left[\frac{1}{\cancel{\sin x}} \cdot \frac{\cancel{\sin x}}{\cos x}\right] = 1 + 1 = 2. \tag{5}$$

2. Find a value for c for which the function given below is continuous everywhere:

$$f(x) = \begin{cases} 3x - 2; & \text{if } x \le 1 \\ \\ cx + 3; & \text{if } 1 < x. \end{cases}$$

   Be sure to explain the reasoning, based on the definition of continuity, that leads to the solution.

   Solution: By the Limit Laws, $\lim_{x\to a}(3x - 2) = 3a - 2$ when $a < 1$, and $\lim_{x\to a} cx + 3 = ca + 3$, no matter what c may be, when $a > 1$. Thus, $\lim_{x\to a} f(x) = f(a)$ for all $a \neq 1$. Thus, the only point where continuity can depend on c is at $x = 1$. If we are to ensure continuity at $x = 1$, we must adjust the value of c so that the limit of $f(x)$ as $x \to 1$ exists and equals $f(1) = 1$. But

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3x - 2) = 1 = f(1), \text{while} \tag{6}$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (cx + 3) = c + 3. \tag{7}$$

   We will therefore need to fix the value of c so that $1 = c + 3$. We can accomplish this only by taking $c = -2$, so we conclude that f is everywhere continuous just when $c = -2$.

3. *Use the definition of the derivative as a limit* to find $f'(2)$ when f is given by:

$$f(x) = 3x^2 + x - 2.$$

Solution:

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{[3(2+h)^2 + (\cancel{2} + h) - \cancel{2}] - [3 \cdot 2^2 + \cancel{2 - 2}]}{h} \tag{8}$$

$$= \lim_{h\to 0} \frac{\cancel{3\cdot 4} + 3 \cdot 4h + 3h^2 + h - \cancel{3 \cdot 4}}{h} \tag{9}$$

$$= \lim_{h\to 0} \frac{(13 + 3h)\cancel{h}}{\cancel{h}} = 13. \tag{10}$$

We conclude that $f'(2) = 13$.

4. *Use the definition of the derivative* to find $f'(x)$ when f is given by:

$$f(x) = \frac{1}{\sqrt{x}}.$$

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \left[\frac{1}{h}\left(\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right)\right] \tag{11}$$

$$= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} = \lim_{h\to 0} \frac{\cancel{x} - (\cancel{x} + \cancelto{1}{h})}{\cancelto{1}{h}(\sqrt{x} + \sqrt{x+h})\sqrt{x}\sqrt{x+h}} \tag{12}$$

$$= -\frac{1}{2x^{3/2}}. \tag{13}$$

Thus, $f'(x) = -\dfrac{1}{2x^{3/2}}$.

5. Use whatever method seems to you to be wisest and simplest to find $f'(x)$ given that:

   (a) $f(x) = 3x^4 - 2x^3 + 7x^2 + x - 8$.

   (b) $f(x) = (x^3 - x^2 + x)(3x^2 + 2x - 9)$.

   (c) $f(x) = \dfrac{x^2 - 3x + 2}{2x^2 + x + 1}$.

   (d) $f(x) = (3x^2 - 4x + 3)^{12}$.

   Solution:

   (a) $f'(x) = 12x^3 - 6x^2 + 14x + 1$.

   (b) $f'(x) - (3x^2 - 2x + 1)(3x^2 + 2x - 9) + (x^3 - x^2 + x)(6x + 2)$.

   (c) $f'(x) = \dfrac{(2x - 3)(2x^2 + x + 1) - (x^2 - 3x + 2)(4x + 1)}{(2x^2 + x + 1)^2}$.

   (d) $f'(x) = 12(3x^2 - 4x + 3)^{11}(6x - 4)$.

6. Find the equation of the line tangent to the curve

$$y = \frac{x-1}{x+1}$$

at the point corresponding to $x = 1$.

Solution:

$$y' = \frac{d}{dx}\left[\frac{x-1}{x+1}\right] = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}, \text{ so} \tag{14}$$

$$y'\Big|_{x=2} = \frac{2}{9}. \tag{15}$$

Also, when $x = 2$, $y = \dfrac{2-1}{2+1} = \dfrac{1}{3}$. The desired tangent line is thus the line through the point $(2, 1/3)$ with slope $2/9$. An equation for that line is

$$y = \frac{1}{3} + \frac{2}{9}(x-2). \tag{16}$$

7. An object shot upward has height $x = -5t^2 + 30t$ meters after t seconds of flight. Compute its velocity after $\frac{3}{2}$ seconds, its maximum height, and the speed with which it strikes the ground.

Solution: If height of an object traveling straight upward in free flight is $x = -5t^2 + 30t$ meters after t seconds of flight, then its velocity at that time is $v = x' = -10t + 30$ meters per second. When $t = 3/2$, then, velocity is $v = -10 \cdot \dfrac{3}{2} + 30 = 15$ meters per second. The object reaches its maximum height when $v = 0$, or when $t = 3$ seconds. The object strikes the ground at the instant $t > 0$ when $x = 0$, or when $t > 0$ and $-5t^2 + 30t = 0$. Solving, we find that this instant is $t = 6$ seconds. When $t = 6$, $v = -10 \cdot 6 + 30 = -30$, so the object strikes the ground at 30 meters per second.

8. For a certain function f.

$$f(0) = 5;$$
$$f'(0) = -2.$$

If $g(x) = [f(x)]^2$, and $h(x) = [f(x)]^3$, find $g'(0)$ and $h'(0)$.

Solution: If $g(x) = [f(x)]^2$, then, by the Chain Rule, $g'(0) = 2f(0)f'(0) = 2 \cdot 5 \cdot (-2) = -20$. If $h(x) = [f(x)]^3$, then, also by the Chain Rule, $h'(0) = 3[f(0)]^2 f'(0) = 3 \cdot 5^2 \cdot (-2) = -150$.

MTH 141, Calculus I, Exam II **July 3, 1991**

Instructions: Work the following problems on your own paper. The way in which you present your thinking is at least as important as the answers you get. You must hand in your paper by 9:00pm.

1. Find $\dfrac{dy}{dx}$. (You need not simplify your answers.)

   (a) $$y = \sec^5 x$$

   (b) $$y = \sqrt{\frac{1-x}{1+x^2}}$$

   (c) $$x^{2/3} + y^{2/3} = a^{2/3}$$

   (d) $$x = t^2 - 1$$
   $$y = 3t^4 - t^2.$$

2. Find the linearizaiton , $L(x)$ of the function $y = \sqrt{1+x}$ at $x = 0$, and use it to find an approximate value for $\sqrt{\dfrac{21}{20}}$.

3. Use Newton's Method to find the positive root of the equation

$$\cos x - x = 0. \tag{1}$$

   Use $x_0 = 1$ as your initial guess, and continue until you are reasonably certain of three (3) digits to the right of the decimal point. Be sure to show the equations that govern your calculations. (And be sure that your calculator is in radian mode!)

4. The radius of a sphere is measured to an accuracy of 1%. Use differentials to estimate the maximum relative error in the computed volume of the sphere. (Volume of a sphere is given by $V = \dfrac{4}{3}\pi R^3$.

5. Locate all of the critical points of the function

$$f(x) = (x^2 - 3)^3 = x^6 - 9x^4 + 27x^2 - 27. \tag{2}$$

   Classify each of the points you have located as a local minimum, a local maximum, or neither, being sure to explain the basis for each classification.

6. Let $y = f(x)$ be the function whose graph appears in Figure 1. Which of the other graphs, shown in Figures 2–5, is the graph of $f'(x)$? Explain why each of the choices you have rejected can't be the correct choice.

7. Let $y = g(x)$ be the function whose graph appears in Figure 6. Which of the other graphs, shown in Figures 7–10, is the graph of $g''(x)$? Explain why each of the choices you have rejected can't be the correct choice.

8. Use the following information to sketch the graph of $y = f(x)$:

   (a) There are no easily identified symmetries.

   (b) $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = 1$.

   (c) $f(x) \to \infty$ as $x \to 4^+$ and as $x \to -2^-$.

   (d) $f(x) \to -\infty$ as $x \to 4^-$ and as $x \to -2^+$.

   (e) $f(0) = \frac{3}{8}$ and $f(1) = \frac{4}{9}$.

(f) $f'(x)$ exists for all x in $(-\infty, -2) \cup (-2, 4) \cup (4, \infty)$.

(g) $f'(x) > 0$ on $(-\infty, -2)$ and on $(-2, 1)$.

(h) $f'(x) < 0$ on $(1, 4)$ and on $(4, \infty)$.

(i) $f'(x) = 0$ only when $x = 1$.

(j) $f'(0) = \frac{5}{32}$.

(k) $f''(x) > 0$ on $(-\infty, -2)$ and on $(4, \infty)$.

(l) $f''(x) < 0$ on $(4, \infty)$.

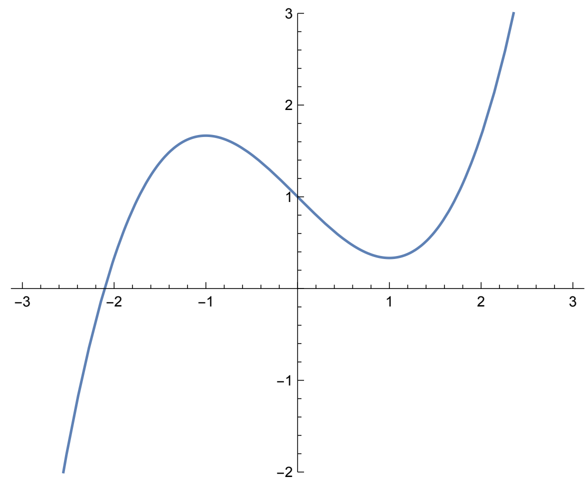


Figure 1: The function $y = f(x)$

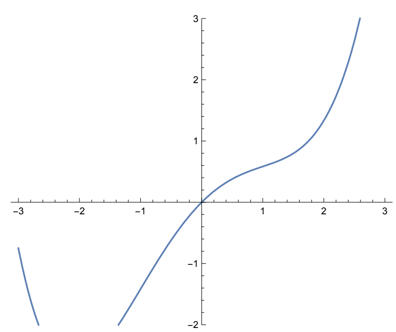


Figure 2: Is this the derivative?

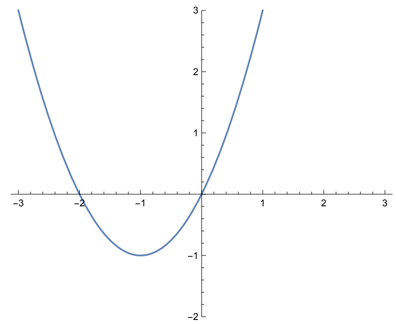


Figure 3: Maybe this is the derivative.

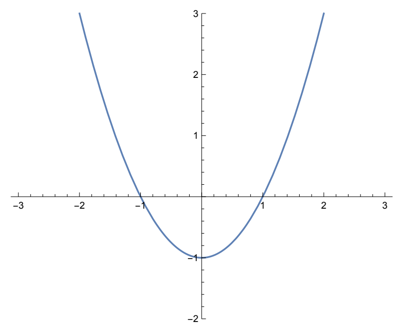


Figure 4: Could it be this one?

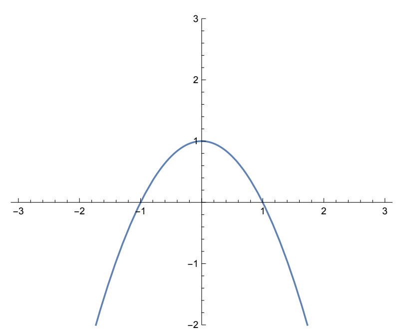


Figure 5: Or, maybe, this one?

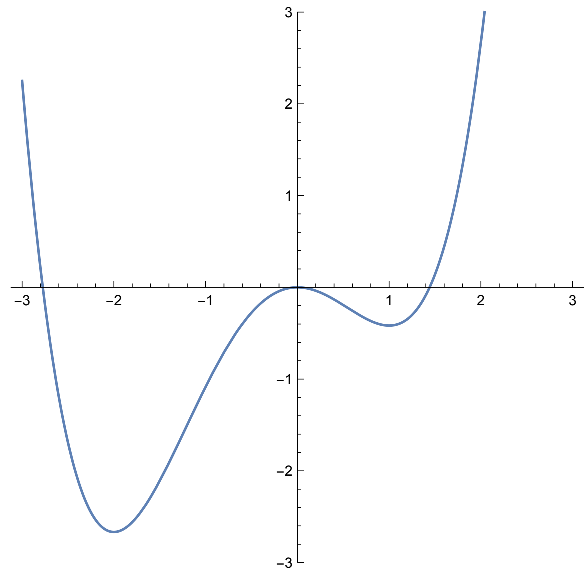


Figure 6: The function $y = g(x)$.

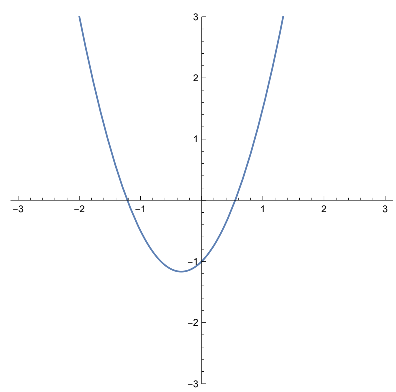


Figure 7: Is this the second derivative?

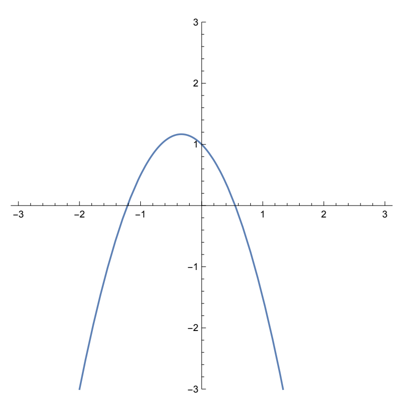


Figure 8: Maybe this is the second derivative.

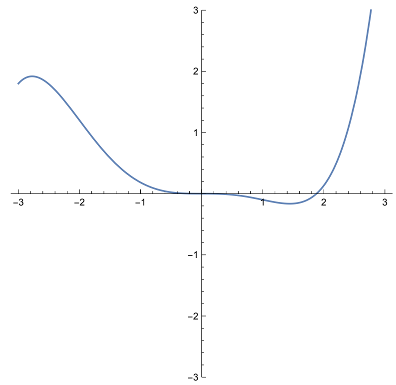


Figure 9: Could it be this one?

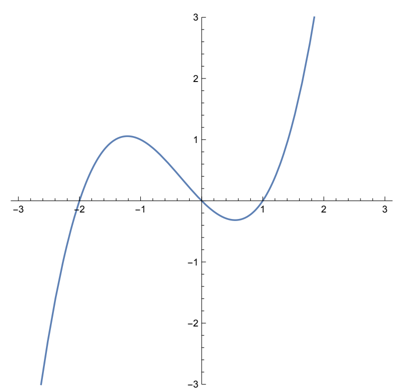


Figure 10: Or, maybe, this one?

MTH 141, Calculus I, Exam II **July 3, 1991**

Instructions: Work the following problems on your own paper. The way in which you present your thinking is at least as important as the answers you get. You must hand in your paper by 9:00pm.

1. Find $\frac{dy}{dx}$. (You need not simplify your answers.)

   (a)
   $$y = \sec^5 x$$

   (b)
   $$y = \sqrt{\frac{1-x}{1+x^2}}$$

   (c)
   $$x^{2/3} + y^{2/3} = a^{2/3}$$

   (d)
   $$x = t^2 - 1$$
   $$y = 3t^4 - t^2.$$

   Solution:

   (a) $y' = 5\sec^4 x \cdot \sec x \tan x = 5\sec^5 x \tan x.$

   (b) $y' = \frac{1}{2}\left(\frac{1-x}{1+x^2}\right)^{-1/2} \cdot \frac{-(1+x^2)-(1-x)\cdot 2x}{(1+x^2)^2}.$

   (c) $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}.$

   (d) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{12t^3 - 2t} = \frac{1}{6t^2-1}.$

2. Find the linearization , $L(x)$ of the function $y = \sqrt{1+x}$ at $x = 0$, and use it to find an approximate value for $\sqrt{\frac{21}{20}}$.

   Solution: If $y = \sqrt{1+x}$, then $y' = \frac{1}{2} \cdot \frac{1}{\sqrt{1+x}}$, so $L(x)$, the linearization of y at $x = 0$ is

   $$L(x) = \sqrt{1+0} + \frac{1}{2\sqrt{1+0}}(x-0) = 1 + \frac{1}{2}x, \tag{1}$$

   so a good approximation for the value of $\sqrt{\frac{21}{20}}$, which is the value of y when $x = \frac{1}{20}$, is

   $$L\left(\frac{1}{20}\right) = 1 + \frac{1}{2} \cdot \frac{1}{20} = \frac{41}{40} = 1.025. \tag{2}$$

3. Use Newton's Method to find the positive root of the equation

   $$\cos x - x = 0. \tag{3}$$

   Use $x_0 = 1$ as your initial guess, and continue until you are reasonably certain of three (3) digits to the right of the decimal point. Be sure to show the equations that govern your calculations. (And be sure that your calculator is in radian mode!)

Solution: Newton's Method for approximating a solution for an equation $F(x) = 0$ requires that we make an initial guess x_0 at the value of the solution and then iterate the calculation

$$x_{k+1} = x_k - \frac{F(x_k)}{F(x_k)} \tag{4}$$

until we are satisfied with the accuracy of the result. When $F(x) = \cos x - x$, we have $F'(x) = -\sin x - 1$, so we begin with $x_0 = 1$ and we compute

$$\begin{aligned} x_{k+1} &= x_k - \frac{\cos x_k - x_k}{-\sin x_k - 1} && (5) \\ &= \frac{x_k \sin x_k - \cancel{x_k} - \cos x_k + \cancel{x_k}}{-\sin x_k - 1} && (6) \\ &= \frac{\cos x_k - x_k \sin x_k}{1 + \sin x_k} && (7) \end{aligned}$$

Thus,

$$\begin{aligned} x_0 &= 1 && (8) \\ x_1 &= \frac{\cos 1 - \sin 1}{1 + \sin 1} \sim 0.750364; && (9) \\ x_2 &= \frac{\cos x_1 - \sin x_1}{1 + \sin x_1} \sim 0.739113 && (10) \\ x_3 &= \frac{\cos x_2 - \sin x_2}{1 + \sin x_2} \sim 0.739085 && (11) \\ x_4 &= \frac{\cos x_3 - \sin x_3}{1 + \sin x_3} \sim 0.739085 && (12) \end{aligned}$$

Further iteration appears to be unlikely to alter the first three digits to the right of the decimal, so our approximation, to three decimal places, is $x_3 = 0.739$.

4. The radius of a sphere is measured to an accuracy of 1%. Use differentials to estimate the maximum percentage error in the computed volume of the sphere. (Volume of a sphere is given by $V = \frac{4}{3}\pi R^3$.

   Solution: If $V = \frac{4}{3}\pi R^3$, then $dV = 4\pi R^2 dR$, so

$$\begin{aligned} \frac{dV}{V} &= \frac{\cancel{4}\cancel{\pi}\cancelto{1}{R^2}\, dR}{\cancel{\frac{4}{3}}\cancel{\pi}\cancelto{1}{R^3}} && (13) \\ &= 3\frac{dR}{R}. && (14) \end{aligned}$$

   It is given that $\left|\frac{dR}{R}\right| \leq \frac{1}{100}$, so we conclude that

$$\left|\frac{dV}{V}\right| = 3\left|\frac{dR}{R}\right| \leq \frac{3}{100}, \tag{15}$$

   or percentage error in V is at most 3%.

5. Locate all of the critical points of the function

$$f(x) = (x^2 - 3)^3 = x^6 - 9x^4 + 27x^2 - 27. \tag{16}$$

   Classify each of the points you have located as a local minimum, a local maximum, or neither, being sure to explain the basis for each classification.

Solution: The function f is everywhere differentiable, so the critical points of f are those where $f'(x) = 0$, or where

$$0 = 6x(x^2 - 3)^2 = 6x\left(x - \sqrt{3}\right)^2\left(x + \sqrt{3}\right)^2. \tag{17}$$

Thus, the critical points of f are at $x = -\sqrt{3}$, at $x = 0$, and at $x = \sqrt{3}$. Taking into account the signs of the factors of $f'(x)$ in (17), we see that $f'(x)$ is negative immediately to either side of of $x = -\sqrt{3}$, changes sign from negative to positive at $x = 0$, and is positive immediately to either side of $x = \sqrt{3}$. We conclude that the critical point at $x = -\sqrt{3}$ is neither a maximum nor a minimum, that the critical point at $x = 0$ gives a local minimum, and that the critical point at $x = \sqrt{3}$ gives neither a local minimum nor a local maximum.

6. Let $y = f(x)$ be the function whose graph appears in Figure 1.

   Which of the other graphs, shown in Figures 2–5, is the graph of $f'(x)$? Explain why each of the choices you have rejected can't be the correct choice.

   Solution: Figures 2, 3, and 5 all show $f'(x) > 0$ on the interval $(0, 1)$, where the graph of $y = f(x)$ is downward sloping and the derivative must be negative. Hence, Figure 4 is the correct choice.

7. Let $y = g(x)$ be the function whose graph appears in Figure 6.

   Which of the other graphs, shown in Figures 7–10, is the graph of $g''(x)$? Explain why each of the choices you have rejected can't be the correct choice.

   Solution: The curves of Figures 8–10 all lie above the x-axis in the interval $(-1, 0)$, where the function g is concave downward—so none of these curves can be the graph of $g''(x)$, which must therefore be the curve of Figure 7.

8. Use the following information to sketch the graph of $y = f(x)$:

   (a) There are no easily identified symmetries.

   (b) $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = 1$.

   (c) $f(x) \to \infty$ as $x \to 4^+$ and as $x \to -2^-$.

   (d) $f(x) \to -\infty$ as $x \to 4^-$ and as $x \to -2^+$.

   (e) $f(0) = \frac{3}{8}$ and $f(1) = \frac{4}{9}$.

   (f) $f'(x)$ exists for all x in $(-\infty, -2) \cup (-2, 4) \cup (4, \infty)$.

   (g) $f'(x) > 0$ on $(-\infty, -2)$ and on $(-2, 1)$.

   (h) $f'(x) < 0$ on $(1, 4)$ and on $(4, \infty)$.

   (i) $f'(x) = 0$ only when $x = 1$.

   (j) $f'(0) = \frac{5}{32}$.

   (k) $f''(x) > 0$ on $(-\infty, -2)$ and on $(4, \infty)$.

   (l) $f''(x) < 0$ on $(4, \infty)$.

   Solution: The curve has a horizontal asymptote at $y = 1$ and vertical asymptotes at $x = -2$ and at $x = 4$. It is increasing and concave upward on $(-\infty, -2)$, Increasing and concave downward on $(-2, 1)$, decreasing and concave downward on $(1, 4)$, and decreasing and concave upward on $(4, \infty$, It has a local maximum at $x = 1$, but no other local extremes. See Figure 11.

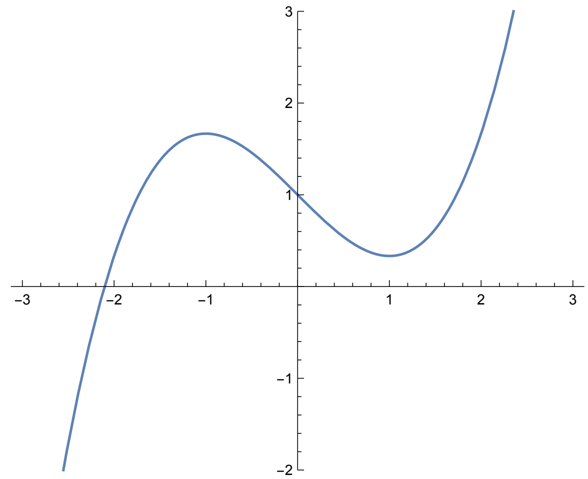


Figure 1: The function $y = f(x)$

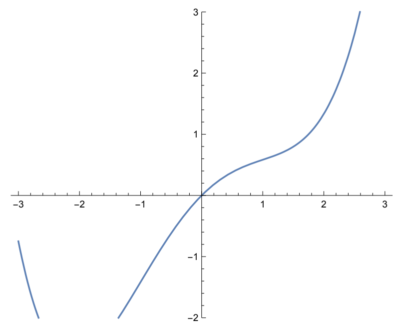


Figure 2: Is this the derivative?

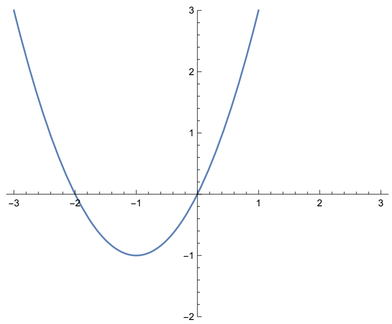


Figure 3: Maybe this is the derivative.

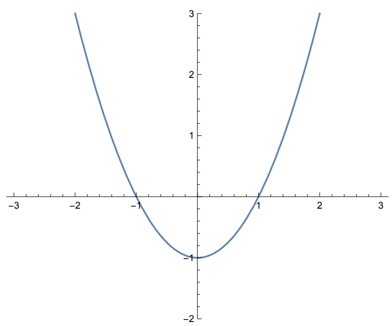


Figure 4: Could it be this one?

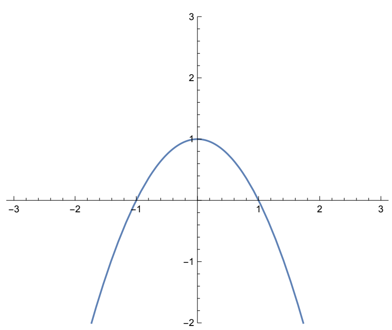


Figure 5: Or, maybe, this one?

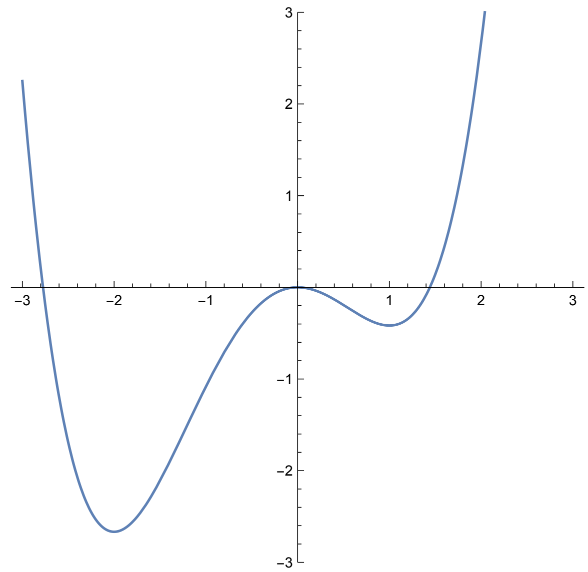


Figure 6: The function $y = g(x)$.

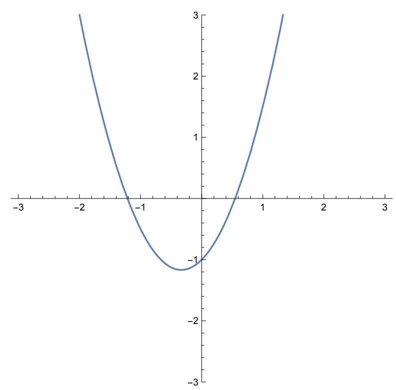


Figure 7: Is this the second derivative?

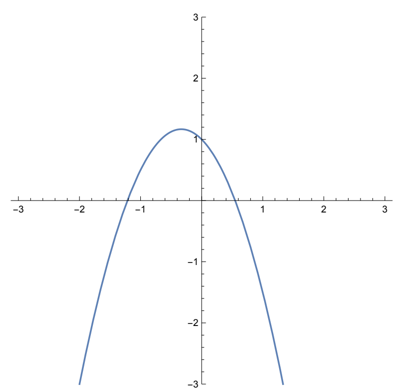


Figure 8: Maybe this is the second derivative.

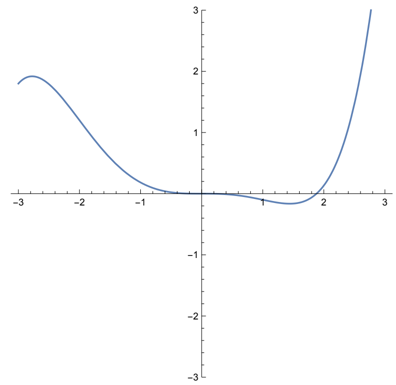


Figure 9: Could it be this one?

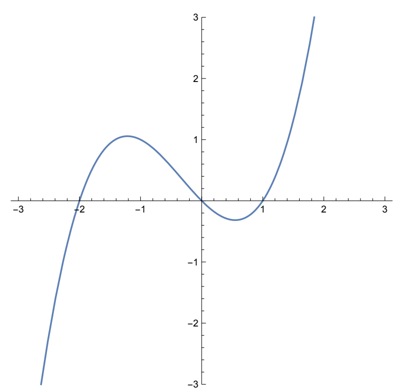


Figure 10: Or, maybe, this one?

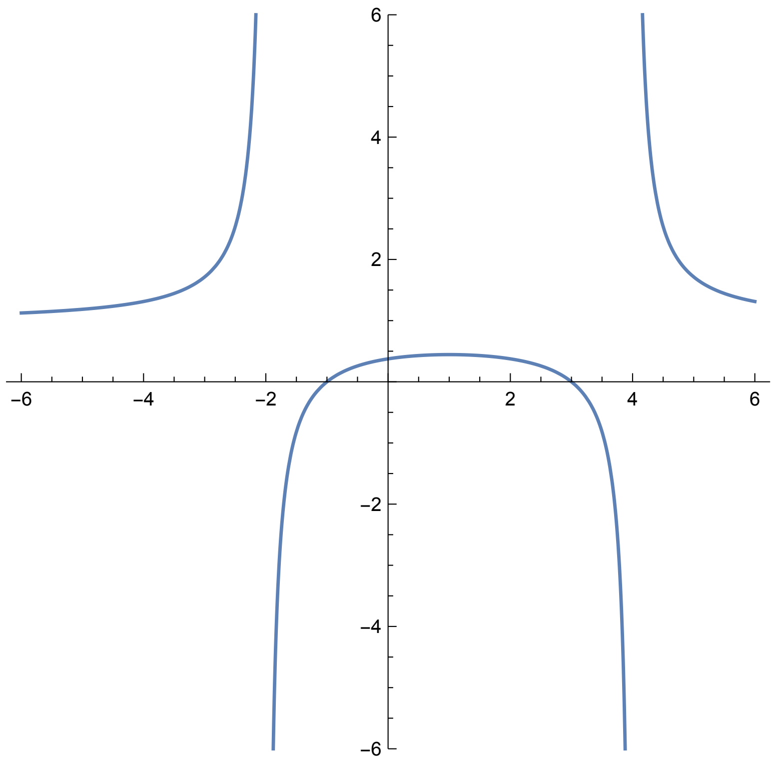


Figure 11: Problem 8.

MTH 141, Calculus I—Exam 3 **July 18, 1991**

Instructions: Work the following problems on your own paper. The way in which you present your thinking is a least as important as the answers you get. You must hand in your paper by 9:00pm.

1. Suppose that at time t, the position of a particle moving on the x-axis is

$$x = (t-2)(t-4).$$

   (a) When is the particle at rest?

   (b) During what time interval does the particle move to the left?

   (c) What is the fastest the particle goes while moving to the left?

2. The strength of a rectangular beam is proportional to the product of the width and the square of the depth, both measurements being made on the cross-section taken at right angles to the length of the beam. Find the dimensions of the strongest beam that can be cut from a circular cylindrical log of radius R.

3. A boat is pulled into a dock by a rope with one end attached to the bow of the boat, the other end passing through a ring attached to the dock at a point 6 feet higher than the bow of the boat. If the rope is pulled in at the rate of 2 feet/second, how fast is the boat approaching the dock when there are 10 feet of rope between the bow of the boat and the ring on the dock?

4. Find

   (a) $\displaystyle\lim_{x\to 1} \frac{x^3-1}{4x^3-x-3}$.

   (b) $\displaystyle\lim_{x\to 0} \frac{\sqrt{a(a+x)}-a}{x}$, given that $a>0$.

5. Estimate the maximum error introduced in replacing $f(x) = x\sqrt{x}$ with its linearization at $x=1$ on the interval $0.9 \le x \le 1.1$.

6. If $\dfrac{dy}{dx} = x\sqrt{y}$, and $y=1$ when $x=0$, find the relationship between x and y.

7. Find:

   (a) $\displaystyle\int \frac{s\,ds}{(s^2+1)^4}$.

   (b) $\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$.

8. Find:

   (a) $\displaystyle\int \cos(3t-1)\,dt$.

   (b) $\displaystyle\int \frac{\cos x\,dx}{\sin^3 x}$

MTH 141, Calculus I—Exam 3 Solutions **July 18, 1991**

Instructions: Work the following problems on your own paper. The way in which you present your thinking is a least as important as the answers you get. You must hand in your paper by 9:00pm.

1. Suppose that at time $t \geq 0$, the position of a particle moving on the x-axis is

$$x = (t-2)(t-4).$$

   (a) When is the particle at rest?

   (b) During what time interval does the particle move to the left?

   (c) What is the fastest the particle goes while moving to the left?

   Solutions:

   (a) The particle is at rest when $0 = x' = (t-4)+(t-2) = 2t-6$, or when $t = 3$.

   (b) The particle is moving to the left when $2t-6 = x' < 0$, or when $0 \leq t < 3$.

   (c) The particle's speed is $|x'| = |2t-6| = \sqrt{(2t-6)^2}$, and

$$\frac{d}{dt}|x'| = \frac{d}{dt}\left[(2t-6)^2\right]^{1/2} = \frac{1}{2}\left[(2t-6)^2\right]^{-1/2} \cdot 2(2t-6) \cdot 2 \tag{1}$$

$$= \frac{2(2t-6)}{\sqrt{(2t-6)^2}} = \frac{2(2t-6)}{|2t-6|}. \tag{2}$$

   This latter quantity is negative when $t < 3$, *i.e.,* when the particle moves to the left, so speed is decreasing during that time. Hence the particle goes the fastest ,while moving left, when $t = 0$.

2. The strength of a rectangular beam is proportional to the product of the width and the square of the depth, both measurements being made on the cross-section taken at right angles to the length of the beam. Find the dimensions of the strongest beam that can be cut from a circular cylindrical log of radius R.

   Solution: Place the origin of an xy-coordinate system at the center of a cross-section of the log so that the width of the log is parallel with the x-axis, and let (x, y) be the coordinates of the point where the corner of the beam in the first quadrant touches the edge of the log. Then $x^2 + y^2 = R^2$, the width of the beam is then $2x$, and its depth is then $2y$. The strength, S of the beam is then $S = kxy^2$, with $0 \leq x \leq R$, for some unknown constant k that depends on the units in use. We treat y as a function of x, and we observe that

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}R^2, \text{ or} \tag{3}$$

$$2x + 2y\frac{dy}{dx} = 0, \text{ whence} \tag{4}$$

$$\frac{dy}{dx} = -\frac{x}{y}. \tag{5}$$

   Now we expect that S will be maximal when $\frac{dS}{dx} = 0$. But we have

$$\frac{dS}{dx} = \frac{d}{dx}(kxy^2) = ky^2 + 2kxy\frac{dy}{dx}. \tag{6}$$

We combine (5) with (6) and equate the result with zero to find that

$$\not{k}y^2 + 2\not{k}x\not{y}\left(-\frac{x}{\not{y}}\right) = 0, \text{ or} \tag{7}$$

$$y^2 = 2x^2. \tag{8}$$

We combine this last equation with the equation for the surface of the log to find that

$$x^2 + 2x^2 = R^2; \tag{9}$$

$$x = \frac{R}{\sqrt{3}}, \tag{10}$$

where we have taken the positive square root because we want (x, y) to be in the first quadrant. Now, from the relation $y^2 = 2x^2$, and again requiring (x, y) to be in the first quadrant, we find that $y = \sqrt{\frac{2}{3}}R$. We conclude that the width of the beam must be $R/\sqrt{3}$, while its depth should be $\sqrt{2}R/\sqrt{3}$.

3. A boat is pulled into a dock by a rope with one end attached to the bow of the boat, the other end passing through a ring attached to the dock at a point 6 feet higher than the bow of the boat. If the rope is pulled in at the rate of 2 feet/second, how fast is the boat approaching the dock when there are 10 feet of rope between the bow of the boat and the ring on the dock?

   Solution: Let x be the distance from the boat to a vertical line through the ring on the deck, and let L denote the length of rope between the bow of the boat and the ring. From what is given, we see that x and L are two functions of time that always satisfy $x^2 + 36 = L^2$. Thus

$$\not{2}x\frac{dx}{dt} = \not{2}L\frac{dL}{dt}. \tag{11}$$

   We seek $\frac{dx}{dt}$ when $L = 10$ and $\frac{dL}{dt} = -2$. When $L = 10$ we have $x^2 + 36 = 100$, or $x = 8$. Thus

$$8\frac{dx}{dt} = 10(-2), \text{ or} \tag{12}$$

$$\frac{dx}{dt} = -\frac{5}{2} \text{ feet/second,} \tag{13}$$

   and the boat is moving toward that dock at two-and-a-half feet per second.

4. Find

   (a) $\lim_{x\to 1} \frac{x^3 - 1}{4x^3 - x - 3}$.

   (b) $\lim_{x\to 0} \frac{\sqrt{a(a+x)} - a}{x}$, given that $a > 0$.

   Solution:

   (a)

$$\lim_{x\to 1} \frac{x^3 - 1}{4x^3 - x - 3} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2 + x + 1)}{\cancel{(x-1)}(4x^2 + 4x + 3)} = \frac{3}{11}. \tag{14}$$

   (b)

$$\lim_{x\to 0} \frac{\sqrt{a(a+x)} - a}{x} = \lim_{x\to 0} \frac{\left[\sqrt{a(a+x)} - a\right]\left[\sqrt{a(a+x)} + a\right]}{x\left[\sqrt{a(a+x)} + a\right]} \tag{15}$$

$$= \frac{\cancel{a^2} + a\cancel{x} - \cancel{a^2}}{\cancel{x}\left[\sqrt{a(a+x)} + a\right]} = \frac{\not{a}}{2\not{a}} = \frac{1}{2}. \tag{16}$$

5. Estimate the maximum error introduced in replacing $f(x) = x\sqrt{x}$ with its linearization at $x = 1$ on the interval $0.9 \leq x \leq 1.1$.

Solution: We know that if L is the linearization of f at $x = x_0$, then

$$|f(x) - L(x)| \leq \frac{M}{2}(x - x_0)^2 \tag{17}$$

for all x in $I = [x_0 - \epsilon, x_0 + \epsilon]$, provided that $M \geq |f''(x)|$ for all x in I. Now $f''(x) = \frac{3}{4}x^{-1/2}$, which is a positive, decreasing function on $[0.9, 1.1]$. Consequently,

$$|f''(x)| = f''(x) \leq \frac{3}{4\sqrt{0.9}} < 0.8 \tag{18}$$

on $[0.9, 1.1]$, and we may take $M = 0.8$. Moreover, on that interval we certainly have $(x - x_0)^2 = (x-1)^2 \leq 0.01$. Putting all of these things together, we conclude that

$$|f(x) - L(x)| \leq \frac{1}{2} \cdot (0.8) \cdot (0.01) = 0.004, \tag{19}$$

so that we estimate that the desired maximum error is at most 0.004.

6. If $\dfrac{dy}{dx} = x\sqrt{y}$, and $y = 1$ when $x = 0$, find the relationship between x and y.

Solution: Suppose that the desired relation is given by $y = f(x)$, and choose an open interval, J, centered at 0. Because $y' = x\sqrt{y}$, we may write, for any t in J,

$$f'(t) = t\sqrt{f(t)}; \tag{20}$$

$$[f(t)]^{-1/2} f'(t) = t. \tag{21}$$

Thus, if x lies in J, we also have

$$\int_0^x [f(t)]^{-1/2} f'(t)\, dt = \int_0^x t\, dt. \tag{22}$$

In the integral on the left, we substitute $u = f(t)$. Then $du = f'(t)\, dt$, $u = f(0) = 1$ when $x = 0$, and $u = f(x)y$ when $u = x$. Thus, we have

$$\int_1^y u^{-1/2}\, du = \int_0^x t\, dt \tag{23}$$

$$2u^{1/2}\Big|_1^y = \frac{t^2}{2}\Big|_0^x \tag{24}$$

$$2\sqrt{y} = \frac{x^2}{2} + 2 \tag{25}$$

$$y = \left(\frac{x^2}{4} + 1\right)^2. \tag{26}$$

7. Find:

(a) $\displaystyle\int \frac{s\, ds}{(s^2+1)^4}$.

(b) $\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\, dx$.

Solution:

(a) We let $u = s^2 + 1$. Then $du = 2s\,ds$, or $s\,ds = \frac{1}{2}\,du$. Thus

$$\int \frac{s\,ds}{(s^2+1)^4} = \frac{1}{2}\int u^{-4}\,du = -\frac{1}{6}u^{-3} + c = -\frac{1}{6(s^2+1)^3} + c. \tag{27}$$

(b) Let $u = 1 + \sqrt{x}$. Then $du = \dfrac{dx}{2\sqrt{x}}$, or $\dfrac{dx}{\sqrt{x}} = 2\,du$. Thus,

$$\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx = \int \frac{1}{(1+\sqrt{x})^2}\frac{dx}{\sqrt{x}} \tag{28}$$

$$= 2\int u^{-2}\,du = -2u^{-1} + c = -\frac{2}{1+\sqrt{x}} + c. \tag{29}$$

8. Find:

(a) $\displaystyle\int \cos(3t-1)\,dt$.

(b) $\displaystyle\int \frac{\cos x\,dx}{\sin^3 x}$

Solution:

(a) Let $u = 3t - 1$. Then $du = 3\,dt$ so that $dt = \frac{1}{3}\,du$ and

$$\int \cos(3t-1)\,dt = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + c = \frac{1}{3}\sin(3t-1) + c. \tag{30}$$

(b) Let $u = \sin x$. Then $du = \cos x\,dx$, so

$$\int \frac{\cos x\,dx}{\sin^3 x} = \int u^{-3}\,du = -\frac{u^{-2}}{2} + c = -\frac{1}{2\sin^2 x} + c. \tag{31}$$

MTH 141, Calculus I, Final Exam **August 1, 1991**

Instructions: Work the following problems on your own paper. The way in which you present your thinking is at least as important as the answers you get. You must hand in your paper by 9:00pm.

1. Evaluate:

   (a) $\lim_{x\to 2} \dfrac{x^3 - 3x + 2}{2x^2 + x + 1}$

   (b) $\lim_{x\to 3} \dfrac{x - 3}{2(x^2 - 9)}$

   (c) $\lim_{x\to 1} \dfrac{x^3 - 2x^2 - 5x + 6}{2x^3 - 7x + 5}$

   (d) $\lim_{x\to 0} \dfrac{x + \tan x}{\sin x}$

2. *Use the definition of the derivative as a limit* to find $f'(2)$ when f is given by:

$$f(x) = 3x^2 + x - 2.$$

3. Find $\dfrac{dy}{dx}$. (You need not simplify your answers.)

   (a)
   $y = \sec^5 x$

   (b)
   $y = \sqrt{\dfrac{1 - x}{1 + x^2}}$

   (c)
   $x^{2/3} + y^{2/3} = a^{2/3}$

   (d)
   $x = t^2 - 1$
   $y = 3t^4 - t^2.$

4. Locate all of the critical points of the function

$$f(x) = (x^2 - 3)^3 = x^6 - 9x^4 + 27x^2 - 27. \qquad (1)$$

   Classify each of the points you have located as a local minimum, a local maximum, or neither, being sure to explain the basis for each classification.

5. A boat is pulled into a dock by a rope with one end attached to the bow of the boat, the other end passing through a ring attached to the dock at a point 6 ft higher than the bow of the boat. If the rope is pulled in at the rate of 2 ft/sec, how fast is the boat approaching the dock when there are 10 ft of rope between the bow of the boat and the ring on the dock?

6. Evaluate:

   (a) $\displaystyle\int_0^2 \frac{u\,du}{\sqrt{9 - u^2}}.$

   (b) $\displaystyle\int_0^{\pi/2} \sin^2 x \cos x\,dx.$

7. Find the area of the bounded region between the curves $y = x + 1$ and $y = x^2 - 4x + 5$.

8. Find the volume generated when the region bounded by $x = 0$, $x = 4$, the x-axis, and $y = 4x - x^2$ is revolved about the y-axis.

MTH 141, Calculus I, Final Exam Solutions **August 1, 1991**

Instructions: Work the following problems on your own paper. The way in which you present your thinking is at least as important as the answers you get. You must hand in your paper by 9:00pm.

1. Evaluate:

   (a) $\lim_{x\to 2} \frac{x^3-3x+2}{2x^2+x+1}$

   (b) $\lim_{x\to 3} \frac{x-3}{2(x^2-9)}$

   (c) $\lim_{x\to 1} \frac{x^3-2x^2-5x+6}{2x^3-7x+5}$

   (d) $\lim_{x\to 0} \frac{x+\tan x}{\sin x}$

   Solution:

   (a)

$$\lim_{x\to 2} \frac{x^3-3x+2}{2x^2+x+1} = \frac{8-6+2}{8+2+1} = \frac{4}{11}. \tag{1}$$

   (b)

$$\lim_{x\to 3} \frac{x-3}{2(x^2-9)} = \lim_{x\to 3}\left[\frac{1}{2}\cdot\frac{\cancel{x-3}}{\cancel{(x-3)}(x+3)}\right] = \frac{1}{12}. \tag{2}$$

   (c)

$$\lim_{x\to 0} \frac{x+\tan x}{\sin x} = \lim_{x\to 0}\left[\frac{x}{\sin x}+\frac{\tan x}{\sin x}\right] \tag{3}$$

$$= \lim_{x\to 0}\frac{x}{\sin x} + \lim_{x\to 0}\left[\frac{1}{\cancel{\sin x}}\cdot\frac{\cancel{\sin x}}{\cos x}\right] = 1+1 = 2. \tag{4}$$

2. *Use the definition of the derivative as a limit* to find $f'(2)$ when f is given by:

$$f(x) = 3x^2 + x - 2.$$

   Solution:

$$f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{]3(2+h)^2+(\cancel{2}+h)-\cancel{2}] - [3\cdot 2^2 + \cancel{2-2}]}{h} \tag{5}$$

$$= \lim_{h\to 0}\frac{\cancel{3\cdot 4}+3\cdot 4h+3h^2+h-\cancel{3\cdot 4}}{h} \tag{6}$$

$$= \lim_{h\to 0}\frac{(13+3h)\cancel{h}}{\cancel{h}} = 13. \tag{7}$$

   We conclude that $f'(2) = 13$.

3. Find $\frac{dy}{dx}$. (You need not simplify your answers.)

   (a)

   $y = \sec^5 x$

   (b)

   $y = \sqrt{\frac{1-x}{1+x^2}}$

(c)
$$x^{2/3} + y^{2/3} = a^{2/3}$$

(d)
$$x = t^2 - 1$$
$$y = 3t^4 - t^2.$$

Solution:

(a) $y' = 5\sec^4 x \cdot \sec x \tan x = 5\sec^5 x \tan x$.

(b) $y' = \dfrac{1}{2}\left(\dfrac{1-x}{1+x^2}\right)^{-1/2} \cdot \dfrac{-(1+x^2)-(1-x)\cdot 2x}{(1+x^2)^2}$.

(c) $\dfrac{2}{3}x^{-1/3} + \dfrac{2}{3}y^{-1/3}\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{y^{1/3}}{x^{1/3}}$.

(d) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t}{12t^3 - 2t} = \dfrac{1}{6t^2-1}$.

4. Locate all of the critical points of the function

$$f(x) = (x^2-3)^3 = x^6 - 9x^4 + 27x^2 - 27. \tag{8}$$

Classify each of the points you have located as a local minimum, a local maximum, or neither, being sure to explain the basis for each classification.

Solution: The function f is everywhere differentiable, so the critical points of f are those where $f'(x) = 0$, or where

$$0 = 6x(x^2-3)^2 = 6x\left(x-\sqrt{3}\right)^2\left(x+\sqrt{3}\right)^2. \tag{9}$$

Thus, the critical points of f are at $x = -\sqrt{3}$, at $x = 0$, and at $x = \sqrt{3}$. Taking into account the signs of the factors of $f'(x)$ in (9), we see that $f'(x)$ is negative immediately to either side of of $x = -\sqrt{3}$, changes sign from negative to positive at $x = 0$, and is positive immediately to either side of $x = \sqrt{3}$. We conclude that the critical point at $x = -\sqrt{3}$ is neither a maximum nor a minimum, that the critical point at $x = 0$ gives a local minimum, and that the critical point at $x = \sqrt{3}$ gives neither a local minimum nor a local maximum.

5. A boat is pulled into a dock by a rope with one end attached to the bow of the boat, the other end passing through a ring attached to the dock at a point 6 ft higher than the bow of the boat. If the rope is pulled in at the rate of 2 ft/sec, how fast is the boat approaching the dock when there are 10 ft of rope between the bow of the boat and the ring on the dock?

Solution: Let L denote the length of rope between the bow of the boat and the ring on the dock. Let x denote the distance from the boat to the vertical line that passes through the ring. Then we are given that $\dfrac{dL}{dt} = -2$ feet per second, and it follows from the Pythagorean theorem that $L^2 = x^2 + 36$. Differentiating this latter equation implicitly with respect to time, we get

$$2L\frac{dL}{dt} = 2x\frac{dx}{dt}. \tag{10}$$

Solving and substituting what has been given,

$$\frac{dx}{dt} = \frac{L}{x}\cdot\frac{dL}{dt} \tag{11}$$
$$= \frac{10}{\sqrt{10^2-35}}(-2) = -\frac{20}{8} = -\frac{5}{2}. \tag{12}$$

and we conclude that the boat is approaching the dock at $\dfrac{5}{2}$ feet per second at the instant when there are 10 feet of rope between the bow of the boat and the ring on the dock.

6. Evaluate:

   (a) $\displaystyle\int_0^2 \frac{u\,du}{\sqrt{9-u^2}}$.

   (b) $\displaystyle\int_0^{\pi/2} \sin^2 x \cos x\,dx$.

   Solution:

   (a) Let $v = 9-u^2$. Then $dv = -2u\,du$, or $u\,du = -\dfrac{dv}{2}$, Moreover, $v = 9$ when $u = 0$, and $v = 5$ when $u = 2$. Thus,

$$\int_0^2 \frac{u\,du}{\sqrt{9-u^2}} = -\frac{1}{2}\int_9^5 v^{-1/2}\,dv \tag{13}$$

$$= -v^{1/2}\Big|_9^5 = 3-\sqrt{5}. \tag{14}$$

   (b) Let $u = \sin x$. Then $du = \cos x\,dx$, $u = 0$ when $x = 0$, and $u = 1$ when $x = \pi/2$. So

$$\int_0^{\pi/2} \sin^2 x \cos x\,dx = \int_0^1 u^2\,du = \frac{u^3}{3}\Big|_0^1 = \frac{1}{3}. \tag{15}$$

7. Find the area of the bounded region between the curves $y = x+1$ and $y = x^2-4x+5$.

   Solution: The two curves cross when

$$x^2-4x+5 = x+1, \tag{16}$$

$$x^2-5x+4 = 0, \tag{17}$$

$$(x-1)(x-4) = 0, \tag{18}$$

   or at $x = 1$ and at $x = 4$, when $y = 2$ and $y = 5$, respecitively. The quadratic curve is a parabola opening upward, and the other curve is a straight line. Therefore the line lies above the parabola on $(1,4)$, and the required area is

$$\int_1^4 \left(-x^2+5x-4\right)\,dx = \left(-\frac{x^3}{3}+\frac{5x^2}{2}-4x\right)\Big|_1^4 \tag{19}$$

$$= \left(-\frac{64}{3}+\frac{80}{2}-16\right)-\left(-\frac{1}{3}+\frac{5}{2}-4\right) = \frac{9}{2}, \tag{20}$$

8. Find the volume generated when the region bounded by $x = 0$, $x = 4$, the x-axis, and $y = 4x-x^2$ is revolved about the y-axis.

   Solution: We use the method of washers, which are what we get when we intersect planes perpendicular to the y-axis with the solid. First, we solve for x in terms of y:

$$y = 4x-x^2; \tag{21}$$

$$x^2-4x+y = 0; \tag{22}$$

$$x = \frac{4\pm\sqrt{16-4y}}{2} = 2\pm\sqrt{4-y}, \tag{23}$$

with $0 \leq y \leq 4$. Hence the outer radius of the washer formed by such an intersection is $2+\sqrt{4-y}$, while the inner radius of $2-\sqrt{4-y}$. So the required volume is

$$\pi \int_0^4 [(2+\sqrt{4-y})^2 - (2-\sqrt{4-y})^2)]\,dy = \pi \int_0^4 [(4+4\sqrt{4-y}+y^2) - (4-4\sqrt{4-y}+y^2)]\,dy \quad (24)$$

$$= 8\pi \int_0^4 \sqrt{4-y}\,dy = -\frac{16}{3}\pi(4-y)^{3/2}\Big|_0^4 = \frac{128}{3}\pi. \quad (25)$$

The required volume is thus $\frac{128}{3}\pi$.

Alternate Solution: The problem can also be solved by the method of cylindrical shells. The necessary integral is $2\pi \int_0^4 x(4x-x^2)\,dx$. The details of the integration are left to the reader, whom we assure that the outcome is the same.

MTH 141, Calculus I—Exam I **Feb. 17, 1995**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions.* Your paper is due at 12:55 pm.

1. Here (Figure 1)is the graph of a function $y = f(x)$:

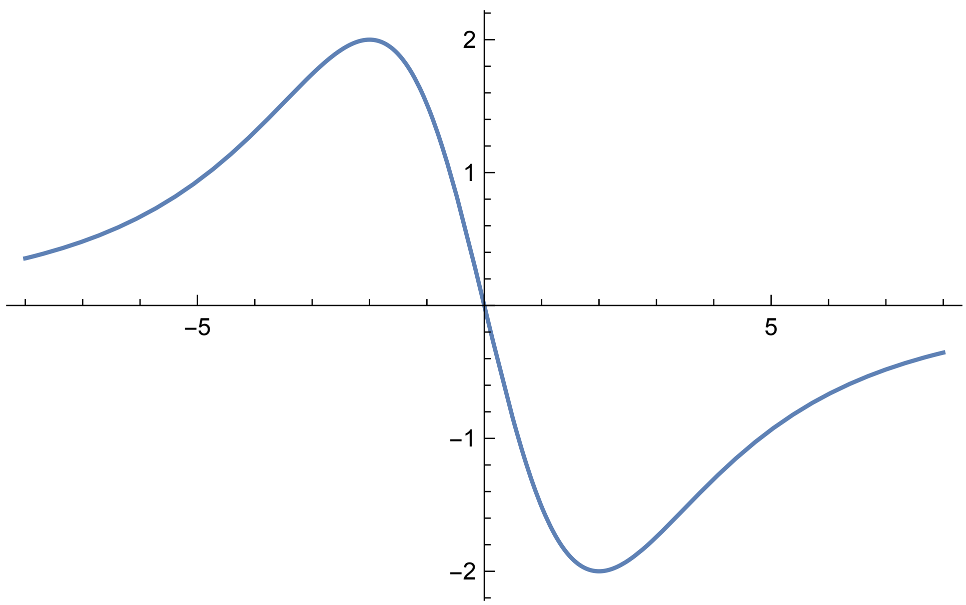


Figure 1: $y = f(x)$

Sketch the graphs of the following:

(a) $y = f(x - 2)$

(b) $y = f(2x)$

(c) $y = \frac{3}{2} f(x)$

(d) $y = f(x) - 3$

2. Show how to evaluate the following limits without using a calculator.

(a) $\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3}$

(b) $\lim_{x \to -2} \frac{x^3 + 2x^2 + 4x + 8}{x^2 + 8x + 12}$

3. Find $f'(x)$ by evaluating a limit of the form

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

when

(a) $f(x) = 3x^2 - x$

(b) $f(x) = \dfrac{1}{x^2}$

4. Find $f'(x)$ when

(a) $f(x) = x^5 - 3x^4 + 12x^2 - 3x + 5.$

(b) $f(x) = (x^3 - 2x^2 + 7x - 3)(x^4 + 2x^2 - 11x - 8).$

5. Find an equation for the line tangent to the curve $y = x^3 - 2x^2 + 3x - 1$ at the point $(1, 1)$.

MTH 141, Calculus I—Exam I **Feb. 17, 1995**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions.* Your paper is due at 12:55 pm.

1. Here (Figure 1)is the graph of a function $y = f(x)$:

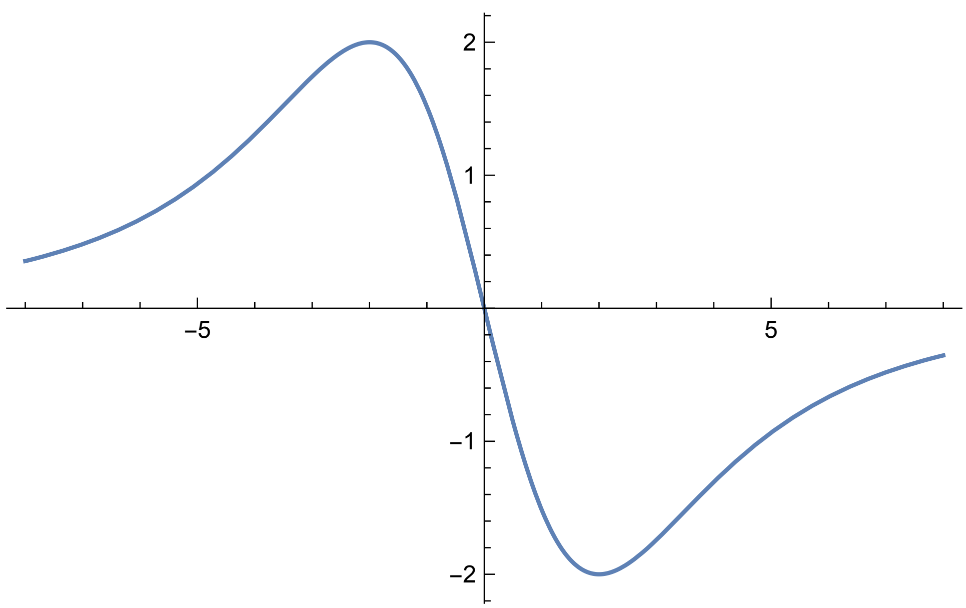


Figure 1: $y = f(x)$

Sketch the graphs of the following:

(a)

$$y = f(x - 2)$$

(b)

$$y = f(2x)$$

(c)

$$y = \frac{3}{2}f(x)$$

(d)

$$y = f(x) - 3$$

Solution:

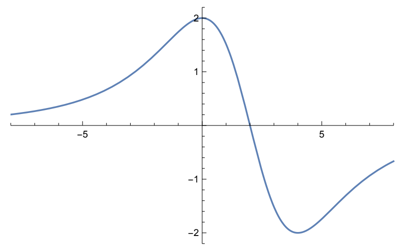


Figure 2: $y = f(x-2)$

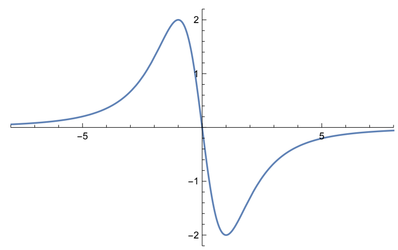


Figure 3: $y = f(2x)$

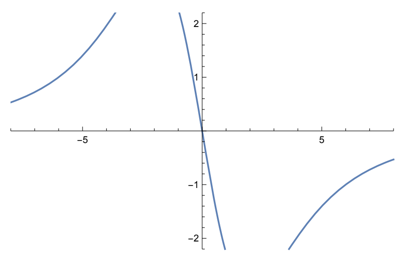


Figure 4: $y = \frac{3}{2}f(x)$

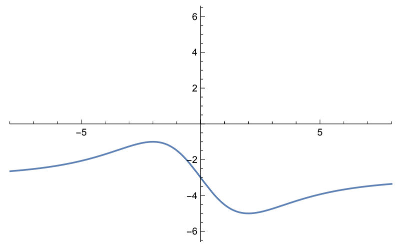


Figure 5: $y = f(x) - 3$

2. Show how to evaluate the following limits without using a calculator.

   (a)

$$\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3}$$

   (b)

$$\lim_{x \to -2} \frac{x^3 + 2x^2 + 4x + 8}{x^2 + 8x + 12}$$

   Solution:

   (a)

$$\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3} = \lim_{x \to 3} \frac{(\cancel{(x - 3)}(x - 2)}{\cancel{x - 3}} \tag{1}$$

$$= \lim_{x \to 3} (x - 2) = 1 \tag{2}$$

   (b)

$$\lim_{x \to -2} \frac{x^3 + 2x^2 + 4x + 8}{x^2 + 8x + 12} = \lim_{x \to -2} \frac{\cancel{(x + 2)}(x^2 + 4)}{\cancel{(x + 2)}(x + 6)} \tag{3}$$

$$= \frac{8}{4} = 2. \tag{4}$$

3. Find $f'(x)$ by evaluating a limit of the form

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

   when

(a)

$$f(x) = 3x^2 - x$$

(b)

$$f(x) = \frac{1}{x^2}$$

Solution:

(a)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && (5)\\
&= \lim_{h\to 0} \frac{3(x+h)^2 - (x+h) - [3x^2 - x]}{h} && (6)\\
&= \lim_{h\to 0} \frac{\cancel{3x^2} + 6x\cancel{h} + h^{\cancelto{1}{2}} - \cancel{x} - \cancelto{1}{h} - \cancel{3x^2} + \cancel{x}}{\cancelto{1}{h}} && (7)\\
&= \lim_{h\to 0} (6x + h - 1) = 6x - 1. && (8)
\end{aligned}$$

(b)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && (9)\\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(x+h)^2} - \frac{1}{x^2}\right] && (10)\\
&= \lim_{h\to 0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2} = \lim_{h\to 0} \frac{\cancel{x^2} - \cancel{x^2} - 2x\cancelto{1}{h} - h^{\cancelto{1}{2}}}{\cancelto{h}{h^2} x^2 (x+h)^2} \\
&= \lim_{h\to 0} \frac{-2x - h}{x^2(x+h)^2} = -\frac{2\cancelto{1}{x}}{x^{\cancelto{3}{4}}} = -\frac{2}{x^3}. && (11)
\end{aligned}$$

4. Find $f'(x)$ when

(a)

$$f(x) = x^5 - 3x^4 + 12x^2 - 3x + 5.$$

(b)

$$f(x) = (x^3 - 2x^2 + 7x - 3)(x^4 + 2x^2 - 11x - 8).$$

Solution:

(a) $f'(x) = 5x^4 - 12x^3 + 24x - 3.$

(b) $f'(x) = (x^4 + 2x^2 - 11x - 8) \cdot D_x[x^3 - 2x^2 + 7x - 3] +$
$(x^3 - 2x^2 + 7x - 3) \cdot D_x[x^4 + 2x^2 - 11x - 8] =$
$(x^4+2x^2-11x-8)(3x^2-4x+7)+(x^3-2x^2+7x-3)(4x^3+4x-11).$

5. Find an equation for the line tangent to the curve $y = x^3-2x^2+3x-1$ at the point $(1, 1)$.

Solution: $f'(x) = 3x^2 - 4x + 3$ so $f'(1) = 3 \cdot 1^2 - 4 \cdot 1 + 3 = 2$. Consequently the desired tangent line is the line that passes through the point $(1, 1)$ with slope 2. An equation for this line is

$$y = 1 + 2(x - 1), \text{ or} \quad (12)$$
$$y = 2x - 1. \quad (13)$$

MTH 141, Calculus I—Exam II **Mar. 31, 1995**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions*. Your paper is due at 12:55 pm.

1. Each of Figures 1 through 5 displays a pair of curves. In each case, one curve is that determined by a function f while the other is the curve determined by the derivative f' of the same function. For each of the five figures, explain which curve is f and which curve is f'. (No credit without the explanations.)

2. Each of Figures 6 through 10 displays a pair of curves. In each case, one curve is that determined by a function f while the other is the curve determined by the second derivative f'' of the same function. For each of the five figures, explain which curve is f and which curve is f''. (No credit without the explanations.)

3. Find the largest and smallest values taken on by the function

$$f(x) = x^3 - 9x^2 + 3x + 45$$

when x is subject to the restriction $0 \leq x \leq 8$. Be sure to explain your reasoning.

4. Take $x_1 = 0$ as your initial approximation to the solution of the equation

$$x^3 + 2x - 1 = 0$$

and use Newton's Method to find x_2 and x_3. Either work with exact fractions or maintain a minimum accuracy of five digits to the right of the decimal point.

5. (a) Find $\left.\dfrac{dy}{dx}\right|_{(3,2)}$ if

$$x^2 + xy + y^3 = 23. \qquad (1)$$

(b) Write the equation of the line tangent to the curve defined by equation (1) at the point $(3, 2)$.

(c) Explain how to use the result of Problem 5a or Problem 5b to find an approximate value of y in Equation (1) when $x = 3.100$. What approximate value for y does your method lead to? (Give at least three digits to the right of the decimal.)

Figure 1

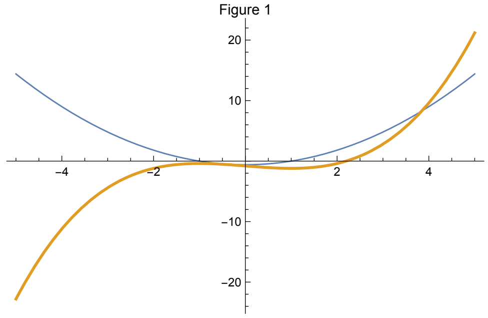


Figure 2

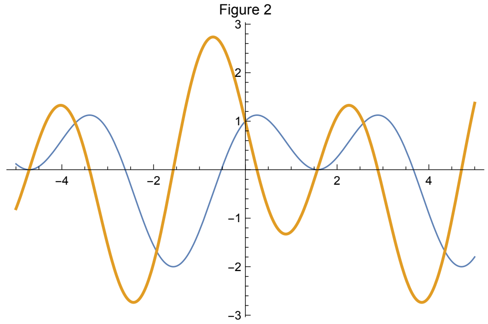


Figure 3

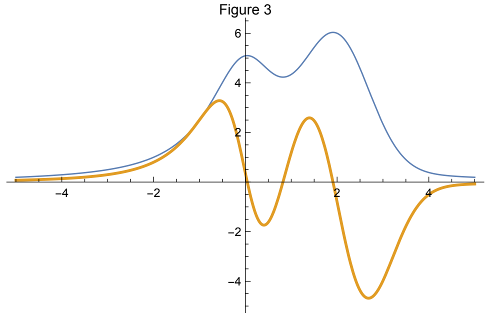


Figure 4

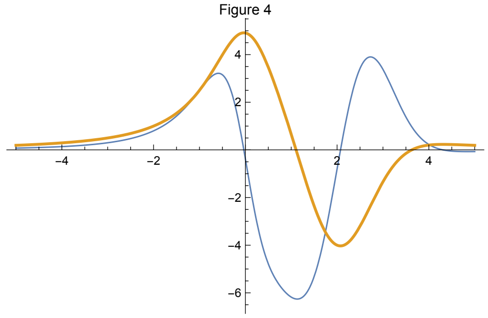


Figure 5

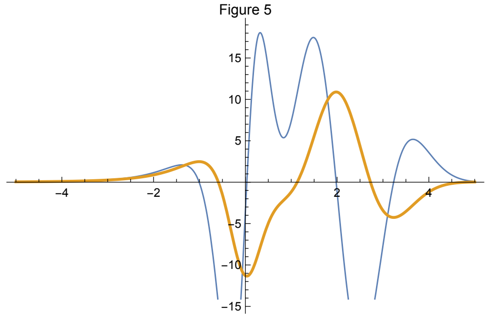


Figure 6

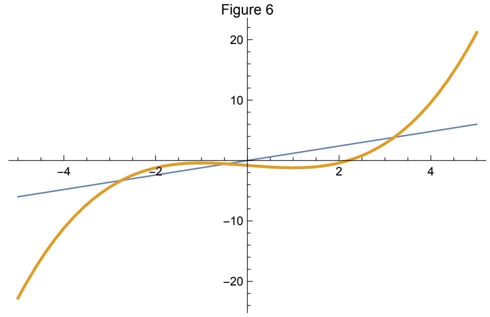


Figure 7

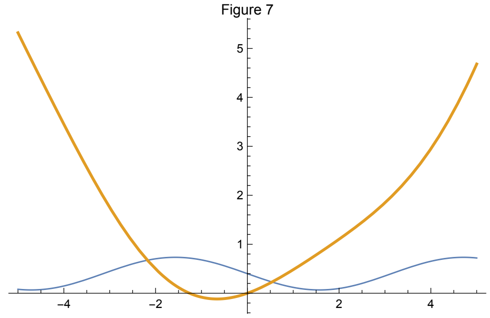


Figure 8

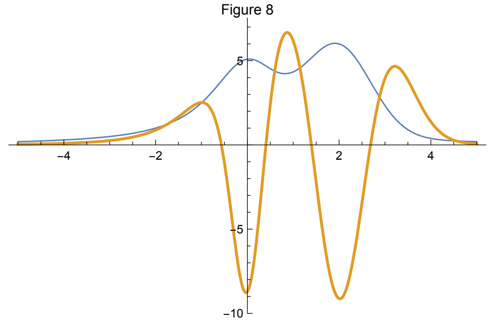


Figure 9

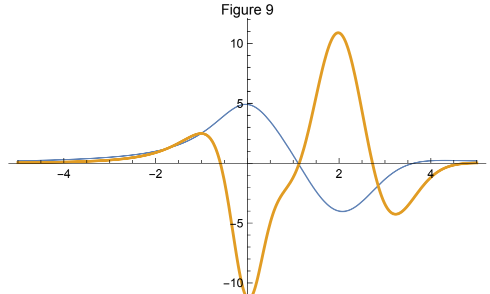
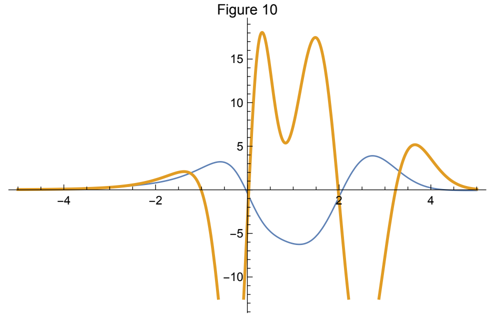


Figure 10



MTH 141, Calculus I—Exam II **Mar. 31, 1995**

Don't forget to update the exam header. **Instructions:** Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions*. Your paper is due at 12:55 pm.

1. Each of Figures 1 through 5 displays a pair of curves. In each case, one curve is that determined by a function f while the other is the curve determined by the derivative f' of the same function. For each of the five figures, explain which curve is f and which curve is f'. (No credit without the explanations.)

   Solution:

Figure 01: The orange curve is f, and the blue curve is f' because the orange curve

i. slopes upward where the blue curve is above the x-axis,
ii. has a horizontal tangent at the points where the blue curve crosses the x axis,
iii. slopes downward where the blue curve is below the x-axis.

Figure 02: The blue curve is f, and the orange curve is f' because the blue curve

i. slopes upward where the orangee curve is above the x-axis,
ii. has a horizontal tangent at the points where the orange curve crosses the x axis,
iii. slopes downward where the orange curve is below the x-axis.

Figure 03: The blue curve is f, and the orange curve is f' because the blue curve

i. slopes upward where the orange curve is above the x-axis,
ii. has a horizontal tangent at the points where the orange curve crosses the x axis,
iii. slopes downward where the orange curve is below the x-axis.

Figure 04: The orange curve is f, and the blue curve is f' because the orange curve

i. slopes upward where the blue curve is above the x-axis,
ii. has a horizontal tangent at the points where the blue curve crosses the x axis,
iii. slopes downward where the blue curve is below the x-axis.

Figure 05: The orange curve is f, and the blue curve is f' because the orange curve

i. slopes upward where the blue curve is above the x-axis,
ii. has a horizontal tangent at the points where the blue curve crosses the x axis,
iii. slopes downward where the blue curve is below the x-axis.

2. Each of Figures 6 through 10 displays a pair of curves. In each case, one curve is that determined by a function f while the other is the curve determined by the second derivative f'' of the same function. For each of the five figures, explain which curve is f and which curve is f''. (No credit without the explanations.)

Solution:

Figure 06: The orange curve is f, and the blue curve is f'' because the orange curve

i. is concave upward where the blue curve is above the x-axis,

ii. is concave downward where the blue curve is below the x-axis.

Figure 07: The orange curve is f, and the blue curve is f'' because the orange curve is concave upward everywhere, and the blue curve is always above the x-axis. However, the blue curve is concave upward in a region (at the far right of the graph) where the orange curve is positive.

Figure 08: The blue curve is f, and the orange curve is f'' because the blue curve

i. is concave upward where the orange curve is above the x-axis,

ii. is concave downward where the orange curve is below the x-axis.

Figure 09: The blue curve is f, and the orange curve is f'' because the blue curve

i. is concave upward where the orange curve is above the x-axis,

ii. is concave downward where the orange curve is below the x-axis.

Figure 10: The blue curve is f, and the orange curve is f'' because the blue curve

i. is concave upward where the orange curve is above the x-axis,

ii. is concave downward where the orange curve is below the x-axis.

3. Find the largest and smallest values taken on by the function

$$f(x) = x^3 - 9x^2 + 3x + 45$$

when x is subject to the restriction $0 \leq x \leq 8$. Be sure to explain your reasoning.

Solution: The given function is differentiable, and the absolute extremes of a differentiable function in a closed interval are to be found at points which are either critical points of the function or endpoints of the interval. We have

$$f'(x) = 3x^2 - 18x + 3, \tag{1}$$

By the Quadratic Formula, $f'(x) = -$ when $x = 3 \pm 2\sqrt{2}$. Computing, we find that

$$f(0) = 45; \tag{2}$$
$$f(3 - 2\sqrt{2}) = 32\sqrt{2} \sim 45.2548^+; \tag{3}$$
$$f(3 + 2\sqrt{2}) = -32\sqrt{2}] \sim -45.2548^- \tag{4}$$
$$f(8) = 5. \tag{5}$$

The largest value taken on by $f(x)$ in $[0, 8]$ is thus $f(3 + 2\sqrt{2}) = 32\sqrt{2}] \sim 45.2548^+$, while the smallest is $f(3 + 2\sqrt{2}) = -32\sqrt{2}] \sim -45.2548^-$.

4. Take $x_1 = 0$ as your initial approximation to the solution of the equation

$$x^3 + 2x - 1 = 0$$

and use Newton's Method to find x_2 and x_3. Either work with exact fractions or maintain a minimum accuracy of five digits to the right of the decimal point.

Solution: We let $x_1 = 0$, and apply the Newton iteration

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^3 + 2x_k - 1}{3x_k^2 + 2} \tag{6}$$

to find that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{-1}{2} = \frac{1}{2} = 0.5; \tag{7}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{1}{2} - \frac{\frac{1}{8} + 1 - 1}{\frac{3}{4} + 2} = \frac{5}{11} \sim 0.454545^+. \tag{8}$$

5. (a) Find $\left.\frac{dy}{dx}\right|_{(3,2)}$ if

$$x^2 + xy + y^3 = 23. \tag{9}$$

(b) Write the equation of the line tangent to the curve defined by equation (9) at the point $(3, 2)$.

(c) Explain how to use the result of Problem 5a or Problem 5b to find an approximate value of y in Equation (9) when $x = 3.100$. What approximate value for y does your method lead to? (Give at least three digits to the right of the decimal.)

Solution:

(a) We treat y as a function of f and differentiate implicitly:

$$\frac{d}{dx}(x^2 + xy + y^3) = \frac{d}{dx}23; \tag{10}$$

$$2x + y + xy' + 3y^2y' = 0; \tag{11}$$

$$y' = -\frac{2x + y}{x + 3y^2}. \tag{12}$$

Thus

$$\left.y'\right|_{(3,2)} = -\frac{8}{15}. \tag{13}$$

(b) We need the equation of the line through the point $(3, 2)$ with slope $-8/15$. The required equation is thus

$$y = 2 - \frac{8}{15}(x - 3). \tag{14}$$

(c) The ordinates of points on a tangent line whose abscissas are near x_0 are good approximations to the corrsponding ordinates of points on the curve whose tangent line we use. Thus, when $x = 3.100$, the ordinate of the corresponding point on the curve is close to

$$y = 2 - \frac{8}{15}(3.100 - 3.000) \tag{15}$$

$$= 2 - \frac{4}{75} = \frac{146}{75} \sim 1.94667. \tag{16}$$

Figure 1

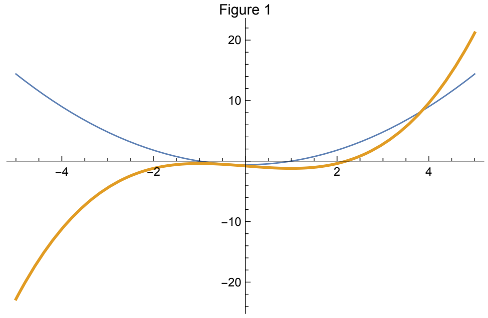


Figure 2

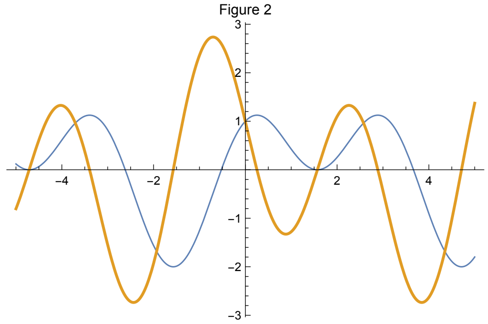


Figure 3

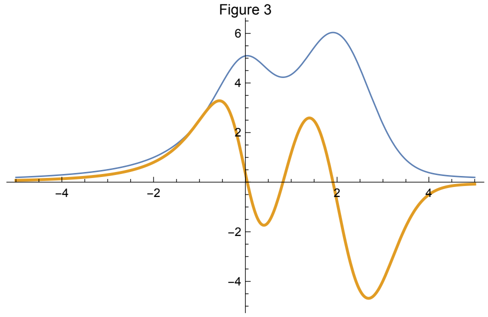


Figure 4

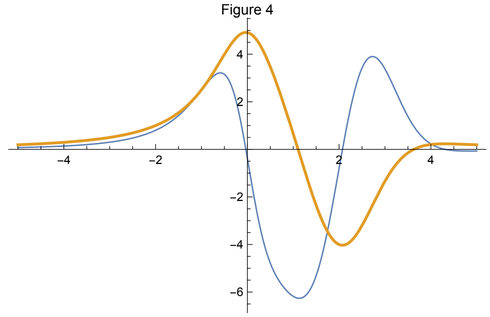


Figure 5

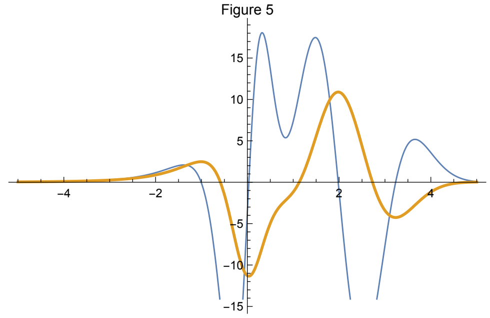


Figure 6

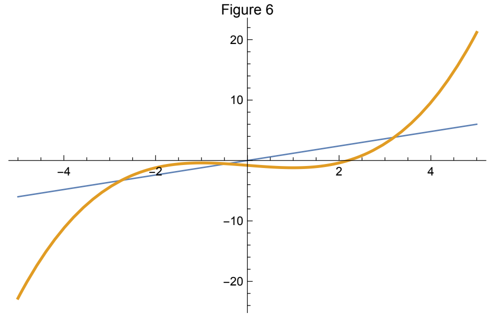


Figure 7

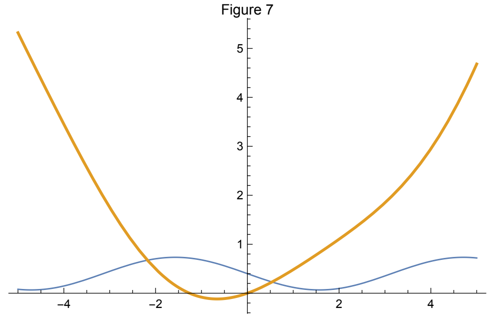


Figure 8

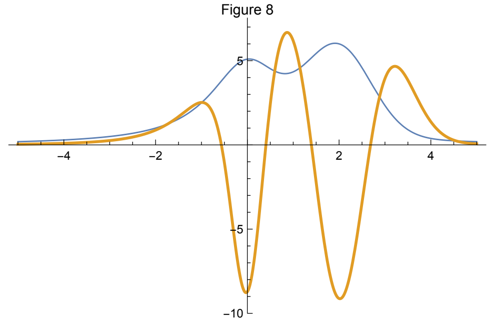


Figure 9

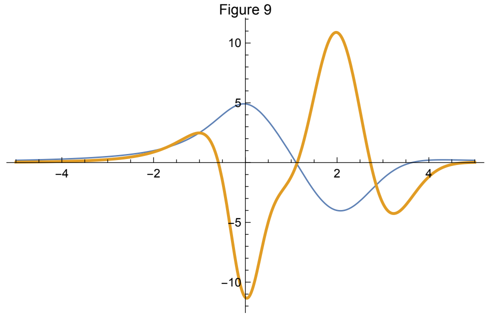
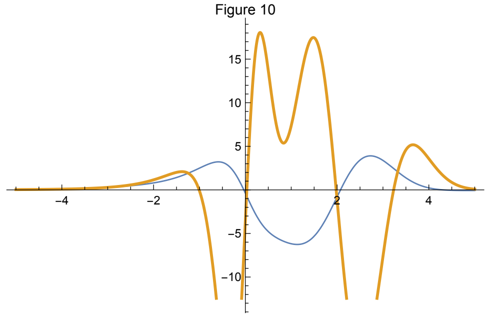


Figure 10



MTH 141, Calculus I—Exam III **Apr. 19, 1995**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions*. Your paper is due at 12:55 pm.

1. Evaluate:

   (a)

$$\sum_{j=1}^{10}(j+5)$$

   (b)

$$\sum_{n=5}^{10}3^n$$

   Solution:

   (a)

$$\sum_{j=1}^{10}(j+5) = (1+5)+(2+5)+\cdots+(10+5) \tag{1}$$

$$= 6+7+8+\cdots+15 = 105. \tag{2}$$

   (b)

$$\sum_{n=5}^{10}3^n = 3^5+3^6+\cdots+3^9+3^{10} = 88452. \tag{3}$$

2. Evaluate:

   (a)

$$\int_0^2 (t^4 - t)\,dt$$

   (b)

$$\int_{-\pi/4}^{\pi/6} \sec^2 u\,du$$

   Solution:

   (a)

$$\int_0^2 (t^4 - t)\,dt = \left(\frac{t^5}{5} - \frac{t^2}{2}\right)\Bigg|_0^4 \tag{4}$$

$$= \left(\frac{32}{5} - \frac{4}{2}\right) - \left(\frac{0}{5} - \frac{9}{2}\right) = \frac{22}{5}. \tag{5}$$

(b)

$$\int_{-\pi/4}^{\pi/6} \sec^2 u\, du = \tan u\Big|_{-\pi/4}^{\pi/6} = \frac{\sqrt{3}}{3} - (-1) = \frac{3+\sqrt{3}}{3}. \tag{6}$$

3. Use five rectangles of equal width and whose heights are calculated at the midpoints of their bases to find an approximate value (correct to at least three digits to the right of the decimal) for the integral

$$\int_1^2 \sqrt{1+x^2}\, dx.$$

Solution:

$$\int_1^2 \sqrt{1+x^2}\, dx \sim \frac{1}{5}\sum_{k=1}^{5} \sqrt{1+\left(1+\frac{2k-1}{10}\right)^2} \tag{7}$$

$$\sim \frac{1}{5}\left(\sqrt{1+\frac{121}{100}} + \sqrt{1+\frac{169}{100}} + +\sqrt{1+\frac{225}{100}} + \sqrt{1+\frac{289}{100}} + \sqrt{1+\frac{361}{100}}\right) \tag{8}$$

$$\sim \frac{1}{50}\left(\sqrt{221} + \sqrt{269} + \sqrt{325} + \sqrt{389} + \sqrt{361}\right) \tag{9}$$

$$\sim 1.80978. \tag{10}$$

Note: The actual value of the integral is $\frac{1}{2}(2\sqrt{5}+\ln[2+\sqrt{5}]) - \frac{1}{2}(\sqrt{2}+\ln[1+\sqrt{2}])$.

4. Evaluate the following integrals. Show your reasoning explicitly.

(a)

$$\int_1^2 (t^2-1)^{10} t\, dt$$

(b)

$$\int_1^{3/2} \left(1-\frac{1}{t^2}\right)\sqrt{\left(t+\frac{1}{t}\right)}\, dt$$

Solution:

(a) let $u = t^2 - 1$. Then $du = 2t\, dt$, $u = 3$ when $t = 2$, and $u = 0$ when $t = 1$. Therefore

$$\int_1^2 (t^2-1)^{10} t\, dt = \frac{1}{2}\int_0^3 u^{10}\, du = \frac{u^{11}}{22}\Big|_0^3 = \frac{177147}{22} \tag{11}$$

(b) Let $u = t + \frac{1}{t}$. Then $du = \left(1 - \frac{1}{t^2}\right) dt$. Moreover, $u = 2$ when $t = 1$ and $u = 13/6$ when $t = 3/2$. Thus,

$$\int_1^{3/2} \left(1 - \frac{1}{t^2}\right) \sqrt{\left(t + \frac{1}{t}\right)}\, dt = \int_2^{13/6} u^{1/2}\, du \tag{12}$$

$$= \frac{2}{3} u^{3/2} \Big|_2^{13/6} = \frac{2}{3}\left[\left(\frac{13}{6}\right)^{3/2} - 2^{3/2}\right] \tag{13}$$

5. Let F be the function given by $F(x) = \int_a^x f(t)\, dt$, where f is a continuous function on some open interval that contains the interval $[a, b]$ and x lies in $[a, b]$. Explain why $F'(x) = f(x)$.

Solution: We compute

$$F'_+(x) = \lim_{h \to 0^+} \frac{1}{h} \frac{F(x + h) - F(x)}{h} \tag{14}$$

$$= \lim_{h \to 0^+} \frac{1}{h} \left[\int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt\right] \tag{15}$$

$$= \lim_{h \to 0^+} \frac{1}{h} \int_x^{x+h} f(t)\, dt \tag{16}$$

But f is continuous throughout some open interval that contains $[a, b]$ and $[x, x + h]$ is a subinterval of $[a, b]$. Consequently there is a number ξ_h in $[x, x + h]$ such that $\int_x^{x+h} f(t)\, dt = f(\xi_h)h$. Thus,

$$F'_+(x) = \lim_{h \to 0^+} \frac{1}{h} \int_x^{x+h} f(t)\, dt \tag{17}$$

$$= \lim_{h \to 0^+} \frac{1}{\not{h}} f(\xi_h)\not{h} \tag{18}$$

But $\lim_{h \to 0^+} \xi_h = x$ (because $x \le \xi_h \le x + h$), and f is continuous, so it follows that

$$F'_+(x) = \lim_{h \to 0^+} f(\xi_h) = f(x), \tag{19}$$

A similar argument shows that $F'_-(x) = f(x)$, and we conclude that $F'(x) = f(x)$.

MTH 141, Calculus I—Final Exam **May 12, 1995**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions*. Your paper is due at 1:55 pm.

1. Here is the graph of a function $y = f(x)$:
   Sketch the graphs of the following:

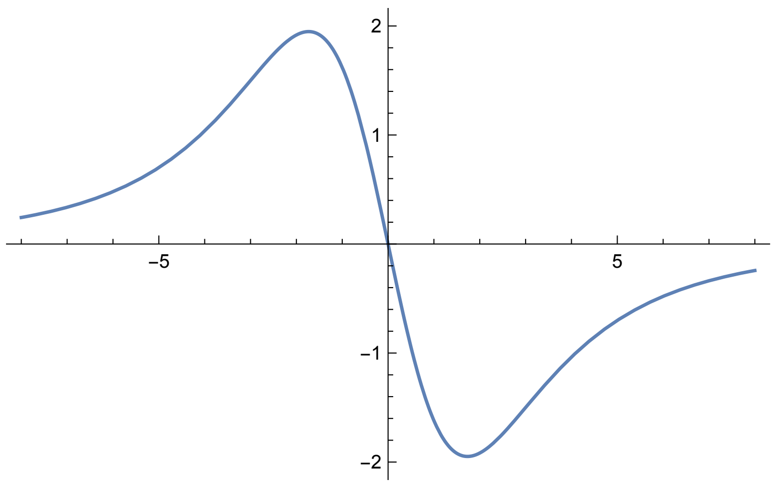


Figure 1: example caption

   (a) $y = f(x - 3)$
   (b) $y = f(x/2)$
   (c) $y = f'(x)$
   (d) $y = \int_0^x f(t)\,dt$

2. (a) What limit must one evaluate in order to calculate $f'(3)$ from the definition of the derivative if f is the function given by
$$f(x) = \frac{1}{x-1}?$$
   (b) Show how to evaluate the limit you identified in part (a) of this problem.

3. (a) Find an equation for the line tangent to the curve $y = x + \sin x$ at the point (π, π).
   (b) On what intervals is the curve
$$y = x^4 + 4x^3 + x - 3$$
   concave upward? Give an explanation that does not depend upon a graph.

4. (a) Find $\left.\frac{dy}{dx}\right|_{(1,1)}$ if
$$y^3 + x^2y + x^2 - 3y^2 = 0. \tag{1}$$

(b) Write an equation for the line tangent to the curve defined by equation (1) at the point $(1, 1)$.

(c) Explain how to use these results to find an approximate value of y in Equation (1) when $x = 1.0231$. What approximate value does your method yield for y? (Give at least four digits to the right of the decimal.)

5. Find

(a) $\displaystyle\int_0^3 (1 - x^2)dx$

(b) $\displaystyle\int_{-1}^1 \frac{x^2\,dx}{\sqrt{4 + x^3}}$

6. Find the volume generated when the first-quadrant region bounded by $y^2 = 4x$, the x-axis, and the line $x = 4$ is revolved about the x-axis. Explain.

7. Find the volume obtained by revolving the area between $y = x^2$ and $x = y^2$ about the line $y = 3$. Explain.

8. Let F be the function given by $F(x) = \displaystyle\int_a^x f(t)\,dt$. Explain why $F'(x) = f(x)$.

MTH 141, Calculus I—Final Exam **May 12, 1995**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions*. Your paper is due at 1:55 pm.

1. Here is the graph of a function $y = f(x)$:
   Sketch the graphs of the following:

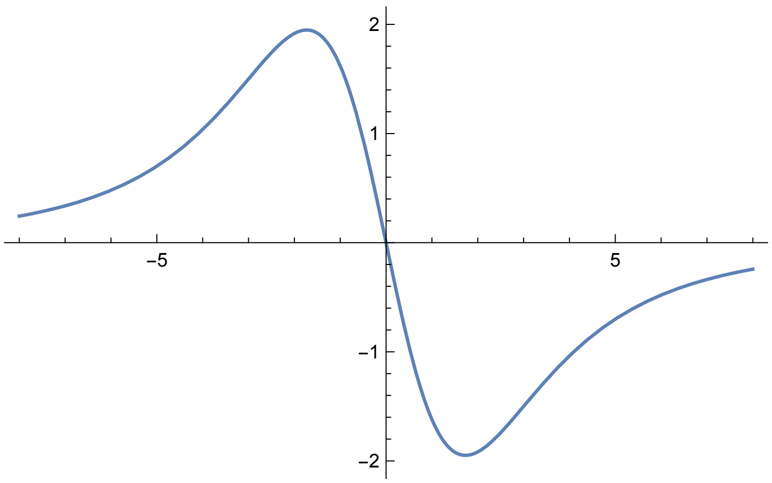


Figure 1: example caption

   (a) $y = f(x - 3)$
   (b) $y = f(x/2)$
   (c) $y = f'(x)$
   (d) $y = \int_0^x f(t)\,dt$

2. (a) What limit must one evaluate in order to calculate $f'(3)$ from the definition of the derivative if f is the function given by

$$f(x) = \frac{1}{x-1}?$$

   (b) Show how to evaluate the limit you identified in part (a) of this problem.

   Solution;

   (a)

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \tag{1}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{1}{(3+h) - 1} - \frac{1}{3-1} \right] \tag{2}$$

(b)

$$f'(3) = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(3+h)-1} - \frac{1}{3-1}\right] \tag{3}$$
$$= \frac{1}{h}\left[\frac{1}{2+h} - \frac{1}{2}\right] \tag{4}$$
$$= \lim_{h\to 0} \frac{\cancel{2} - (\cancel{2} + \cancel{h})}{2\cancel{h}(2+h)} \tag{5}$$
$$= \lim_{h\to 0} \frac{-1}{2(2+h)} = -\frac{1}{4}. \tag{6}$$

3. (a) Find an equation for the line tangent to the curve $y = x + \sin x$ at the point (π, π).
   (b) On what intervals is the curve

$$y = x^4 + 4x^3 + x - 3$$

   concave upward? Give an explanation that does not depend upon a graph.

Solution:

(a) We have

$$f'(x) = 1 + \cos x, \text{ so that} \tag{7}$$
$$f'(\pi) = 1 + (-1) = 0. \tag{8}$$

Consequently, the tangent line to $y = f(x)$ at (π, π) is the line through that point with slope $f'(\pi) = 0$, or the line $y = \pi$.

(b) If $y = x^4+4x^3+x-3$, then $y' = 4x^3+12x^2+1$, and $y'' = 12x^2+24x = 12x(x+2)$. This latter quantity is positive just where its two factors x and $(x+2)$ are either both postive or both negative—that is, where $x < -2$ and where $x > 0$. Consequently, the given curve is concave upward on the intervals $(-\infty, -2)$ and $(0, \infty)$.

4. (a) Find $\left.\frac{dy}{dx}\right|_{(1,1)}$ if

$$y^3 + x^2y + x^2 - 3y^2 = 0. \tag{9}$$

   (b) Write an equation for the line tangent to the curve defined by equation (9) at the point $(1, 1)$.
   (c) Explain how to use these results to find an approximate value of y in Equation (9) when $x = 1.0231$. What approximate value does your method yield for y? (Give at least four digits to the right of the decimal.)

Solution:

(a) We differentiate implicitly, treating y as a function of x:

$$\frac{d}{dx}(y^3 + x^2y + x^2 - 3y^2) = \frac{d}{dx}0 \tag{10}$$
$$3y^2y' + 2xy + x^2y' + 2x - 6yy' = 0. \tag{11}$$

When $x = 1$ and $y = 1$, this latter equation becomes

$$3y' + 2 + 4y' + 2 - 6y' = 0, \text{ or} \tag{12}$$

$$y' = -4. \tag{13}$$

Thus, $\left.\dfrac{dy}{dx}\right|_{(1,1)} = -4.$

(b) The desired tangent line is the line through $(1, 1)$ with slope -4, so an equation for this line is

$$y = 1 - 4(x - 1). \tag{14}$$

(c) The tangent line lies close to the curve near the point of tangency, so the y-coordinate of the point on the tangent line corresponding to the value $x = 1.0231$ is a good approximation to the y-value of the point near $(1, 1)$ on the curve where $x = 1.0231$. Thus, our approximation is

$$y \sim 1 - 4(1 - 1.0231) = 0.9076. \tag{15}$$

5. Find

(a) $\displaystyle\int_0^3 (1 - x^2)dx$

(b) $\displaystyle\int_{-1}^1 \frac{x^2\,dx}{\sqrt{4 + x^3}}$

Solution:

(a)

$$\int_0^3 (1 - x^2)dx = \left(x - \frac{x^3}{3}\right)\Bigg|_0^3 = \left(3 - \frac{27}{3}\right) - 0 = -6. \tag{16}$$

(b) Let $u = 4 + x^3$. Then $du = 3x^2\,dx$, or $x^2\,dx = \dfrac{1}{3}\,du$. When $x = -1$, $u = 3$; and when $x = 1$, $u = 5$. Thus,

$$\int_{-1}^1 \frac{x^2\,dx}{\sqrt{4 + x^3}} = \frac{1}{3}\int_3^5 u^{-1/2}\,du = \frac{2}{3}u^{1/2}\Bigg|_3^5 = \frac{2}{3}\left(\sqrt{5} - \sqrt{3}\right). \tag{17}$$

6. Find the volume generated when the first-quadrant region bounded by $y^2 = 4x$, the x-axis, and the line $x = 4$ is revolved about the x-axis. Explain.

Solution: We note that a cross-section of the region described in the problem along any horizontal line $y = t$ is the segment from $\left(\dfrac{t^2}{4}, t\right)$ to $(4, t)$. When this segment is revolved about the x-axis it generates a cylinder of height $\left(4 - \dfrac{t^2}{4}\right)$ and radius t. We must consider all such cylinders for $0 \le t \le 4$, so the required volume is

$$2\pi \int_0^4 t\left(4 - \frac{t^2}{4}\right)\,dt = 2\pi \int_0^4 \left(4t - \frac{t^3}{4}\right)\,dt = 2\pi\left(2t^2 - \frac{1}{16}t^4\right)\Bigg|_0^4 \tag{18}$$

$$= 2\pi(32 - 16) = 32\pi. \tag{19}$$

7. Find the volume obtained by revolving the area between $y = x^2$ and $x = y^2$ about the line $y = 3$. Explain.

Solution: The curves intersect at $(0, 0)$ and $(1, 1)$, and the region in question extends over the interval $0 \le x \le 1$. A vertical line $x = t$ intersects the region in the line segment determined by the points (t, t^2) and $(t, \sqrt{t})$, and when such a line segment is revolved about the line $y = 3$ it generates a washer whose inner radius is $3 - \sqrt{t}$ and whose outer radius is $3 - t^2$. The area is such a washer is

$$\pi\left[(3 - t^2)^2 - (3 - \sqrt{t})^2\right] = \pi\left(9 - 6t^2 + t^4 - 9 + 6\sqrt{t} - t\right) \tag{20}$$

The desired volume is therefore

$$\pi \int_0^1 \left(t^4 - 6t^2 - t + 6t^{1/2}\right) dt = \pi\left(\frac{t^5}{5} - 2t^3 - \frac{t^2}{2} + 4t^{3/2}\right)\Big|_0^1 = \frac{17}{10}\pi. \tag{21}$$

8. Let F be the function given by $F(x) = \displaystyle\int_a^x f(t)\, dt$. Explain why $F'(x) = f(x)$.

Solution: We compute

$$F'_+(x) = \lim_{h\to 0^+} \frac{1}{h}\frac{F(x+h) - F(x)}{h} \tag{22}$$

$$= \lim_{h\to 0^+} \frac{1}{h}\left[\int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt\right] \tag{23}$$

$$= \lim_{h\to 0^+} \frac{1}{h}\int_x^{x+h} f(t)\, dt \tag{24}$$

But f is continuous throughout some open interval that contains $[a, b]$ and $[x, x + h]$ is a subinterval of $[a, b]$. Consequently there is a number ξ_h in $[x, x + h]$ such that $\int_x^{x+h} f(t)\, dt = f(\xi_h)h$. Thus,

$$F'_+(x) = \lim_{h\to 0^+} \frac{1}{h}\int_x^{x+h} f(t)\, dt \tag{25}$$

$$= \lim_{h\to 0^+} \frac{1}{\not{h}} f(\xi_h)\not{h} \tag{26}$$

But $\lim_{h\to 0^+} \xi_h = x$ (because $x \le \xi_h \le x + h$), and f is continuous, so it follows that

$$F'_+(x) = \lim_{h\to 0^+} f(\xi_h) = f(x), \tag{27}$$

A similar argument shows that $F'_-(x) = f(x)$, and we conclude that $F'(x) = f(x)$.

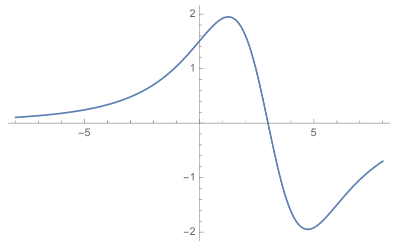


Figure 2: Problem 1(a)

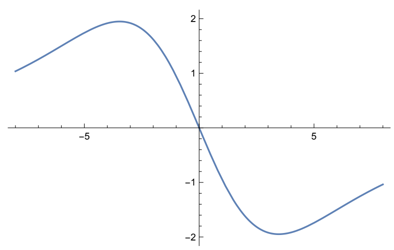


Figure 3: Problem 1(b)

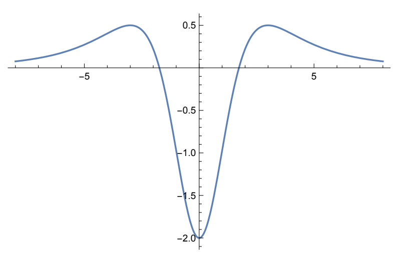


Figure 4: Problem 1(c)

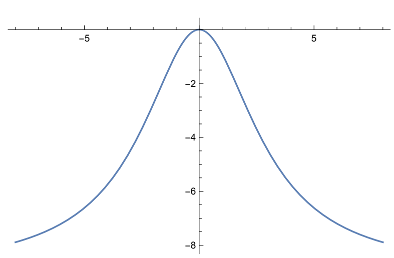


Figure 5: Problem 1(d)

MTH141, Calculus I, Exam 1 Unrecorded date in 1997

Instructions: Work the following problems on your ow paper, unless otherwise indicated. Show enough detail to support your conclusions. Your paper is due at 12:50 pm.

1. Explain how to use the Limit Laws to evaluate:

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{2x^2 + x + 1}$$

2. Let f be the function whose graph is: Use the sets of coordinate axes provided on the last

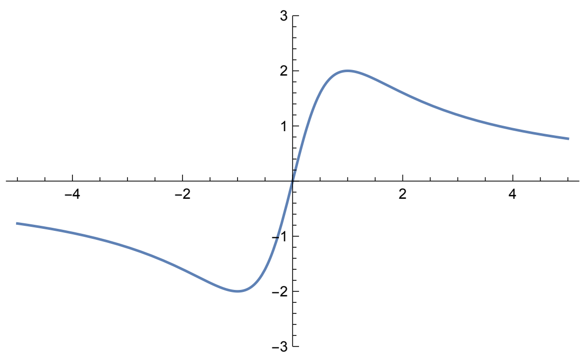


Figure 1: example caption

page of the exam to sketch the graphs of

   (a) $y = f(x - 2)$

   (b) $y = f(2x)$

3. Find the equation of the straight line

   (a) through $(3, 8)$ with growth rate $\frac{5}{4}$.

   (b) through $(2, 5)$ and perpendicular to the line whose equation is $3x - y = 12$.

4. Solve the inequality:

$$\left| x^2 - 7x + 5 \right| < 5.$$

   Explain your reasoning.

5. Use the meaning of the term *derivative* to find the derivative of f when $f(x) = \dfrac{1}{5\sqrt{x}}$.

6. An object moves so that its height, x, above the ground at time t is

$$x = -5t^2 + 30t. \qquad (1)$$

   Compute both its velocity when $t = 1.5$ and the speed with which it strikes the ground.

7. Find the equation of the line tangent to the curve

$$y = \frac{x + 1}{x - 1} \qquad (2)$$

   at the point where $x = 2$. [Hint: $y' = -\dfrac{2}{(x-1)^2}$.]

8. Let f and g be functions, such that $f(2) = 4$, $g(2) = 3$, $f'(2) = 2$, and $g'(2) = -2$. Find $P'(2)$ and $Q'(2)$, where P and Q are the functions defined by

$$P(x) = f(x)g(x), \tag{3}$$

$$Q(x) = \frac{g(x)}{f(x)}. \tag{4}$$

9. Find $f'(x)$ when

   (a) $f(x) = \cos x - \tan x$.

   (b) $f(x) = e^x(\cos x - \sin x)$.

   (c) $f(x) = \cos^2 x + \sin^2 x$.

MTH141, Calculus I, Exam 1 Unrecorded date in 1997

Instructions: Work the following problems on your ow paper, unless otherwise indicated. Show enough detail to support your conclusions. Your paper is due at 12:50 pm.

1. Explain how to use the Limit Laws to evaluate:

$$\lim_{x\to 2} \frac{x^2 - 3x + 2}{2x^2 + x + 1}$$

Solution:

$$\lim_{x\to 2} \frac{x^2 - 3x + 2}{2x^2 + x + 1} = \frac{\lim_{x\to 2}(x^2 - 3x + 2)}{\lim_{x\to 2}(2x^2 + x + 1)}, \tag{1}$$

provided both of the limits on the right side of the equation exist and the limit in the denominator is not zero. But

$$\lim_{x\to 2}(x^2 - 3x + 2) = \lim_{x\to 2} x^2 - \lim_{x\to 2} 3x + \lim_{x\to 2} 2, \tag{2}$$

again provided all of the limits on the right side of the equation exist. We have

$$\lim_{x\to 2} x^2 = \left[\lim_{x\to 2} x\right]^2 = 2^2 = 4, \tag{3}$$

$$\lim_{x\to 2} 3x = 3 \lim_{x\to 2} x = 3 \cdot 2 = 6, \text{ and} \tag{4}$$

$$\lim_{x\to 2} 2 = 2. \tag{5}$$

Thus, the limit in the numerator of (1) is $4 - 6 + 2 = 0$. Similar reasoning shows that the limit in the denominator of (1) is $8 + 2 + 1 = 11$. We conclude that

$$\lim_{x\to 2} \frac{x^2 - 3x + 2}{2x^2 + x + 1} = \frac{0}{11} = 0. \tag{6}$$

2. Let f be the function whose graph is shown in Figure 1. Use the sets of coordinate axes

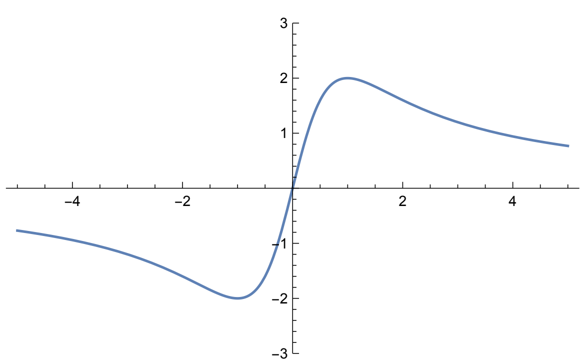


Figure 1: $y = f(x)$

provided on the last page of the exam to sketch the graphs of

(a) $y = f(x - 2)$

(b) $y = f(2x)$

Solution: See Figure 2 and Figure 2.

3. Find the equation of the straight line

   (a) through $(3, 8)$ with growth rate $\frac{5}{4}$.

   (b) through $(2, 5)$ and perpendicular to the line whose equation is $3x - y = 12$.

   Solution:

   (a) An equation for the line through (x_0, y_0) with slope m is $y = y_0 + m(x - x_0)$, so an equation for the line through $(3, 8)$ with slope = growth rate = $\frac{5}{4}$ is $y = 8 + \frac{5}{4}(x - 3)$.

   (b) The line $3x - y = 12$ has slope 3, and lines perpendicular to a given non-vertical, non-horizontal, line have for their slope the negative of the reciprocal of the slope of the given line. Therefore, the line through $(2, 5)$ perpendicular to the line $3x - y = 12$ has equation $y = 5 - \frac{1}{3}(x - 2)$.

4. Solve the inequality:

$$\left|x^2 - 7x + 5\right| < 5.$$

   Explain your reasoning.

   Solution: We know that $\left|x^2 - 7x + 5\right| < 5$ if, and only if,

$$-5 < x^2 - 7x + 5 < 5 \text{ or, equivalently,} \tag{7}$$

$$0 < x^2 - 7x + 10 \text{ and } x^2 - 7x < 0. \tag{8}$$

   The first of these inequalities may be written $(x - 2)(x - 5) < 0$, which is true just when both factors have the same sign—which happens just when x lies in $(-\infty, 2) \cup (5, \infty)$. The second inequality can be rewritten $x(x - 7) < 0$, and this is true just when both factors have opposite signs—which means that x must lie in $(0, 7)$. But the compound inequality (7) is true just when both of the inequalities of (8) are true, and this happens when x is in both $(-\infty, 2) \cup (5, \infty)$ and $(0, 7)$. We conclude that the solution of the original inequality is $(0, 2) \cup (5, 7)$.

5. Use the meaning of the term *derivative* to find the derivative of f when $f(x) = \frac{1}{5\sqrt{x}}$.

   Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{1}{h}\left[\frac{1}{5\sqrt{x+h}} - \frac{1}{5\sqrt{x}}\right] \tag{9}$$

$$= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{5h\sqrt{x}\sqrt{x+h}} = \lim_{h \to 0} \frac{\cancel{x} - (\cancel{x} + h)}{5h\sqrt{x}\sqrt{x+h}\left[\sqrt{x} + \sqrt{x+h}\right]} \tag{10}$$

$$= \lim_{h \to 0} \frac{\cancelto{-1}{-h}}{5\cancelto{1}{h}\sqrt{x}\sqrt{x+h}\left[\sqrt{x} + \sqrt{x+h}\right]} = -\frac{1}{10x^{3/2}}. \tag{11}$$

6. An object moves so that its height, x, above the ground at time t is

$$x = -5t^2 + 30t. \tag{12}$$

   Compute both its velocity when $t = 1.5$ and the speed with which it strikes the ground.

   Solution: If altitude is $x(t) = -5t^2 + 30t$, then velocity is $v(t) = x'(t) = -10t + 30$, so velocity when $t = 1.5$ is $v(1.5) = -10 \cdot 1.5 + 30 = 15$. The object strikes the ground when $t > 0$ but $x(t) = 0$—that is, when $0 = -5t^2 + 30t$, which is when $t = 6$. When $t = 6$, velocity is

$v(6) = -10 \cdot 6 + 30 = -30$. Speed is the magnitude of velocity, so speed at impact is 30. (**Note:** No units are given in the statement of the problem, so it is inappropriate to include them in the solution.)

7. Find the equation of the line tangent to the curve

$$y = \frac{x+1}{x-1} \tag{13}$$

at the point where $x = 2$. [Hint: $y' = -\frac{2}{(x-1)^2}$.]

Solution: When $x = 2$, $y = 3$ and $y' = -2$. Hence an equation for the desired line is

$$y = 3 - 2(x-2). \tag{14}$$

8. Let f and g be functions, such that $f(2) = 4$, $g(2) = 3$, $f'(2) = 2$, and $g'(2) = -2$. Find $P'(2)$ and $Q'(2)$, where P and Q are the functions defined by

$$P(x) = f(x)g(x), \tag{15}$$

$$Q(x) = \frac{g(x)}{f(x)}. \tag{16}$$

Soluion:

$$P'(2) = f'(2)g(2) + f(2)g'(2) = 2 \cdot 3 + 4 \cdot (-2) = -2. \tag{17}$$

$$Q'(2) = \frac{g'(2)f(2) - f'(2)g(2)}{[f(2)]^2} = \frac{(-2) \cdot 4 - 2 \cdot 3}{4^2} = -\frac{7}{8}. \tag{18}$$

9. Find $f'(x)$ when

(a) $f(x) = \cos x - \tan x$.

(b) $f(x) = e^x(\cos x - \sin x)$.

(c) $f(x) = \cos^2 x + \sin^2 x$.

Solution:

(a) $f'(x) = -\sin x - \sec^2 x$.

(b) $f'(x) = e^x(\cos x - \sin x) + e^x(-\sin x - \cos x)$.

(c) $f'(x) = \frac{d}{dx}(\cos^2 x + \sin^2 x) = \frac{d}{dx}(1) = 0$

Alternate Calculation: $\frac{d}{dx}(\cos^2 x + \sin^2 x) = -2\cos x \sin x + 2 \sin x \cos x = 0$.

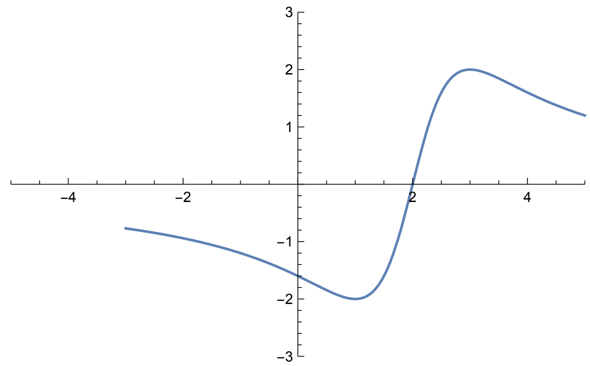


Figure 2: $y = f(x - 2)$

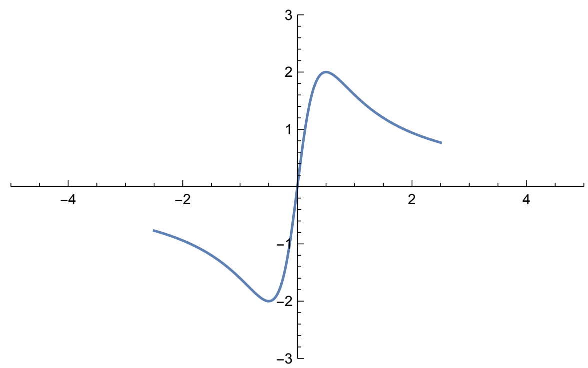


Figure 3: $y = f(2x)$

MTH 141, Calculus I—Exam II **Apr. 17, 1997**

Instructions: Work the following problems on your own paper, unless otherwise indicated. Give explanations as required. Your paper is due at 12:50 pm.

1. One of the curves in Figure 1 is the graph of a function $y = f(x)$, and the other is the graph of its derivative. Explain which is which.

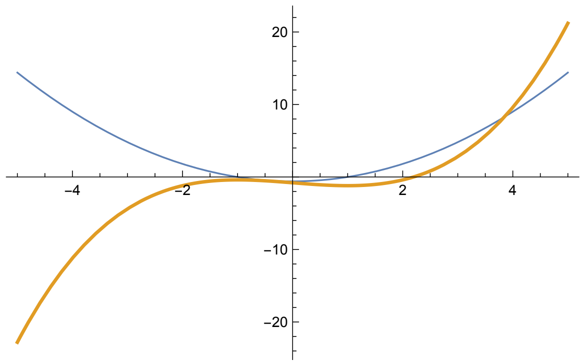


Figure 1: Problem 1

2. One of the curves in Figure 2 is the graph of a function $y = f(x)$, and the other is the graph of its second derivative. Explain which is which.

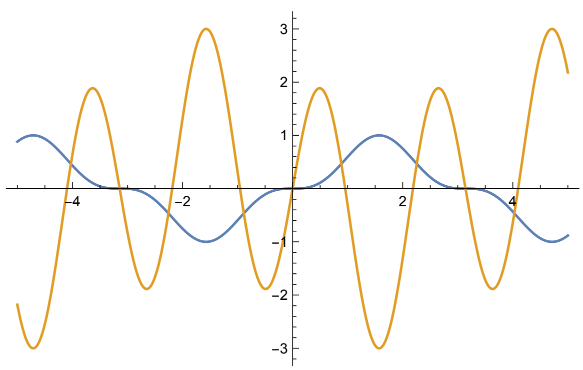


Figure 2: Problem 2

3. Use two iterations of Newton's Method, with an initial guess of $x_1 = 1.5$, to estimate the root of the equation

$$\ln x = \frac{1}{2}.$$

   Show your calculations to at least four digits to the right of the decimal.

4. Let g be the function given by

$$g(x) = \frac{x^2 + 1}{x^2 + x + 1}.$$

Then

$$g'(x) \quad = \quad \frac{x^2 - 1}{(x^2 + x + 1)^2},$$

so that g has critical points at $x = \pm 1$. Tell whether each of these critical points is a local maximum, a local minimum, or neither. Explain the basis for your decisions.

5. Let $f(x) = x^2 + x$.

   (a) Find $\displaystyle\sum_{k=1}^{4} \left[f\left(\frac{k}{2}\right) \cdot \frac{1}{2} \right]$.

   (b) Draw a picture showing how to estimate $\displaystyle\int_0^2 f(x)\, dx$ by using four rectangles of equal width all of whose upper right corners lie on the curve. What numerical value does this estimate yield?

   (c) Do you expect the estimate of problem 5b to be too large or too small? Explain.

6. Find two numbers so that

   (a) The sum of the first number and twice the second is 100.

   (b) The product of the two numbers is a large as possible.

   Explain your reasoning.

7. The curve whose equation is

$$x^3 - 2x^2y = 3y^3 - 3$$

passes through the point $(2, 1)$.

(a) Find the slope of the line tangent to this curve at $(2, 1)$.

(b) Write an equation for the line whose slope you found in problem 7a.

(c) Use the results of problems 7a and/or 7b to find an approximate value for the y-coordinate of the point on the curve near $(2, 1)$ where $x = 2.1$.

Explain your reasoning.

8. A point moves along the curve $y = x^2$ so that its projection on the x-axis moves along that axis at 2 units per second. How quickly is the point's projection on the y-axis moving along that axis when $x = 3$?

MTH 141, Calculus I—Exam II Solutions **Apr. 17, 1997**

Instructions: Work the following problems on your own paper, unless otherwise indicated. Give explanations as required. Your paper is due at 12:50 pm.

1. One of the curves in Figure 1 is the graph of a function $y = f(x)$, and the other is the graph of its derivative. Explain which is which.

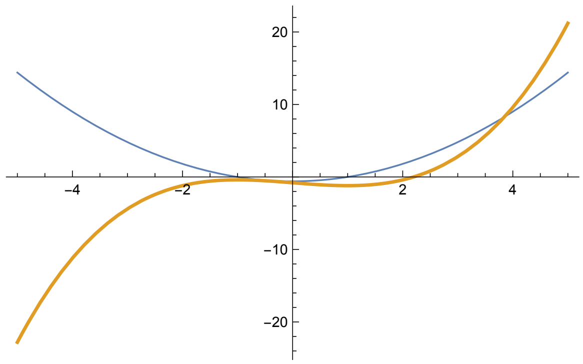


Figure 1: Problem 1

Solution: The blue curve

(a) lies above the x-axis in exactly the intervals where the orange curve slopes upward,

(b) touches the x-axis exactly where the orange curve has a horizontal tangent, and

(c) lies below the x-axis in exactly the intervals where the orange curve slopes downward.

Therefore, the orange curve is the graph of f and the blue curve is the graph of f'.

2. One of the curves in Figure 2 is the graph of a function $y = f(x)$, and the other is the graph of its second derivative. Explain which is which.

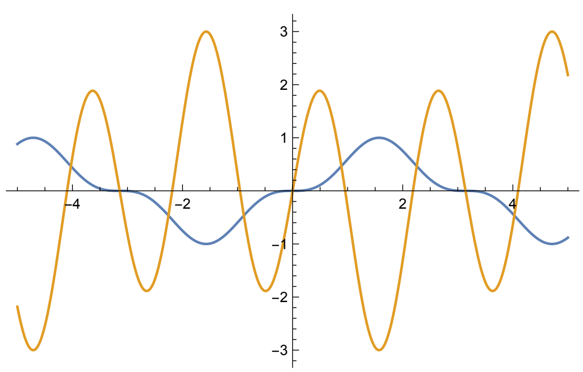


Figure 2: Problem 2

Solution: The orange curve

(a) lies above the x-axis in exactly the intervals where the blue curve is concave upward,

(b) lies below the x-axis in exactly the intervals where the blue curve is concave downward.

Therefore, the blue curve is the graph of f and the orange curve is the graph of f''.

3. Use two iterations of Newton's Method, with an initial guess of $x_1 = 1.5$, to estimate the root of the equation

$$\ln x = \frac{1}{2}.$$

Show your calculations to at least four digits to the right of the decimal.

Solution: We set $f(x) = \ln x - 0.5$. We seek a zero of f. We apply the Newton iteration

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \tag{1}$$

beginning with $x_1 = 1.5$. This gives

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5 - 1.5(\ln 1.5 - 0,5) \sim 1.6418023; \tag{2}$$

$$x_3 \sim x_2 - \frac{f(x_2)}{f'(x_2)} \tag{3}$$

$$\sim 1.6418023 - 1.6418023(\ln 1.6418023 - 1.6418023) \sim 1.648707 \tag{4}$$

4. Let g be the function given by

$$g(x) = \frac{x^2+1}{x^2+x+1}.$$

Then

$$g'(x) = \frac{x^2-1}{(x^2+x+1)^2},$$

so that g has critical points at $x = \pm 1$. Tell whether each of these critical points is a local maximum, a local minimum, or neither. Explain the basis for your decisions.

Solution: We note that the denominator of the fraction $g'(x)$, which is the square of a quadratic that has no zeros, is always positive. Thus the derivative has the same sign as its numerator $x^2 - 1 = (x-1)(x+1)$, which is positive on both of the intervals $(-\infty, -1)$ and $(1, \infty)$, but negative on the interval $(-1, 1)$. Thus, $g'(x)$ changes sign from positive to negative at $x = -1$ and changes sign from negative to positive at $x = 1$. We conclude, by the First Derivative Test, that g has a local maximum at $x = -1$ and has a local minimum at $x = 1$.

5. Let $f(x) = x^2 + x$.

(a) Find $\sum_{k=1}^{4} \left[f\left(\frac{k}{2}\right) \cdot \frac{1}{2} \right]$.

(b) Draw a picture showing how to estimate $\int_0^2 f(x)\, dx$ by using four rectangles of equal width all of whose upper right corners lie on the curve. What numerical value does this estimate yield?

(c) Do you expect the estimate of problem 5b to be too large or too small? Explain.

Solution:

(a) We begin by noting that

$$\sum_{k=1}^{4} \left[f\left(\frac{k}{2}\right) \cdot \frac{1}{2} \right] = f\left(\frac{1}{2}\right) \cdot \frac{1}{2} + f\left(\frac{2}{2}\right) \cdot \frac{1}{2} + f\left(\frac{3}{2}\right) \cdot \frac{1}{2} + f\left(\frac{4}{2}\right) \cdot \frac{1}{2}. \quad (5)$$

But $f(x) = x^2 + x$, so

$$f\left(\frac{1}{2}\right) = \frac{3}{4}; \quad (6)$$

$$f\left(\frac{2}{2}\right) = 2; \quad (7)$$

$$f\left(\frac{3}{2}\right) = \frac{15}{4}; \quad (8)$$

$$f\left(\frac{4}{2}\right) = 6. \quad (9)$$

Therefore,

$$\sum_{k=1}^{4} \left[f\left(\frac{k}{2}\right) \cdot \frac{1}{2} \right] = \frac{3}{8} + 1 + \frac{15}{8} + 3 = \frac{25}{4}. \quad (10)$$

(b) The numerical value given by this approximation scheme is precisely $\frac{25}{4}$. See Figure 3.

(c) The function f is increasing on $[0, 2]$, so the area inside each rectangle is greater than the area between the curve and the base of that rectangle. Thus the estimate is too large.

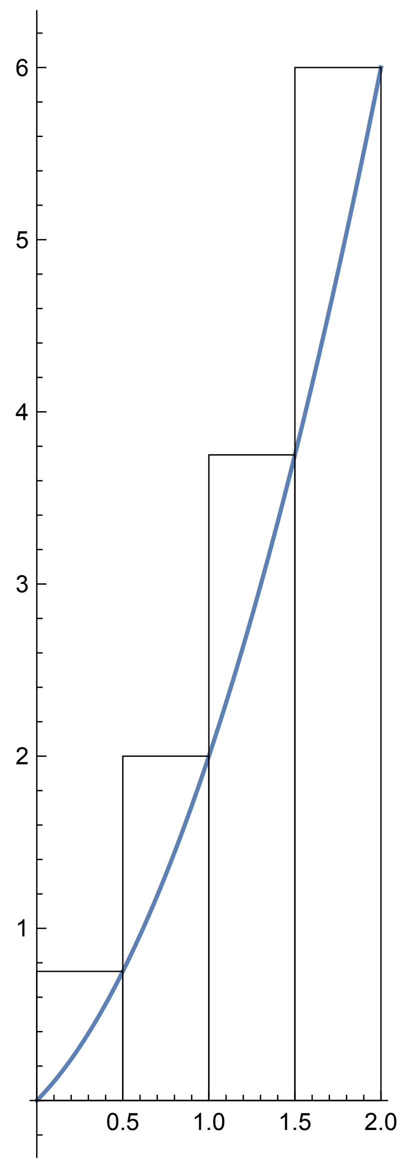


Figure 3: The Rectangles

6. Find two numbers so that

   (a) The sum of the first number and twice the second is 100.

   (b) The product of the two numbers is a large as possible.

   Explain your reasoning.

   Solution: Let x denote the first number, y the second. We want to maximize the product $P = xy$ given that $x + 2y = 100$, with $0 \le x \le 100$. We treat y as a function of x, so that we can think of P, also, as a function of x whose domain is $[0, 100]$. The maximum value of P must then be found at a critical point—that is, a point where $P' = 0$. (The endpoints of the domain of P give $P = 0$, which is not the maximum because P is positive interior to its domain.) Thus, when P is maximal we have

   $$P' = y + xy' = 0, \text{ or} \tag{11}$$

   But $x + 2y = 100$, so, differentiating again, we see that $1 + 2y' = 0$, or $y' = -\frac{1}{2}$. Combining this observation with equation (11), we find that $y - \frac{x}{2} = 0$, or $y = \frac{x}{2}$. The condition $x + 2y = 100$ now becomes $2x = 100$, or $x = 50$. Thus, $y = 25$. This being the only critical point, we conclude that it gives the maximum we need. The two numbers required are therefore 50 and 25; the sum of the first and twice the second is 100, and $P = 1250$ gives the largest possible value of the product under those conditions.

7. The curve whose equation is

   $$x^3 - 2x^2 y = 3y^3 - 3$$

   passes through the point $(2, 1)$.

   (a) Find the slope of the line tangent to this curve at $(2, 1)$.

   (b) Write an equation for the line whose slope you found in problem 7a.

   (c) Use the results of problems 7a and/or 7b to find an approximate value for the y-coordinate of the point on the curve near $(2, 1)$ where $x = 2.1$.

   Explain your reasoning.

   Solution:

   (a) Implicit differerentiation with respect to x gives

   $$3x^2 - 4xy - 2x^2 y' = 9y^2 y', \text{ or} \tag{12}$$

   $$y' = \frac{3x^2 - 4xy}{2x^2 + 9y^2}, \tag{13}$$

   and it follows that

   $$y'\Big|_{(2,1)} = \frac{4}{17}. \tag{14}$$

   Thus, the required slope is $\frac{4}{17}$.

(b) An equation for the line of slope m passing through the point (x_0, y_0) is $y = y_0 + m(x - x_0)$, so the required equation can be written

$$y = 1 + \frac{4}{17}(x - 2). \tag{15}$$

(c) A tangent line approximates the curve well near the point of tangency, so an approximate value for y in the equation $x^3 - 2x^2y = 3y^3 - 3$ when $x = 2.1$ is

$$y \sim 1 + \frac{4}{17}(2.1 - 2) = \frac{87}{85} \sim 1.024. \tag{16}$$

8. A point moves along the curve $y = x^2$ so that its projection on the x-axis moves along that axis at 2 units per second. How quickly is the point's projection on the y-axis moving along that axis when $x = 3$?

Solution: If $y = x^2$, then

$$\frac{dy}{dt} = 2x\frac{dx}{dt}, \tag{17}$$

If $x = 3$ and $\dfrac{dx}{dt} = 2$, as given, then

$$\frac{dy}{dt} = 2x\frac{dx}{dt} = 2 \cdot 3 \cdot 2 = 12, \tag{18}$$

so the point's projection on the y-axis is moving upward at 12 units per second.

MTH 141, Calculus I—Exam I **Sept. 24, 1998**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions.* Your paper is due at 3:55 pm.

1. Figure 1 on the last page of this exam cdisplays the complete graph of a certain function f. Answer the following questions about f:
   (a) What is the domain of f? Explain.
   (b) What is the range of f? Explain.
   (c) What is the largest *interval* on which f is a decreasing function? Explain.
   (d) Is f a one-to-one function? Explain.
   (e) Detach the page with the figure on it and *put your name on the page.* Then sketch the graph of $y = f(|x|) - 6$ on the figure and turn it in with your exam.

2. Find the following limits without using a calculator. Show all of your calculations.
   (a) $\lim_{x \to 3} \frac{x^2 - 5x + 8}{x - 2}$
   (b) $\lim_{x \to -2} \frac{x^3 + 2x^2 + 4x + 8}{x^2 + 8x + 12}$

3. When a camera flash goes off, the batteries immediately begin to recharge the flash's capacitor, which stores charge $Q(t)$ according to the equation

$$Q(t) = Q_0(1 - e^{-t/a}).$$

   (The maximum possible charge is Q_0 and t is measured in seconds; a is a certain positive constant that depends upon the battery, and the circuitry.)
   (a) Find the inverse of this function and explain its meaning.
   (b) How long does it take to recharge the capacitor to 90% of capacity if $a = 2$?

4. Give examples to show that when $\lim_{x \to a^+} N(x) = 0$ and $\lim_{x \to a^+} D(x) = 0$, we may have

$$\lim_{x \to a^+} \frac{N(x)}{D(x)} = L,$$

   where
   (a) $L = \infty$,
   (b) $L = -\infty$,
   (c) $L = 0$, or
   (d) L is some non-zero real number.

   Explain why each of your examples exhibits the behavior you say it does.

5. The position of an object moving along the x-axis is given by

$$x(t) \quad = \quad 30 + 60t - t^3,$$

where $x(t)$ is in meters and t is in seconds. Give answers to the following questions in simplest algebraic form.

(a) Where is the object when $t = 5$?

(b) Where is the object when $t = 6$?

(c) Where is the object when $t = 5 + h$?

(d) How far did the object travel during the interval $5 \leq t \leq 6$?

(e) How far did the object travel during the interval $5 \leq t \leq 5 + h$?

(f) What was the average velocity of the object for the interval $5 \leq t \leq 6$?

(g) Assume that $h > 0$. What was the average velocity of the object for the interval $5 \leq t \leq 5 + h$?

6. Show how to use an appropriate limit to find an equation for the line tangent to the curve $y = x^3 - 2x^2 + 3x - 1$ at the point $(1, 1)$.

7. A scientist has some data points

$$\{\,(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\,\}.$$

When she took the natural logarithm of each of the y-values and plotted the points

$$\{\,(x_1, \ln y_1), (x_2, \ln y_2), \ldots, (x_n, \ln y_n)\,\}$$

on a graph, she found that all of these points lie on a single straight line. When she saw this, she decided that there must be constants A and r so that her *original* data points all lie on the curve $y = Ae^{rx}$.

(a) Explain the reasoning that supports her conclusion.

(b) Explain how she could use measurements involving the straight line to determine the values of the constants A and r for her data.

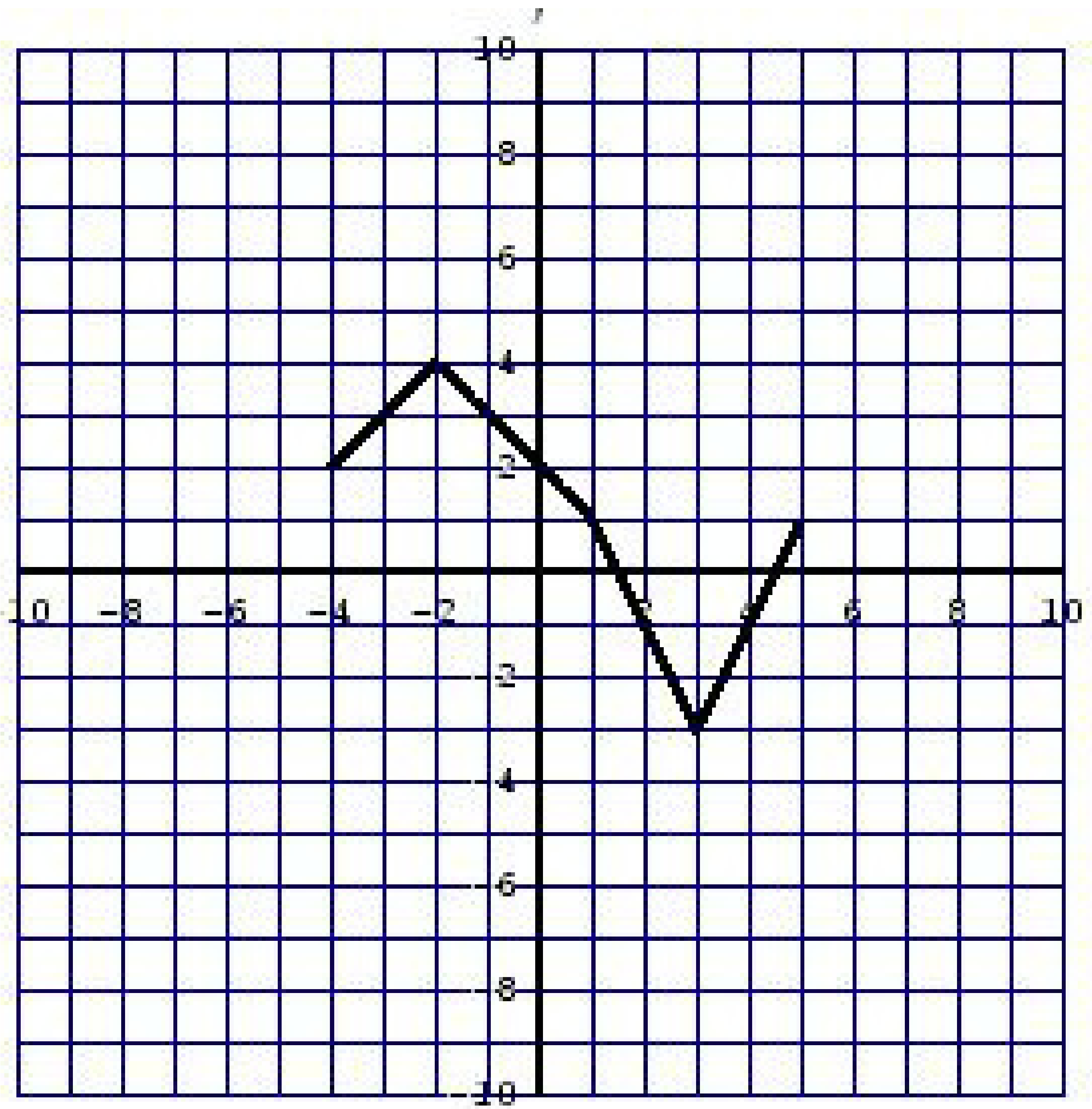


Figure 1:

MTH 141, Calculus I—Exam I **Sept. 24, 1998**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions.* Your paper is due at 3:55 pm.

1. The last page of this exam contains a figure which is the complete graph of a certain function f. Answer the following questions about f:

   (a) What is the domain of f? Explain.

   Solution: The graph extends horizontally from $x = -4$ to $x = 5$; hence the domain of f is $\{ x : -4 \le x \le 5 \}$.

   (b) What is the range of f? Explain.

   Solution: The graph extends vertically from $y = -3$ to $y = 4$; hence the range of f is $\{ y : -3 \le y \le 4 \}$.

   (c) What is the largest *interval* on which f is a decreasing function? Explain.

   Solution: A function is decreasing on any interval where its graph slopes downward to the right. The largest interval on which the graph of f slopes downward to the right is $\{ x : -2 \le x \le 3 \}$.

   (d) Is f a one-to-one function? Explain.

   Solution: f is not one-to-one because $f(2) = -1 = f(4)$. In order for f to be one-to-one, it would have to satisfy $f(a) \ne f(b)$ whenever $a \ne b$.

   (e) Detach the page with the figure on it and *put your name on the page.* Then sketch the graph of $y = f(\,|x|\,) - 6$ on the figure and turn it in with your exam.

   Solution: See last page.

2. Find the following limits without using a calculator. Show all of your calculations.

   (a) $\lim\limits_{x \to 3} \dfrac{x^2 - 5x + 8}{x - 2}$

   Solution:

$$\begin{aligned}
\lim_{x \to 3} \frac{x^2 - 5x + 8}{x - 2} &= \frac{\lim\limits_{x \to 3}(x^2 - 5x + 8)}{\lim\limits_{x \to 3}(x - 2)} \\
&= \frac{\lim\limits_{x \to 3} x^2 - \lim\limits_{x \to 3} 5x + \lim\limits_{x \to 3} 8}{\lim\limits_{x \to 3} x - \lim\limits_{x \to 3} 2} \\
&= \frac{9 - 15 + 8}{3 - 2} \\
&= 2.
\end{aligned}$$

   The calculation works because the limit in the denominator is non-zero.

(b) $\lim_{x\to -2} \frac{x^3+2x^2+4x+8}{x^2+8x+12}$

Solution:

$$\begin{aligned}
\lim_{x\to -2} \frac{x^3+2x^2+4x+8}{x^2+8x+12} &= \lim_{x\to -2} \frac{(x+2)(x^2+4)}{(x+2)(x+6)} \\
&= \lim_{x\to -2} \frac{x^2+4}{x+6} \\
&= \frac{4+4}{-2+6} \\
&= 2
\end{aligned}$$

3. When a camera flash goes off, the batteries immediately begin to recharge the flash's capacitor, which stores charge $Q(t)$ according to the equation

$$Q(t) = Q_0(1-e^{-t/a}).$$

(The maximum possible charge is Q_0 and t is measured in seconds; a is a certain positive constant that depends upon the battery, and the circuitry.)

(a) Find the inverse of this function and explain its meaning.

Solution: Writing only Q in place of $Q(t)$, we have

$$\begin{aligned}
Q &= Q_0(1-e^{-t/a}); \\
\frac{Q}{Q_0} &= 1-e^{-t/a}; \\
e^{-t/a} &= 1-\frac{Q}{Q_0}; \\
e^{-t/a} &= \frac{Q_0-Q}{Q_0}; \\
-\frac{t}{a} &= \ln\left(\frac{Q_0-Q}{Q_0}\right); \\
t &= a\ln\left(\frac{Q_0}{Q_0-Q}\right).
\end{aligned}$$

The last equation gives the inverse function, which tells us how long (t) it takes for the capacitor to reach a specified charge (Q).

(b) How long does it take to recharge the capacitor to 90% of capacity if $a = 2$?

Solution: We put $a = 2$ and $Q = 0.90Q_0$ in the last equation of the previous part of this problem, and we obtain

$$\begin{aligned}
t &= 2\ln\left(\frac{Q_0}{Q_0-0.90Q_0}\right) \\
&= 2\ln\left(\frac{Q_0}{0.10Q_0}\right) \\
&= \ln 100 \simeq 4.605 \text{ seconds.}
\end{aligned}$$

4. Give examples to show that when $\lim_{x\to a^+} N(x) = 0$ and $\lim_{x\to a^+} D(x) = 0$, we may have

$$\lim_{x\to a^+} \frac{N(x)}{D(x)} = L,$$

where

(a) $L = \infty$,
Solution: Put $N(x) = x$, $D(x) = x^2$, $a = 0$. Then $\lim_{x\to a+} N(x) = 0$, $\lim_{x\to a^+} D(x) = 0$, and

$$\begin{aligned}\lim_{x\to a^+} \frac{N(x)}{D(x)} &= \lim_{x\to 0^+} \frac{x}{x^2} \\ &= \lim_{x\to 0^+} \frac{1}{x} = \infty.\end{aligned}$$

(b) $L = -\infty$,
Solution: Put $N(x) = -x$, $D(x) = x^2$, $a = 0$. Then $\lim_{x\to a+} N(x) = 0$, $\lim_{x\to a^+} D(x) = 0$, and

$$\begin{aligned}\lim_{x\to a^+} \frac{N(x)}{D(x)} &= \lim_{x\to 0^+} \frac{-x}{x^2} \\ &= -\lim_{x\to 0^+} \frac{1}{x} = -\infty.\end{aligned}$$

(c) $L = 0$, or
Solution: Put $N(x) = x^2$, $D(x) = x$, $a = 0$. Then $\lim_{x\to a+} N(x) = 0$, $\lim_{x\to a^+} D(x) = 0$, and

$$\begin{aligned}\lim_{x\to a^+} \frac{N(x)}{D(x)} &= \lim_{x\to 0^+} \frac{x^2}{x} \\ &= \lim_{x\to 0^+} x = 0.\end{aligned}$$

(d) L is some non-zero real number.
Solution: If L is a non-zero real number, put $N(x) = Lx$, $D(x) = x$, $a = 0$. Then $\lim_{x\to a+} N(x) = 0$, $\lim_{x\to a^+} D(x) = 0$, and

$$\begin{aligned}\lim_{x\to a^+} \frac{N(x)}{D(x)} &= \lim_{x\to 0^+} \frac{Lx}{x} \\ &= \lim_{x\to 0^+} L = L.\end{aligned}$$

Explain why each of your examples exhibits the behavior you say it does.

5. The position of an object moving along the x-axis is given by

$$x(t) = 30 + 60t - t^3,$$

where $x(t)$ is in meters and t is in seconds. Give answers to the following questions in simplest algebraic form.

(a) Where is the object when $t = 5$?
Solution: $x(5) = 30 + 60 \cdot 5 - 5^3 = 205$ meters to the right of the origin.

(b) Where is the object when $t = 6$?
Solution: $x(5) = 30 + 60 \cdot 6 - 6^3 = 174$ meters to the right of the origin.

(c) Where is the object when $t = 5 + h$?
Solution:

$$\begin{aligned} x(5+h) &= 30 + 60 \cdot (5+h) - (5+h)^3 \\ &= 30 + 300 + 60h - (5^3 + 3 \cdot 5^2 \cdot h + 3 \cdot 5 \cdot h^2 + h^3) \\ &= 205 - 15h - 15h^2 - h^3. \end{aligned}$$

(d) How far did the object travel during the interval $5 \le t \le 6$?
Solution: During the second $5 \le t \le 6$, the object traveled $x(6) - x(5) = 174 - 205 = -31$ meters, or 31 meters to the left.

(e) How far did the object travel during the interval $5 \le t \le 5 + h$?
Solution: During the interval $5 \le t \le 5+h$, the object traveled $x(5+h) - x(5) = (205 - 15h - 15h^2 - h^3) - 205 = -15h - 15h^2 - h^3$ meters.

(f) What was the average velocity of the object for the interval $5 \le t \le 6$?
Solution: As t varies over $5 \le t \le 6$, average velocity is

$$\frac{\Delta x}{\Delta t} = \frac{-31}{1},$$

or -31 meters per second.

(g) Assume that $h > 0$. What was the average velocity of the object for the interval $5 \le t \le 5 + h$?
Solution: As t varies over the interval $5 \le t \le 5 + h$, average velocity is

$$\frac{\Delta x}{\Delta t} = \frac{-15h - 15h^2 - h^3}{h},$$

or $-15 - 15h - h^2$ meters per second.

6. Show how to use an appropriate limit to find an equation for the line tangent to the curve $y = x^3 - 2x^2 + 3x - 1$ at the point $(1, 1)$.

Solution: Let h be a small positive number, and onsider the secant line to the curve $y = x^3 - 2x^2 + 3x - 1$ that passes through the points $(1, 1)$ and $(1 + h, y(1 + h))$. We have

$$\begin{aligned} y(1+h) &= (1+h)^3 - 2(1+h)^2 + 3(1+h) - 1 \\ &= h^3 + h^2 + 2h + 1 \end{aligned}$$

Hence, the slope of this secant line is

$$\begin{aligned}\frac{y(1+h)-y(1)}{(1+h)-1} &= \frac{(h^3+h^2+2h+1)-1}{h}\\ &= \frac{h^3+h^2+2h}{h}\\ &= h^2+h+2.\end{aligned}$$

The slope of the tangent line at $(1,1)$ is the limiting value of the slope of this secant line as $h \to 0$. Hence the tangent line has slope

$$\begin{aligned}\lim_{h\to 0}\frac{y(1+h)-y(1)}{(1+h)-1} &= \lim_{h\to 0}(h^2+h+2)\\ &= 2.\end{aligned}$$

The tangent line is thus the line through $(1,1)$ with slope 2, and an equation for this line is $y = 1 + 2(x-1)$, or $y - 2x + 1 = 0$.

7. A scientist has some data points

$$\{\,(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)\,\}.$$

When she took the natural logarithm of each of the y-values and plotted the points

$$\{\,(x_1,\ln y_1),(x_2,\ln y_2),\ldots,(x_n,\ln y_n)\,\}$$

on a graph, she found that all of these points lie on a single straight line. When she saw this, she decided that there must be constants A and r so that her *original* data points all lie on the curve $y = Ae^{rx}$.

(a) Explain the reasoning that supports her conclusion.

Solution: Let us write $v = \ln y$. Her data points $\{(x_k, v_k)\}$ all lie on a single straight line, and so there are constants r and b such that her data points all satisfy the equation $v = rx + b$. Hence

$$\begin{aligned}\ln y &= rx+b;\\ y &= e^{rx+b};\\ y &= e^{rx}\cdot e^{b};\\ y &= Ae^{rx},\end{aligned}$$

where $A = e^b$.

(b) Explain how she could use measurements involving the straight line to determine the values of the constants A and r for her data.

Solution: The number r is just the slope of the line in her plot. Also, $A = e^b$, and b is the v-intercept of the line in her plot. Thus, she need only measure the slope and the v-intercept of the line in her plot to determine r and A.

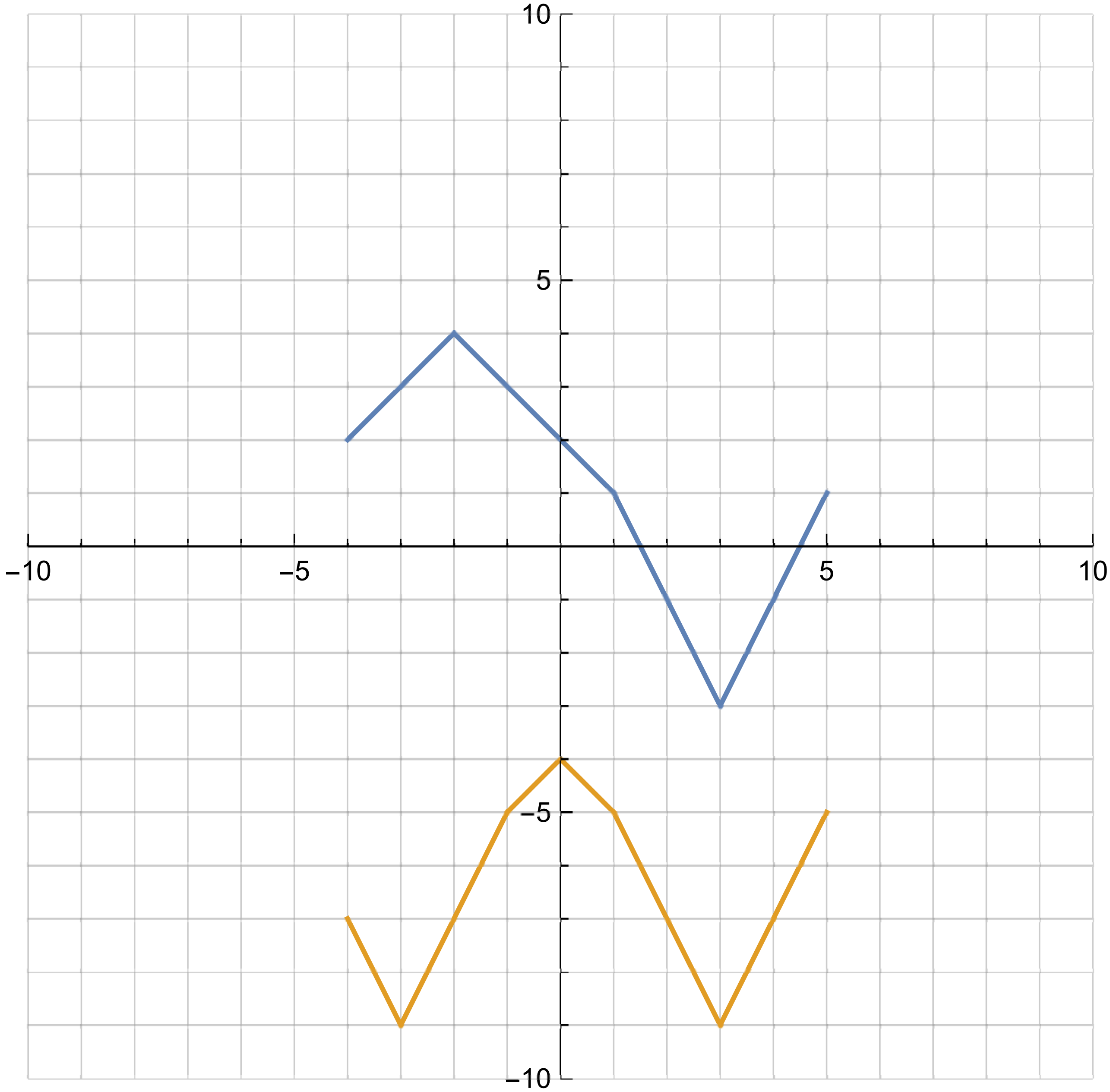


Figure 1:

MTH 141, Calculus I—Final Exam **Dec 17, 1998**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions.* Your paper is due at 3:55 pm.

1. Find the following limits analytically. Show all of your calculations.

   (a) $\lim_{x\to 4} \dfrac{x^2 - 7x + 16}{x - 3}$

   (b) $\lim_{x\to 1} \dfrac{x^3 - 7x^2 + 19x - 13}{x^2 + 2x - 3}$

   Solution:

   (a)

$$\lim_{x\to 4} \frac{x^2 - 7x + 16}{x - 3} = \frac{4^2 - 7\cdot 4 + 16}{4 - 3} = 4. \tag{1}$$

   (b)

$$\lim_{x\to 1} \frac{x^3 - 7x^2 + 19x - 13}{x^2 + 2x - 3} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2 - 6x + 13)}{\cancel{(x-1)}(x+3)} = \frac{1 - 6 + 13}{1 + 3} = 2. \tag{2}$$

2. When $\lim_{x\to a^+} N(x) = 0$ and $\lim_{x\to a^+} D(x) = 0$, we may have any of several behaviors for $\lim_{x\to a^+} \dfrac{N(x)}{D(x)}$. Give examples to show that each of the following is one of the possibilities:

$$\lim_{x\to a^+} \frac{N(x)}{D(x)} \quad = \quad L,$$

   where

   (a) $L = \infty$,

   (b) $L = -\infty$,

   (c) $L = 0$, or

   (d) L is some non-zero real number.

   Explain why each of your examples exhibits the behavior you say it does.

   Solution:

   (a) Let $N(x) = (x-3)$, $D(x) = (x-3)^2$. Then $\lim_{x\to 3^+} N(x) = 0 = \lim_{x\to 3^+} D(x)$, but

$$\lim_{x\to 3^+} \frac{N(x)}{D(x)} = \lim_{x\to 3^+} \frac{\cancel{x-3}^{\,1}}{(x-3)^{\cancel{2}\,1}} = +\infty, \tag{3}$$

   because $(x-3) > 0$ when $x > 3$, while the numerator is the positive number 1.

   (b) Let $D(x)$ be as in Problem 2a, and let $N(x)$ be the negative of the $N(x)$ of Problem 2a. Then a calculation similar to that of Problem 2a leads to

$$\lim_{x\to 3^+} \frac{N(x)}{D(x)} = -\infty, \tag{4}$$

(c) Let $N(x) = (x-3)^2$, $D(x) = (x-3)$. Then $\lim_{x\to 3^+} N(x) = 0 = \lim_{x\to 3^+} D(x)$, but

$$\lim_{x\to 3^+} \frac{N(x)}{D(x)} = \lim_{x\to 3^+} \frac{(x-3)^{\cancel{2}\,1}}{\cancel{x-3}\,1} = 0. \tag{5}$$

(d) Let $N(x) = x^2 - 4x + 3$, and let $D(x) = x^2 - 5x + 6$. Then $\lim_{x\to 3^+} N(x) = 0 = \lim_{x\to 3^+} D(x)$, but

$$\lim_{x\to 3^+} \frac{N(x)}{D(x)} = \lim_{x\to 3^+} \frac{x^2-4x+3}{x^2-5x+6} \tag{6}$$

$$= \lim_{x\to 3^+} \frac{(x-1)\cancel{(x-3)}}{(x-2)\cancel{(x-3)}} = 2 \neq 0. \tag{7}$$

3. Show how to use an appropriate limit to find an equation for the line tangent to the curve $y = 4x^2 - 5x + 2$ at the point $(1, 1)$.

Solution: An equation for the line passing through the point (x_0, y_0) with slope m is

$$y = y_0 + m(x - x_0). \tag{8}$$

and the slope of the line tangent to the curve $y = 4x^2 - 5x + 2$ at the point $(1, 1)$ is given by

$$\lim_{h\to 0} \frac{4(x+h)^2 - 5(x+h) + 2 - (4x^2 - 5x + 2)}{h} = \lim_{h\to 0} \frac{\cancel{4x^2} + 8xh + 4h^2 - \cancel{5x} - 5h + \cancel{2} - \cancel{4x^2} + \cancel{5x} - \cancel{2}}{h} \tag{9}$$

$$= \lim_{h\to 0} \frac{(8x + 4h - 5)\cancel{h}}{\cancel{h}} = 8x - 5 \tag{10}$$

after we set $x = 1$. Thus, the slope of the required tangent line is 3 and we may write

$$y = 1 + 3(x - 1) \tag{11}$$

for an equation of that line.

4. Find the absolute maximum and the absolute minimum for the function f given by

$$f(x) = (6x^2 - 48x + 96)e^x - 96 \tag{12}$$

on the interval $[-1, 5]$. Explain your reasoning.

Solution: The funciton f is differentiable everywhere, so the absolute extremes of $f(x)$ on the interval $[-1, 5]$ will be found at either endpoints of the interval or values of x where $f(x) = 0$. But

$$f'(x) = (12x - 48)e^x + (6x^2 - 48x + 96)e^x = 6(x^2 - 6x + 8)e^x \tag{13}$$

$$= 6(x-2)(x-4)e^x. \tag{14}$$

Now e^x is never zero, so $f'(x) = 0$ only when $x = 2$ or when $x = 4$. We compute:

$$f(-1) \sim -40.818 \tag{15}$$

$$f(2) \sim 81.337; \tag{16}$$

$$f(4) = -96 \tag{17}$$

$$f(5) \sim 794.479. \tag{18}$$

Finally, we conclude that the desired minimum is -96 at $x = 4$, and the desired maximum is about 794.479 at $x = 5$.

5. Find the points on the hyperbola $y^2 - x^2 = 4$ that are closest to the point $(2, 0)$. Explain your reasoning.

Solution: Let (x, y) be a point on the hyperbola $y^2 - x^2 = 4$. Then $y^2 = 4 + x^2$, or $y = \pm\sqrt{4 + x^2}$, and x may be any real number. If D is the distance from the point (x, y) to the point $(2, 0)$, then $D^2 = (x - 2)^2 + (y - 0)^2 = (x - 2)^2 + y^2$. The smallest value of D must be found at a point where $D' = 0$ or D' doesn't exist. But

$$2DD' = 2(x - 2) + 2yy', \tag{19}$$

which is meaningful for all real values of x. Consequently, we want those values of x for which $D' = 0$, or for which $x - 2 + yy' = 0$. Equivalently, we need

$$y' = \frac{2 - x}{y}. \tag{20}$$

From the equation $y^2 - x^2 = 4$, we see that $2yy' - 2x = 0$, or

$$y' = \frac{x}{y}. \tag{21}$$

When we combine (20) and (21), we find that

$$\frac{2 - x}{y} = \frac{x}{y}, \tag{22}$$

whence $y = 0$ or $2 - x = x$. The first of these can't be, because $y^2 = 4 + x^2 \geq 4$, and the second of these is equivalent to $x = 1$. We conclude that the points on the curve $y^2 - x^2 = 4$ that are closest to the point $(2, 0)$ are the points for which $x = 1$, or the points $\left(1, \sqrt{5}\right)$ and $\left(1, -\sqrt{5}\right)$, where $D = \sqrt{1 + 5} = \sqrt{6}$.

6. Use the Fundamental Theorem of Calculus to find:

(a) $\displaystyle \frac{d}{dx} \int_0^x \sqrt[3]{1 + t^3}\, dt$.

(b) $\displaystyle \int_1^2 (3t^2 - 4t + 5)\, dt$

(c) $\displaystyle \int_0^2 t\sqrt{1 + t^2}\, dt$.

Solution:

(a)

$$\frac{d}{dx} \int_0^x \sqrt[3]{1 + t^3}\, dt = \sqrt[3]{1 + x^3}. \tag{23}$$

(b)

$$\int_1^2 (3t^2 - 4t + 5)\, dt = (t^3 - 2t^2 + 5t)\Big|_1^2 = (8 - 8 + 10) - (1 - 2 + 5) = 6. \tag{24}$$

(c) Let $u = 1 + t^2$. Then $du = 2t\, dt$, or $t\, dt = \frac{1}{2}\, du$. In addition, $u = 2$ when $t = 1$ and $u = 5$ when $t = 2$. Therefore

$$\int_0^2 t\sqrt{1 + t^2}\, dt = \int_0^2 \sqrt{1 + t^2}\, t\, dt = \frac{1}{2} \int_2^5 u^{1/2}\, du = \frac{1}{3} u^{3/2}\Big|_2^5 = \frac{1}{3}\left(5\sqrt{5} - 2\sqrt{2}\right). \tag{25}$$

7. Suppose that x and y are related by the equation

$$x^3y^3 - 3x^2y + xy - 4 \quad = \quad 0. \tag{26}$$

(a) Verify that $y = 2$ satisfies equation (26) when $x = 1$.

(b) Find $y' = \dfrac{dy}{dx}$ when $x = 1$ and $y = 2$.

(c) If the value of x changes slightly from $x = 1$ to $x = 0.995$, it is reasonable to suppose that there is a corresponding value of y not too far away from $y = 2$. What is your best estimate for this new value of y? Explain your reasoning.

Solution:

(a) Setting $x = 1$, $y = 2$, on the left side of (26) gives

$$1^3 \cdot 2^3 - 3 \cdot 1^2 \cdot 2 + 1 \cdot 2 - 4 = 8 - 6 + 2 - 4 = 0. \tag{27}$$

The coordinates of the given point satisfy (26), so the point lies on the curve defined by that equation.

(b) Differentiating both sides of (26) with respect to x while treating y as a function of x gives

$$3x^2y^3 + 3x^3y^2y' - 6xy - 3x^2y' + y + xy' = 0, \tag{28}$$

or

$$\frac{dy}{dx} = y' = \frac{6xy - y - 3x^2y^3}{3x^3y^2 - 3x^2 + x}. \tag{29}$$

Therefore,

$$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{12 - 2 - 24}{12 - 3 + 1} = -\frac{7}{5}. \tag{30}$$

(c) If (x_1, y_1) is a point on the line, $y = y_0 + f'(x_0)(x - x_0)$, tangent to the curve $y = f(x)$ at a point (x_0, y_0), where x_1 is near x_0, then y_1 is a good approximation to $f(x_1)$. Hence, the desired value of y is approximately

$$2 - \frac{7}{5}(0.995 - 1) = 2.007. \tag{31}$$

8. Two troopers in a Colorado Highway Patrol car are east-bound at 55 mph on a straight highway out on the eastern plains. They are approaching an intersection with a north-south highway. Another car is northbound on the other highway, also approaching the intersection. At the instant when the patrol car was 400 yards west of the intersection, the other car was 300 yards south of the intersection and one of the troopers used a radar device to determine that the distance between the two vehicles was decreasing at 77 miles per hour. If the speed limit on the north-south highway is 55 miles per hour, should the troopers cite the driver of the other vehicle for speeding? Give your reasoning. (1 mile is 1760 yards.)

Solution: Let x be the distance, measured with east positive, between the patrol car and the intersection, and let y be the distance, measured with north positive, between the other car and the intersection. Then ,D, the distance between the two cars satisfies

$$D^2 = x^2 + y^2, \tag{32}$$

and d, x, and y are all functions of t, time. One mile is 1760 yards, so we have been given, in miles,

$$x = -\frac{400}{1760} = -\frac{5}{22} \tag{33}$$

$$y = -\frac{300}{1760} = -\frac{15}{88} \tag{34}$$

and, in miles per hour

$$\frac{dx}{dt} = 55, \tag{35}$$

$$\frac{dD}{dt} = -77. \tag{36}$$

Substituting our given values of x and y into (32), we find that $D = 25/88$ miles. From equation (32), we see that

$$\cancel{2}D\frac{dD}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} \tag{37}$$

so that

$$y\frac{dy}{dt} = D\frac{dD}{dt} - x\frac{dx}{dt}. \tag{38}$$

Thus

$$\left(-\frac{15}{88}\right) \cdot \frac{dy}{dt} = \frac{25}{88}(-77) - \left(-\frac{5}{22}\right) \cdot (55) \tag{39}$$

$$\frac{dy}{dt} = 55. \tag{40}$$

The other car is approaching the intersection at 55 miles per hour; they should not cite the other driver.

MTH 141, Calculus I—Exam I **Feb. 22, 1999**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions.* Your paper is due at 1:45 pm.

1. The last page of this exam contains a figure which is the complete graph of a certain function f. Answer the following questions about f (If necessary, give your best estimate, but qualify your answer as an estimate):

   (a) What is the domain of f? Explain.

   (b) What is the range of f? Explain.

   (c) What is the largest *interval* on which f is an increasing function? Explain.

   (d) Is f a one-to-one function? Explain.

   (e) Detach the page with the figure on it and put your name on the page. Then sketch the graph of $y = f(|x|) - 3$ on the figure and turn it in with your exam.

2. Show that if the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are drawn on a coordinate grid where the unit of measurement is 1 inch, then at a distance 2 ft to the right of the origin the height of the graph of f is 48 ft but the height of the graph of g is about 265 mi. [Hint: 1 ft = 12 in; 1 mi = 5280 ft.]

3. Use the Limit Laws to evaluate the following limits. You need not cite specific laws as you use them, but you must show all of your calculations.

   (a) $\displaystyle\lim_{x\to -3} \frac{x^2 - x + 12}{x + 3}$

   (b) $\displaystyle\lim_{h\to 0} \frac{(3+h)^{-1} - 3^{-1}}{h}$

4. Show how to use an appropriate limit to find an equation for the line tangent to the curve $y = 2x^3 + 3x^2 + 6x + 3$ at the point $(-1, -2)$. Explain the underlying reasoning.

5. A scientist has some data points

$$\{\,(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\,\}.$$

   When she took the natural logarithms of all of her data and plotted the points

$$\{\,(\ln x_1, \ln y_1), (\ln x_2, \ln y_2), \ldots, (\ln x_n, \ln y_n)\,\}$$

   on a graph, she found that all of these points lie on a single straight line. When she saw this, she decided that there must be constants A and r so that her *original* data points all lie on the curve $y = Ax^r$.

   (a) Explain the reasoning that supports her conclusion.

   (b) Explain how she could use measurements involving the straight line to determine the values of the constants A and r for her data.

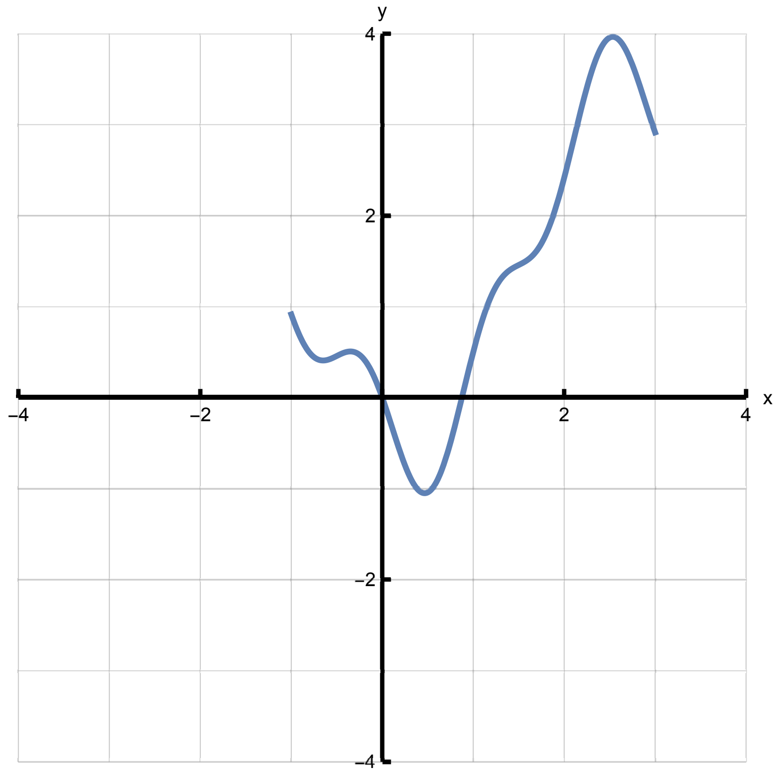


Figure 1: $y = f(x)$

MTH 141, Calculus I—Exam I Solutions **Feb. 22, 1999**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions*. Your paper is due at 1:45 pm.

1. The last page of this exam contains a figure which is the complete graph of a certain function f. Answer the following questions about f (If necessary, give your best estimate, but qualify your answer as an estimate):

   (a) What is the domain of f? Explain.
   Solution: The horizontal extent of the graph is from about $x = -1$ to $x = 3$. Hence the domain of f, which is the set of all admissible inputs to f, is (approximately) the interval $-1 \le x \le 3$.

   (b) What is the range of f? Explain.
   Solution: The vertical extent of the graph is from about $y = -1$ to about $y = 4$. Hence, the range of f, which is the set of all possible outputs from f, is (approximately) the interval $-1 \le y \le 4$.

   (c) What is the largest *interval* on which f is an increasing function? Explain.
   Solution: A function f is increasing on an interval I if it has the property that $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I. (See the definition on page 23 of Stewart.) The largest interval on which the given function has this property is approximately the interval $\frac{1}{2} \le x \le \frac{5}{2}$.

   (d) Is f a one-to-one function? Explain.
   Solution: f is not one-to-one. In order to be one-to-one, f would have to have the property that $f(x_1) \ne f(x_2)$ whenever $x_1 \ne x_2$. (See the definition on page 64 of Stewart.) But f crosses the x-axis at two different points, so there are two numbers $x_1 \ne x_2$ for which $f(x_1) = 0 = f(x_2)$.

   (e) Detach the page with the figure on it and put your name on the page. Then sketch the graph of $y = f(\,|x|\,) - 3$ on the figure and turn it in with your exam.
   Solution: See Figure 1.

2. Show that if the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are drawn on a coordinate grid where the unit of measurement is 1 inch, then at a distance 2 ft to the right of the origin the height of the graph of f is 48 ft but the height of the graph of g is about 265 mi. [Hint: 1 ft = 12 in; 1 mi = 5280 ft.]

   Solution: At a distance 2 ft to the right of the origin, we have $x = 24$. Now $f(24) = 24^2 = 576$, so the point on the curve $y = f(x)$ that corresponds to $x = 24$ is 576 in $= 48$ ft above the x-axis. On the other hand, $g(24) = 2^{24} = 16,777,216$, so that the point on the curve $y = g(x)$ that corresponds to $x = 24$ is $16,777,216$ in above the x-axis. But

$$\begin{aligned} 16,777,216 \text{ in} &= \frac{16,777,216}{12} \text{ ft} \\ &= 1,398,101.33 \text{ ft} \end{aligned}$$

$$\begin{aligned} &= \frac{1,398,101.33}{5280} \text{ mi} \\ &= 264.79 \text{ mi.} \end{aligned}$$

3. Use the Limit Laws to evaluate the following limits. You need not cite specific laws as you use them, but you must show all of your calculations.

   (a) $\displaystyle\lim_{x\to -3} \frac{x^2 - x + 12}{x+3}$

   Solution: We note that

$$\begin{aligned} \lim_{x\to -3}(x^2 - x + 12) &= (-3)^2 - (-3) + 12 \\ &= 24, \end{aligned}$$

   while

$$\begin{aligned} \lim_{x\to -3}(x+3) &= (-3) + 3 \\ &= 0. \end{aligned}$$

   Hence $\displaystyle\lim_{x\to -3} \frac{x^2 - x + 12}{x+3}$ does not exist.

   (b) $\displaystyle\lim_{h\to 0} \frac{(3+h)^{-1} - 3^{-1}}{h}$

   Solution: We have

$$\begin{aligned} \lim_{h\to 0} \frac{(3+h)^{-1} - 3^{-1}}{h} &= \lim_{h\to 0} \left[\left(\frac{1}{h}\right) \cdot \left(\frac{1}{3+h} - \frac{1}{3}\right)\right] \\ &= \lim_{h\to 0} \left[\left(\frac{1}{h}\right) \cdot \left(\frac{3-(3+h)}{3(3+h)}\right)\right] \\ &= \lim_{h\to 0} \frac{-\cancel{h}}{3\cancel{h}(3+h)} \\ &= \lim_{h\to 0} \frac{-1}{3(3+h)} \\ &= -\frac{1}{9}. \end{aligned}$$

4. Show how to use an appropriate limit to find an equation for the line tangent to the curve $y = 2x^3 + 3x^2 + 6x + 3$ at the point $(-1, -2)$. Explain the underlying reasoning.

   Solution: Let $f(x) = 2x^3 + 3x^2 + 6x + 3$, let h be non-zero, and consider the secant line that passes through the points $(-1, -2) = (-1, f(-1))$ and $(-1+h, f(-1+h))$. The slope of this secant line is given by

$$\frac{f(-1+h) - f(-1)}{(-1+h) - (-1)} = \frac{\{2(-1+h)^3 + 3(-1+h)^2 + 6(-1+h) + 3\} - (-2)}{h}$$

$$\begin{aligned} &= \frac{(2h^3 - 3h^2 + 6h - 2) - (-2)}{h} \\ &= \frac{(2h^2 - 3h + 6)\cancel{h}}{\cancel{h}} \\ &= 2h^2 - 3h + 6. \end{aligned}$$

The slope of the tangent line at the point $(-1, -2)$ is the limiting value of the slopes of the secant lines as h goes to zero. Hence, the slope of the required tangent line is

$$\lim_{h \to 0} (2h^2 - 3h + 6) = 6.$$

Consequently, an equation for the line tangent to the curve $y = 2x^3 + 3x^2 + 6x + 3$ at the point $(-1, -2)$ is

$$y = -2 + 6(x + 1).$$

5. A scientist has some data points

$$\{ (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \}.$$

When she took the natural logarithms of all of her data and plotted the points

$$\{ (\ln x_1, \ln y_1), (\ln x_2, \ln y_2), \ldots, (\ln x_n, \ln y_n) \}$$

on a graph, she found that all of these points lie on a single straight line. When she saw this, she decided that there must be constants A and r so that her *original* data points all lie on the curve $y = Ax^r$.

(a) Explain the reasoning that supports her conclusion.

Solution: Let us introduce the variables $u = \ln x$ and $v = \ln y$. Then the condition that the points

$$\{ (\ln x_1, \ln y_1), (\ln x_2, \ln y_2), \ldots, (\ln x_n, \ln y_n) \}$$

all lie on the same straight line can be rewritten as the statement that the points

$$\{ (u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n) \}$$

all lie on the same straight line. Let $v = mu + b$ be an equation for that line. The points

$$\{ (\ln x_1, \ln y_1), (\ln x_2, \ln y_2), \ldots, (\ln x_n, \ln y_n) \}$$

must then all satisfy the equation $\ln y = m \ln x + b$. Applying the natural exponential function $t \mapsto e^t$ to both sides of this equation, we obtain the equivalent equations

$$\begin{aligned} e^{\ln y} &= e^{m \ln x + b}, & (1) \\ y &= e^{m \ln x} \cdot e^b, & (2) \\ y &= e^b \cdot (e^{\ln x})^m, & (3) \\ y &= e^b x^m. & (4) \end{aligned}$$

(We have used the relationship $T = e^{\ln T}$ twice here.) Equation (4) has the form $y = Ax^r$ where $A = e^b$ and $r = m$. Because the points (x_k, y_k), $k = 1, \ldots, n$, all satisfy equation (1), they must also all satisfy equation (4).

(b) Explain how she could use measurements involving the straight line to determine the values of the constants A and r for her data.

Solution: We have seen above that the original data points satisfy an equation of the form $y = Ax^r$, where $A = e^b$ and $r = m$. The values b and m come from the equation $v = mu + b$ of the line on which the points

$$\{ (\ln x_1, \ln y_1), (\ln x_2, \ln y_2), \ldots, (\ln x_n, \ln y_n) \}$$

all lie. Thus she can determine the constants A and r from her line by measuring its slope m, measuring its v-intercept b, and then setting $A = e^b$; $r = m$.

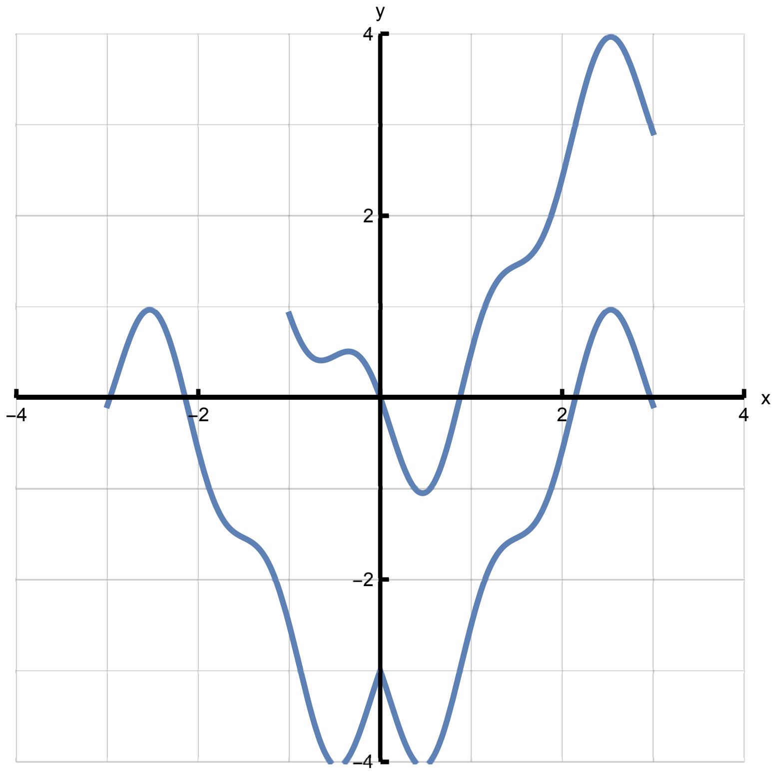


Figure 1: $y = f(x)$

MTH 141, Calculus I—Exam II **Sep. 13, 2001**

Instructions: Answer the following questions as completely as you can. Use your own paper. You must support your answers for full credit. Your paper is due at 3:50 pm. Your paper is due at 3:50 pm.

1. Give an algebraic description of the function f whose graph consists of the straight line segment whose endpoints are $(-5, 3)$ and $(-2, 2)$ together with the lower half of the circle whose radius is 3 and whose center is at $(0, 2)$. Sketch the graph of f. Identify the domain and range of f.

2. A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle passes directly beneath it traveling at a constant speed of 17 ft/sec. Give the distance from the bicycle to the balloon as a function of the time elapsed from the instant when the bicycle is directly below the balloon.

3. suppose that a function f and the graph of the equation $y = f(x)$ are given. Write equations for the graphs that are obtained from the graph of f by
   (a) shifting it upward 3 units.
   (b) shifting it downward 3 units.
   (c) shifting it rightward 3 units.
   (d) shifting it leftward 3 units.
   (e) reflecting it about the x-axis.
   (f) reflecting it about the y-axis.
   (g) stretching it vertically by a factor of 3.
   (h) stretching it horizontally by a factor of 3.
   (i) shifting it 1 unit upward and then stretching the result vertically by a factor of 3.
   (j) replacing the portion of the graph that lies to the left of the y-axis with the reflection (about the y-axis) of the portion that lies to the right of the y-axis.

4. The **Heaviside function** H is give by

$$H(x) = \begin{cases} 0 & : \text{ if } x < 0 \\ 1 & : \text{ if } x > 0. \end{cases}$$

   (a) Sketch the graph of the Heaviside function.
   (b) Sketch the graph of the current $C(t)$ in a circuit if the switch is turned on at time $t = 0$ and a constant current of 3 amps instantaneously begins to flow. (Assume that there is no current when the switch is off.) Write a formula for $C(t)$ in terms of the function H.
   (c) Repeat part (b) if the switch is trne on at time $t = 5$ and turned off at time $t = 10$.

5. Solve for x when $2^{x-5} = 3$. Show all of your reasoning. Give an exact expression for your solution, and then use your calculator to give an approximate solution that that is correct to four digits to the right of the decimal.

6. Find the exponential function $f(x) = Ca^x$ (determine the constants C and a, that is) if the graph of $y = f(x)$ passes through the points $(0, 2)$ and $(2, \frac{2}{9}z0$.

7. Consider the curve whose parametric equations are

$$x = \ln t$$
$$y = \sqrt{t},$$

where $t \geq 1$.

(a) Eliminate the parameter to obtain a Cartesian equation of the form $y = f(x)$ for this curve.

(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced out as the parameter increases.

MTH 141, Calculus I—Exam II **Sep. 13, 2001**

Instructions: Answer the following questions as completely as you can. Use your own paper. You must support your answers for full credit. Your paper is due at 3:50 pm. Your paper is due at 3:50 pm.

1. Give an algebraic description of the function f whose graph consists of the straight line segment whose endpoints are $(-5, 3)$ and $(-2, 2)$ together with the lower half of the circle whose radius is 3 and whose center is at $(0, 2)$. Sketch the graph of f. Identify the domain and range of f.

   Solution: The function f is given by

$$f(x) = \begin{cases} 3 - \dfrac{1}{3}(x+5) & : \text{ if } -5 \le x < -2 \\ 2 - \sqrt{4 - x^2} & : \text{ if } -2 \le x \le 2. \end{cases}$$

   See Figure 1. (The slight break in the curve near $(-2, 2)$ is an artifact of the graphing program, and shouldn't be there.)

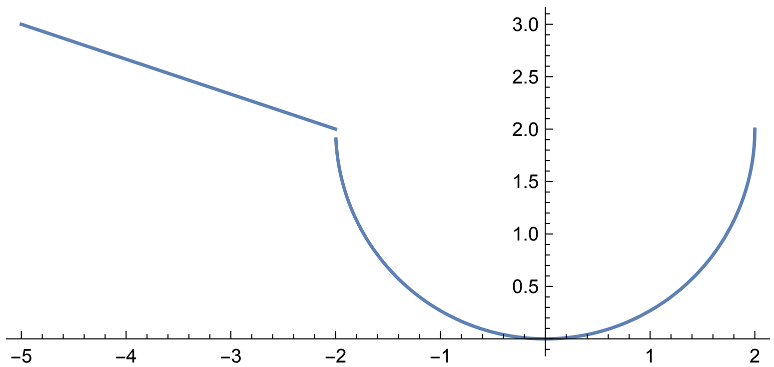


Figure 1: The graph of f

2. A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle passes directly beneath it traveling at a constant speed of 17 ft/sec. Give the distance from the bicycle to the balloon as a function of the time elapsed from the instant when the bicycle is directly below the balloon.

   Solution: At time t seconds after the bicycle passes below the balloon, the balloon will reach an altitude of $65 + t$ feet. At the same time, the bicycle will be $17t$ feet from the point on the road directly below the balloon. By the Pythagorean Theorem, $D(t)$, the distance between the two at time t, will be $D(t) = \sqrt{(65+t)^2 + (17t)^2} = \sqrt{290t^2 + 130t + 4225}$.

3. suppose that a function f and the graph of the equation $y = f(x)$ are given. Write equations for the graphs that are obtained from the graph of f by

   (a) shifting it upward 3 units.

   (b) shifting it downward 3 units.

   (c) shifting it rightward 3 units.

(d) shifting it leftward 3 units.
(e) reflecting it about the x-axis.
(f) reflecting it about the y-axis.
(g) stretching it vertically by a factor of 3.
(h) stretching it horizontally by a factor of 3.
(i) shifting it 1 unit upward and then stretching the result vertically by a factor of 3.
(j) replacing the portion of the graph that lies to the left of the y-axis with the reflection (about the y-axis) of the portion that lies to the right of the y-axis.

Solution:

(a) $y = f(x) + 3$.
(b) $y = f(x) - 3$.
(c) $y = f(x - 3)$.
(d) $y = f(x + 3)$.
(e) $y = -f(x)$.
(f) $y = f(-x)$.
(g) $y = 3f(x)$.
(h) $y = f\left(\frac{x}{3}\right)$.
(i) $y = 3\big[f(x) + 1\big] = 3f(x) + 3$.
(j) $y = f\big(|x|\big)$.

4. The **Heaviside function** H is give by

$$H(x) = \begin{cases} 0 & : \text{ if } x < 0 \\ 1 & : \text{ if } x > 0. \end{cases}$$

(a) Sketch the graph of the Heaviside function.
(b) Sketch the graph of the current $C(t)$ in a circuit if the switch is turned on at time $t = 0$ and a constant current of 3 amps instantaneously begins to flow. (Assume that there is no current when the switch is off.) Write a formula for $C(t)$ in terms of the function H.
(c) Repeat part (b) if the switch is turned on at time $t = 5$ and turned off at time $t = 10$.

Solution:

(a) See Figure 2
(b) See Figure ??
(c) See Figure 4

5. Solve for x when $2^{x-5} = 3$. Show all of your reasoning. Give an exact expression for your solution, and then use your calculator to give an approximate solution that that is correct to four digits to the right of the decimal.

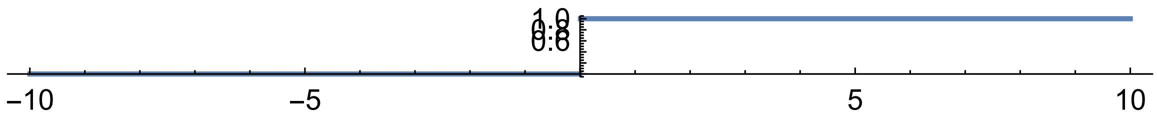


Figure 2: $y = H(x)$

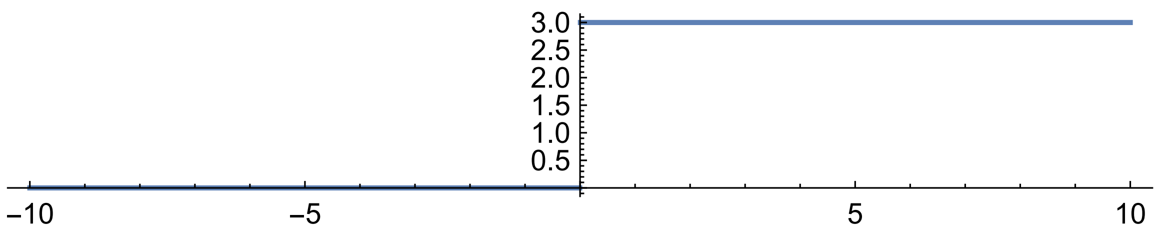


Figure 3: $C(t) = 3H(t)$

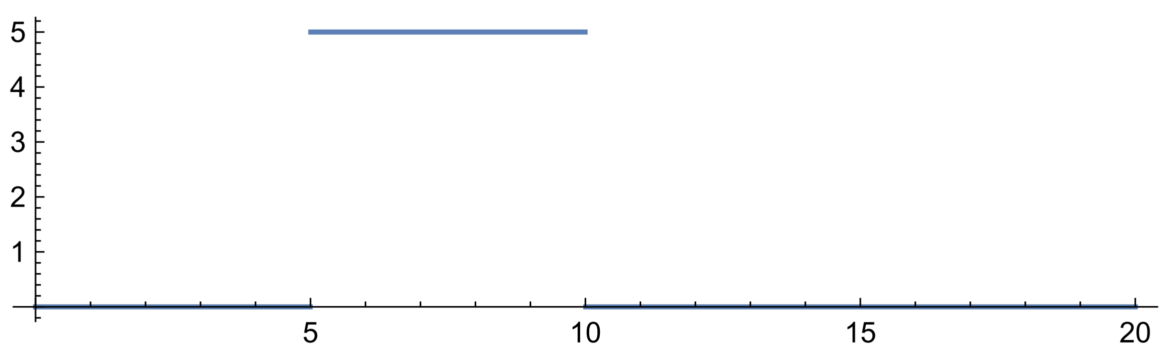


Figure 4: $C(t) = 5H(t-5)H(10-t)$

6. Find the exponential function $f(x) = Ca^x$ (that is, determine the constants C and a) if the graph of $y = f(x)$ passes through the points $(0, 2)$ and $(2, \frac{2}{9})$.

Solution: If $f(0) = 2$, then $2 = Ca^0 = C$, so $C = 2$. But $f(2) = \frac{2}{9}$, so

$$\frac{2}{9} = 2a^2, \text{ whence}$$
$$a = \frac{1}{3}.$$

7. Consider the curve whose parametric equations are

$$x = \ln t$$
$$y = \sqrt{t},$$

where $t \geq 1$.

(a) Eliminate the parameter to obtain a Cartesian equation of the form $y = f(x)$ for this curve.

(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced out as the parameter increases.

Solution:

(a) The equation $x = \ln t$ is equivalent to the equation $t = e^x$. Substituting this latter equation into the equation $y = \sqrt{t}$ gives $y = \sqrt{e^x}$ for the Cartesian equation of the curve. The restriction $t \geq 1$ corresponds to requiring that $x \geq 0$. Hence the curve in question can also by described by the equation $y = \sqrt{e^x}$, where $x \geq 0$.

(b) See Figure 5.

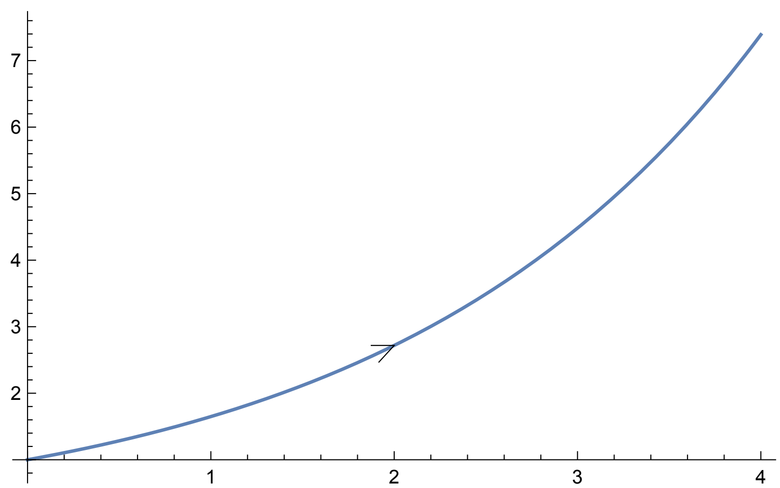


Figure 5: $y = \sqrt{e^x}$

MTH 1410, Calculus I—Exam II **Oct. 11, 2001**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Your paper is due at 3:50 pm.

1. Evaluate the limits:

   (a) $$\lim_{x \to -3} \frac{x^2 + x - 6}{x^2 + 4x + 3}$$

   (b) $$\lim_{x \to -\infty} \frac{x^3 + 3x^2 + 2}{2x^3 + 2x^2 + 2 + 1}$$

2. Find

   $$\lim_{x \to a} \frac{\left(\frac{1}{x^2} - \frac{1}{a^2}\right)}{x - a},$$

   given that $a \neq 0$.

3. Find

   $$\lim_{x \to \infty} \left(\sqrt{x^2 + 6x} - x\right)$$

4. Where is the function f, given by

   $$f(x) = \frac{\sqrt{2 + x} - 2}{x},$$

   continuous? Reasons?

5. A function f is given by

   $$f(x) = \begin{cases} 2cx + 2 & ; \quad x \leq 3 \\ 3 - cx & ; \quad 3 < x. \end{cases}$$

   For what values of the constant c is f continuous on $(-\infty, \infty)$? Be sure to give your reasoning.

6. The altitude y of a falling body is given by

$$y = 1600 + 64t - 16t^2$$

for $t \geq 0$.

(a) What is the body's altitude when $t = 4$?

(b) What is the body's altitude when $t = 4 + h$?

(c) How far did the body travel during the time interval between $t = 4$ and $t = 4 + h$?

(d) What is the body's average velocity over the time interval between $t = 4$ and $t = 4 + h$?

(e) What is the body's velocity at the instant $t = 4$?

7. (a) If $f(x) = \sqrt{x}$ for $0 \leq x$, find $f'[x]$ by directly evaluating an appropriate limit as $h \to 0$.

(b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = \sqrt{x}$ at $x = 1$, at $x = 4$, and at $x = 9$.

MTH 1410, Calculus I—Exam II **Oct. 11, 2001**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Your paper is due at 3:50 pm.

1. Evaluate the limits:

   (a)

$$\lim_{x\to -3} \frac{x^2+x-6}{x^2+4x+3}$$

Solution:

$$\begin{aligned}\lim_{x\to -3} \frac{x^2+x-6}{x^2+4x+3} &= \lim_{x\to -3} \frac{(x+3)(x-2)}{(x+3)(x+1)} \\ &= \lim_{x\to -3} \frac{x-2}{x+1} \\ &= \frac{5}{2}.\end{aligned}$$

   (b)

$$\lim_{x\to -\infty} \frac{x^3+3x^2+2}{2x^3+2x^2+2+1}$$

Solution:

$$\begin{aligned}\lim_{x\to -\infty} \frac{x^3+3x^2+2}{2x^3+2x^2+2+1} &= \lim_{x\to -\infty} \frac{1+(3/x)+(2/x^3)}{2+(2/x)+(3/x^3)} \\ &= \frac{1}{2}.\end{aligned}$$

2. Find

$$\lim_{x\to a} \frac{\left(\dfrac{1}{x^2}-\dfrac{1}{a^2}\right)}{x-a},$$

   given that $a \neq 0$.

Solution:

$$\begin{aligned}
\lim_{x\to a} \frac{\left(\dfrac{1}{x^2}-\dfrac{1}{a^2}\right)}{x-a} &= \lim_{x\to a} \frac{a^2-x^2}{x^2a^2(x-a)} \\
&= \lim_{x\to a} \frac{-\cancel{(x-a)}(x+a)}{x^2a^2\cancel{(x-a)}} \\
&= \lim_{x\to a} \frac{-(x+a)}{x^2a^2} \\
&= -\frac{2a}{a^4} \\
&= -\frac{2}{a^3}.
\end{aligned}$$

3. Find

$$\lim_{x\to\infty} \left(\sqrt{x^2+6x}-x\right)$$

Solution:

$$\begin{aligned}
\lim_{x\to\infty} \left(\sqrt{x^2+6x}-x\right) &= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+6x}-x\right)\left(\sqrt{x^2+6x}+x\right)}{\left(\sqrt{x^2+6x}+x\right)} \\
&= \lim_{x\to\infty} \frac{(x^2+6x)-x^2}{\sqrt{x^2+6x}+x} \\
&= \lim_{x\to\infty} \frac{6x}{\sqrt{x^2+6x}+x} \\
&= \lim_{x\to\infty} \frac{6}{\sqrt{1+(6/x)}+1} \\
&= \frac{6}{\sqrt{1}+1} \\
&= 3.
\end{aligned}$$

4. Where is the function f, given by

$$f(x) = \frac{\sqrt{2+x}-2}{x},$$

continuous? Reasons?

Solution: The square root function $u \mapsto \sqrt{u}$ is continuous on its domain, so the function $x \mapsto \sqrt{2+x}$ is continuous on the interval $[-2, \infty)$. Subtracting 2 has no effect on continuity, so the function in the numerator of f is continuous on the interval $[-2, \infty)$. The denominator of f is the function $x \mapsto x$, and this is continuous everywhere. A quotient of continuous functions is continuous everywhere that its numerator and denominator are both continuous, except where the denominator is zero, so f is thus continuous on $[-2, 0) \cup (0, \infty)$.

5. A function f is given by

$$f(x) = \begin{cases} 2cx + 2 & ; \quad x \leq 3 \\ 3 - cx & ; \quad 3 < x. \end{cases}$$

For what values of the constant c is f continuous on $(-\infty, \infty)$? Be sure to give your reasoning.

Solution: First note that f is continuous at every point with the possible exception of $x = 3$ because f agrees with a polynomial function near every point except $x = 3$. We have

$$\begin{aligned} \lim_{x \to 3^-} f(x) &= \lim_{x \to 3^-} (2cx + 2) \\ &= 6c + 2 \;=\; f(3), \end{aligned}$$

and

$$\begin{aligned} \lim_{x \to 3^+} f(x) &= \lim_{x \to 3^+} (3 - cx) \\ &= 3 - 3c. \end{aligned}$$

Thus, f will be continuous at $x = 3$ if and only if $6c + 2 = 3 - 3c$, which is the same as $9c = 1$ or $c = 1/9$. The function f is therefore continuous on the interval $(-\infty, \infty)$ if, and only if, $c = 1/9$.

6. The altitude y of a certain falling body is given, in feet, by

$$y = 1600 + 64t - 16t^2$$

for $t \geq 0$.

(a) What is the body's altitude when $t = 4$?
Solution:

$$\begin{aligned} y(4) &= 1600 + 64 \cdot 4 - 16 \cdot 4^2 \\ &= 1600 \text{ ft.} \end{aligned}$$

(b) What is the body's altitude when $t = 4 + h$?
Solution:

$$\begin{aligned} y(4+h) &= 1600 + 64(4+h) - 16(4+h)^2 \\ &= (1600 + 64h - 16h^2) \text{ ft.} \end{aligned}$$

(c) How far did the body travel during the time interval between $t = 4$ and $t = 4 + h$?
Solution:

$$\begin{aligned} y(4+h) - y(4) &= (\cancel{1600} + 64h - 16h^2) - \cancel{1600} \\ &= (64h - 16h^2) \text{ ft.} \end{aligned}$$

(d) What is the body's average velocity over the time interval between $t = 4$ and $t = 4 + h$?

Solution:

$$\begin{aligned}\frac{y(4+h)-y(4)}{h} &= \frac{(64\not{h} - 16h^{\not 2}}{\not{h}} \\ &= (64 - 16h) \text{ ft/sec.}\end{aligned}$$

(e) What is the body's velocity at the instant $t = 4$?

Solution:

$$\begin{aligned}\lim_{h\to 0}\frac{y(4+h)-y(4)}{h} &= \lim_{h\to 0}(64 - 16h) \\ &= 64 \text{ ft/sec.}\end{aligned}$$

7. (a) If $f(x) = \sqrt{x}$ for $0 \le x$, find $f'(x)$ by directly evaluating an appropriate limit as $h \to 0$.

Solution:

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}.\end{aligned}$$

(b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = \sqrt{x}$ at $x = 1$, at $x = 4$, and at $x = 9$.

Solution: $f'(1) = 1/(2\sqrt{1}) = 1/2$, so an equation for the line tangent to $y = \sqrt{x}$ at $x = 1$ is

$$y = 1 + (1/2)(x - 1).$$

$f'(4) = 1/(2\sqrt{4}) = 1/4$, so an equation for the line tangent to $y = \sqrt{x}$ at $x = 4$ is

$$y = 2 + (1/4)(x - 4).$$

Finally, $f'(9) = 1/(2\sqrt{9}) = 1/6$, and an equation for the line tangent to the curve $y = \sqrt{x}$ at $x = 9$ is therefore

$$y = 3 + (1/6)(x - 9).$$

MTH 1410, Calculus I—Exam I **Sept. 18, 2002**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Your exam is due at 3:50 pm.

1. (a) Give two distinct parametrizations for the curve that consists of that part of the unit circle that lies in the first quadrant.

   (b) Find a formula for the inverse of the function $f(x) = \dfrac{4x-1}{2x+3}$.

2. Suppose that the graph of a function f is given. Describe how the graphs of the following functions can be obtained from the graph of f:

   (a) $y = f(x) + 8$
   (b) $y = f(x+8)$
   (c) $y = 1 + 2f(x)$
   (d) $y = f(x-2) - 2$
   (e) $y = -f(x)$
   (f) $y = f^{-1}(x)$

3. Use the Limit Laws to find the limits:

   (a) $\displaystyle\lim_{t\to 2} \frac{t^2-4}{t^3-8}$

   (b) $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2+4x-5} - \sqrt{x^2-2x+8}\right)$

4. Let a be a fixed, but unspecified, real number. Find the limit: $\displaystyle\lim_{x\to a} \frac{x^{2/3}-a^{2/3}}{x-a}$.

5. We will find later that when f is given by the equation $f(x) = \tan x$, then f' is given by $f'(x) = \sec^2 x$. Use this fact to write an equation for the line tangent to the curve $y = f(x)$ at the point corresponding to $x = \pi/3$.

6. (a) Use the definition of the derivative to find $f'(2)$ when f is given by $f(x) = x^2 - x$.

   (b) Use the definition of the derivative to find $f'(x)$ when f is given by $f(x) = \dfrac{1}{\sqrt{x}}$.

7. A projectile's height y (in centimeters) at time t (in seconds) is given by

$$y = 3920t - 490t^2$$

   (a) What is the projectile's height when $t = 3$?

   (b) What is the projectile's height when $t = 3 + h$?

   (c) How far did the projectile travel during the time interval between $t = 3$ and $t = 3 + h$?

   (d) What was the projectile's average velocity over the time interval between $t = 3$ and $t = 3 + h$?

   (e) What was the projectile's velocity at the instant $t = 3$?

8. Find all numbers a such that the function f given by

$$f(x) = \begin{cases} x+2 & \text{if } x \le a \\ x^2 & \text{if } x > a \end{cases}$$

   is continuous on $\mathbb{R}$. Explain the reasoning that leads you to your conclusions.

MTH 1410, Calculus I—Exam I **Sept. 18, 2002**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Your exam is due at 3:50 pm.

1. (a) Give two distinct parametrizations for the curve that consists of that part of the unit circle that lies in the first quadrant.

   (b) Find a formula for the inverse of the function $f(x) = \dfrac{4x-1}{2x+3}$.

   Solution:

   (a) Here are a few:

   i.

$$x = t; \tag{1}$$
$$y = \sqrt{1-t^2}; \tag{2}$$

   with $0 \le t \le 1$.

   ii.

$$x = t^2; \tag{3}$$
$$y = \sqrt{1-t^4}; \tag{4}$$

   with $0 \le t \le 1$.

   iii.

$$x = \cos t; \tag{5}$$
$$y = \sin t; \tag{6}$$

   with $0 \le t \le \pi/2$.

   iv.

$$x = \cos^2 t - \sin^2 t \tag{7}$$
$$y = 2 \sin t \cos t; \tag{8}$$

   with $0 \le t \le \pi/4$.

   v.

$$x = \frac{1-t^2}{1+t^2}; \tag{9}$$
$$y = \frac{2t}{1+t^2}; \tag{10}$$

   with $0 \le t \le 1$.

   Observe that we can produce another parametrization from any one of these by interchanging the roles of x and y.

2. Suppose that the graph of a function f is given. Describe how the graphs of the following functions can be obtained from the graph of f:

   (a) $y = f(x) + 8$
   (b) $y = f(x+8)$
   (c) $y = 1 + 2f(x)$
   (d) $y = f(x-2) - 2$
   (e) $y = -f(x)$
   (f) $y = f^{-1}(x)$

Solution:

(a) Translate the graph of $y = f(x)$ upward 8 units.

(b) Translate the graph of $y = f(x)$ leftward 8 units.

(c) Stretch the graph of $y = f(x)$ vertically by a factor of 2 and translate the result upward 1 unit.

(d) Translate the graph of $y = f(x)$ rightward 2 units and then translate the result downward 2 units.

(e) Reflect the graph of $y = f(x)$ about the x-axis.

(f) Reflect the graph of $y = f(x)$ about the line $y = x$.

3. Use the Limit Laws to find the limits:

(a) $\lim_{t \to 2} \frac{t^2 - 4}{t^3 - 8}$

(b) $\lim_{x \to \infty} \left(\sqrt{x^2 + 4x - 5} - \sqrt{x^2 - 2x + 8} \right)$

Solution:

(a)

$$\lim_{t \to 2} \frac{t^2 - 4}{t^3 - 8} = \lim_{t \to 2} \frac{\cancel{(t-2)}(t+2)}{\cancel{(t-2)}(t^2 + 2t + 4)} = \frac{2+2}{4+4+4} = \frac{1}{3}. \qquad (11)$$

(b)

$$\lim_{x \to \infty} \left(\sqrt{x^2 + 4x - 5} - \sqrt{x^2 - 2x + 8} \right) = \lim_{x \to \infty} \frac{(x^2 + 4x - 5) - (x^2 + 2x + 8)}{\left(\sqrt{x^2 + 4x - 5} + \sqrt{x^2 - 2x + 8}\right)} \qquad (12)$$

$$= \lim_{x \to \infty} \frac{2x - 13}{\left(\sqrt{x^2 + 4x - 5} + \sqrt{x^2 - 2x + 8}\right)} \qquad (13)$$

$$= \lim_{x \to \infty} \frac{2 - \frac{13}{x}}{\sqrt{1 + \frac{4}{x} - \frac{5}{x^2}} + \sqrt{1 - \frac{2}{x} + \frac{8}{x^2}}} = 1. \qquad (14)$$

4. Let a be a fixed, but unspecified, real number. Find the limit: $\lim_{x \to a} \frac{x^{2/3} - a^{2/3}}{x - a}$.

Solution:

$$\lim_{x \to a} \frac{x^{2/3} - a^{2/3}}{x - a} = \lim_{x \to a} \frac{\cancel{\left(x^{1/3} - a^{1/3}\right)}\left(x^{1/3} + a^{1/3}\right)}{\cancel{\left(x^{1/3} - a^{1/3}\right)}\left(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}\right)} = \frac{2a^{1/3}}{3a^{2/3}} = \frac{2}{3}a^{-1/3}. \qquad (15)$$

5. We will find later that when f is given by the equation $f(x) = \tan x$, then f' is given by $f'(x) = \sec^2 x$. Use this fact to write an equation for the line tangent to the curve $y = f(x)$ at the point corresponding to $x = \pi/3$.

Solution: If $f'(x) = \sec^2 x$, then $f'\left(\frac{\pi}{3}\right) = \sec^2 \frac{\pi}{3} = 2^2 = 4$. From $f(x) = \tan x$ we see that $f\left(\frac{\pi}{3}\right) = \sqrt{3}$. The required tangent line is therefore the line of slope 4 passing through the point $\left(\frac{\pi}{3}, \sqrt{3}\right)$ with slope 4. An equation for this line is therefore

$$y = \sqrt{3} + 4\left(x - \frac{\pi}{3}\right). \qquad (16)$$

6. (a) Use the definition of the derivative to find $f'(x)$ when f is given by $f(x) = x^2 - x$.

(b) Use the definition of the derivative to find $f'(x)$ when f is given by $f(x) = \frac{1}{\sqrt{x}}$.

Solution:

(a)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^2 - (x+h)] - [x^2 - x]}{h} \tag{17}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x} - h - \cancel{x^2} + \cancel{x}}{h} \tag{18}$$

$$= \lim_{h \to 0} \frac{(2x + h - 1)\cancel{h}}{\cancel{h}} = 2x + 1. \tag{19}$$

(b)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{20}$$

$$= \lim_{h \to 0} \left[\frac{1}{h} \cdot \frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}} \right] \tag{21}$$

$$= \lim_{h \to 0} \left[\frac{1}{h} \cdot \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h}\sqrt{x}} \right] \tag{22}$$

$$= \lim_{h \to 0} \left[\frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} \right] \tag{23}$$

$$= \lim_{h \to 0} \frac{\cancel{x} - (\cancel{x} + h)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x} + \sqrt{x+h}\right)} \tag{24}$$

$$= -\lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\sqrt{x}\sqrt{x+h}\left(\sqrt{x} + \sqrt{x+h}\right)} = -\frac{1}{2x^{3/2}}. \tag{25}$$

7. A projectile's height y (in centimeters) at time t (in seconds) is given by

$$y = 3920t - 490t^2$$

(a) What is the projectile's height when $t = 3$?

(b) What is the projectile's height when $t = 3 + h$?

(c) How far did the projectile travel during the time interval between $t = 3$ and $t = 3 + h$?

(d) What was the projectile's average velocity over the time interval between $t = 3$ and $t = 3 + h$?

(e) What was the projectile's velocity at the instant $t = 3$?

Solution:

(a) When $t = 3$, the projectile's height is $y = 3920 \cdot 3 - 490 \cdot 3^2 = 7350$ cm.

(b) When $t = 3 + h$, the projectile's height is

$$y = 3920 \cdot (3+h) - 490 \cdot (3+h)^2 = 7350 + 980h - 490h^2 \text{ cm.} \tag{26}$$

(c) During the time interval between $t = 3$ and $t = 3 + h$, the projectile traveled

$$[7350 + 980h - 490h^2] - [7350] = 980h - 490h^2 \text{ cm.} \tag{27}$$

(d) The particle's average velocity, V, over the interval from $t = 3$ to $t = 3 + h$ was

$$V = \frac{980h - 490h^2}{h} = 980 - 490h \text{ cm/sec.} \tag{28}$$

(e) The projectile's velocity at time $t = 3$ was

$$\lim_{h \to 0} (980 - 490h) = 980 \text{ cm/sec.} \tag{29}$$

8. Find all numbers a such that the function f given by

$$f(x) = \begin{cases} x+2 & \text{if} \quad x \leq a \\ x^2 & \text{if} \quad x > a \end{cases}$$

is continuous on $\mathbb{R}$. Explain the reasoning that leads you to your conclusions.

Solution: A polynomial function restricted to an open interval is continuous on that open interval, so f is continuous on $(-\infty, a)$ and on (a, ∞). If f is to be continuous on all of $\mathbb{R}$, then, in addition to what we have just observed, we must have

$$\lim_{x \to a} f(x) = f(a). \tag{30}$$

Now according to the definition give for f, we have

$$\lim_{x \to a^-} f(x) = a + 2 = f(a) \text{ and} \tag{31}$$

$$\lim_{x \to a^+} f(x) = a^2. \tag{32}$$

Thus, the equation $\lim_{x \to a} f(x) = f(a)$ will be correct, and f will be continuous on all of $\mathbb{R}$, if and only if $a^2 = a + 2$. But the only solutions of the latter quadratic equation are $a = 2$ and $a = -1$. We conclude that f is continuous on $\mathbb{R}$ in just two cases: when $a = -1$ and when $a = 2$.

MTH 1410, Calculus I—Exam II **Oct. 16, 2002**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Your exam is due at 3:50 pm.

1. The equation of motion of a certain particle is $s = t^3 - 6t^2$.
   (a) Give the velocity and the acceleration as functions of t.
   (b) What is the acceleration when $t = 2$?
   (c) What is the acceleration when velocity is 0?

2. Find $f'(x)$ if
   (a) $f(x) = 4x^8 - 31x^5 + 22x^3 - x^2 + 12$.
   (b) $f(x) = \dfrac{x^3 - x^2}{4x^2 + 1}$.

3. Use the definition of the derivative to show how to find the formula for F', where F is given by $F(x) = f(x) \cdot g(x)$, and f' and g' are both given.

4. When a resistor whose resistance is r ohms is placed in a circuit in parallel with another resistance of 100 ohms, the effective resistance, R, in ohms, in that circuit because of the presence of the two resistors is given by

$$R = \frac{100r}{100 + r}.$$

   Suppose that r is measured as 50 ohms with an error no worse than ± 3 ohms. Use linearization to estimate the maximum error in the computed value for R.

5. Let A and B be constants and let $y = A \sin 3x + Bx \sin 3x$. What values should we assign to A and B in order to have $y'' + 9y = 12 \cos 3x$?

6. (a) Find an equation for the line tangent to the curve $x^3 - 4xy + 2y^3 = 2$ at the point $(2, 1)$.
   (b) Show how to use the result in part (a) to estimate the value of y for the point on the curve near $(2, 1)$ where $x = 1.997$.

7. A function f is to be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \le 1, \\ ax + b & \text{if } x > 1, \end{cases}$$

   where a and b are constants. If $f'(1)$ is to exist, what must values a and b have? What value does $f'(1)$ have?

8. The function f given by $f(x) = x^3 + x$ can be shown to be a one-to-one function carrying the set of all real numbers onto the set of all real numbers. Hence the inverse function, f^{-1}, is also defined for all real numbers. If $u = f^{-1}(v)$, what is $\left.\dfrac{du}{dv}\right|_{v=10}$?

MTH 1410, Calculus I—Exam II Solutions **Oct. 16, 2002**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Your exam is due at 3:50 pm.

1. The equation of motion of a certain particle is $s = t^3 - 6t^2$.

   (a) Give the velocity and the acceleration as functions of t.

   (b) What is the acceleration when $t = 2$?

   (c) What is the acceleration when velocity is 0?

   Solution:

   (a) Velocity, v is the derivative, taken with respect to time, of position, and acceleration is the derivative, taken with respect to time, of velocity. Therefore,

$$v = \frac{d}{dt}s = \frac{d}{dt}(t^3 - 6t^2) = 3t^2 - 12t, \text{ and} \tag{1}$$

$$a = \frac{d}{dt}v = \frac{d}{dt}(3t^2 - 12t) = 6t - 12. \tag{2}$$

2. Find f' if

   (a) $f(x) = 4x^8 - 31x^5 + 22x^3 - x^2 + 12$.

   (b) $f(x) = \dfrac{x^3 - x^2}{4x^2 + 1}$.

   Solution:

   (a)

$$f'(x) = 32x^7 - 155x^4 + 66x^2 - 2x. \tag{3}$$

   (b)

$$f'(x) = \frac{(3x^2 - 2x)(4x^2 + 1) - (x^3 - x^2)(8x)}{(4x^2 + 1)^2}. \tag{4}$$

3. Use the definition of the derivative to show how to find the formula for F', where F is given by $F(x) = f(x) \cdot g(x)$, and f' and g' are both given.

Solution:

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} && (5)\\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} && (6)\\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} && (7)\\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} g(x+h)\right] + \lim_{h\to 0} \left[f(x)\frac{g(x+h) - g(x)}{h}\right] && (8)\\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h}\right] \lim_{h\to 0} g(x+h) + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}. && (9)
\end{aligned}$$

But it is given that g' exists, and this guarantees that g is continuous. This means that $\lim_{h\to 0} g(x+h) = g(x)$. The other two limits are the derivatives of f and g, so

$$F'(x) = f'(x)g(x) + f(x)g'(x). \qquad (10)$$

4. When a resistor whose resistance is r ohms is placed in a circuit in parallel with another resistance of 100 ohms, the effective resistance, R, in ohms, in that circuit because of the presence of the two resistors is given by

$$R \;=\; \frac{100r}{100+r}.$$

Suppose that r is measured as 50 ohms with an error no worse than ± 3 ohms. Use linearization to estimate the maximum error in the computed value for R.

Solution: We begin by noting that $R = \frac{5000}{150} = \frac{100}{3}$ when $r = 50$. Then we note that

$$\begin{aligned}
\frac{dR}{dr} &= \frac{10000}{(100+r)^2} \text{ so that} && (11)\\
\left.\frac{dR}{dr}\right|_{r=50} &= \frac{10000}{150^2} = \frac{4}{9}. && (12)
\end{aligned}$$

Consequently, the linearization function L for R at $r = 50$ is given by

$$L(r) = \frac{100}{3} + \frac{4}{9}(r - 50). \qquad (13)$$

But $L(r)$ is a good approximation for any value of R that corresponds to a value of r close to 50, so $\left|L(r) - R\right|$ is small when $|r - 50|$ is small. Because

$$\left|L(r) - R\right| = \left|\left(\frac{100}{3} - \frac{4}{9}(r-50)\right) - \frac{100}{3}\right| = \frac{4}{9}\left|r - 50\right| \qquad (14)$$

and we know that $\left|r - 50\right| \le 3$, we estimate that error in R under the given circumstances is no more than about $\frac{4}{9} \cdot 3 = \frac{4}{3}$ ohms.

5. Let A and B be constants and let $y = A\sin 3x + Bx\sin 3x$. What values should we assign to A and B in order to have $y'' + 9y = 12\cos 3x$?

Solution: We have

$$\begin{aligned}
y' &= 3A\cos 3x + B\sin 3x + 3Bx\cos 3x && (15)\\
&= (3A + 3Bx)\cos 3x + B\sin 3x; && (16)\\
y'' &= 3B\cos 3x - 3(3A + 3Bx)\sin 3x + 3B\cos 3x && (17)\\
&= 6B\cos 3x - 9(A + B)\sin 3x. && (18)
\end{aligned}$$

consequently,

$$\begin{aligned}
y'' + 9y &= 3B\cos 3x - \cancel{(3A + 3Bx)\sin 3x} + 3B\cos 3x + \cancel{9(A\sin 3x + Bx\sin 3x)} && (19)\\
&= 6B\cos 3x. && (20)
\end{aligned}$$

This is $12\cos 3x$ if, and only if, $B = 2$. The unknown A may take on any real value.

6. (a) Find an equation for the line tangent to the curve $x^3 - 4xy + 2y^3 = 2$ at the point $(2, 1)$.

   (b) Show how to use the result in part (a) to estimate the value of y for the point on the curve near $(2, 1)$ where $x = 1.997$.

Solution:

(a) Applying implicit differentiation with respect to x to the equation

$$x^3 - 4xy + 2y^3 = 2 \tag{21}$$

gives

$$\begin{aligned}
3x^2 - 4y - 4xy' + 6y^2y' &= 0, \text{ or} && (22)\\
y' &= \frac{4y - 3x^2}{6y^2 - 4x}. \text{ Thus} && (23)\\
y'\Big|_{(2,1)} &= \frac{4\cdot 1 - 3\cdot 2^2}{6\cdot 1^2 - 4\cdot 2} = \frac{8}{2} = 4. && (24)
\end{aligned}$$

An equation for the tangent line to the curve at $(2, 1)$ is $y = y(2) + y'(2)(x - 2)$, or

$$y = 1 + 4(x - 2). \tag{25}$$

(b) Using the principle that the y-coordinates of points on the tangent line that correspond to values of x near a value x_0 are good approximations to y-values nearby on the curve corresponding to those same x values we see that we can estimate the value of y on the curve near $(2, 1)$ at $x = 1.997$ by

$$1 + 4(1.997 - 2) = 1 + 4\cdot 1.0003 = 0.988. \tag{26}$$

7. A function f is to be defined by

$$f(x) \quad = \quad \begin{cases} x^2 & \text{if} \quad x \le 1, \\ ax + b & \text{if} \quad x > 1, \end{cases}$$

where a and b are constants. If $f'(1)$ is to exist, what must values a and b have? What value does $f'(1)$ have?

Solution: For $f'(1)$ to exist, f must be continuous at $x = 1$ and we will need $f'_-(1) = f'_+(1)$. But $f(1) = 1$ and $\lim_{x\to 1^-} f(x) = 1$. Therefore, we must have $\lim_{x\to 1^+} f(x) = 1$. But. whatever a and b may be, we know that

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (ax + b) = a + b. \tag{27}$$

Thus, continuity of f at $x = 1$ requires that $a + b = 1$. Let us assume, for the moment, that $a + b = 1$—so that this condition is met. Then in order for $f'_-(1) = f'_+(1)$, we will need to know that

$$f'_-(1) = \lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h}, \text{ or} \tag{28}$$

$$\lim_{h\to 0^-} \frac{(1+h)^2 - 1}{h} = \lim_{h\to 0^-} \frac{1 + 2h + h^2 - 1}{h} = 2, \text{ while} \tag{29}$$

$$f'_+(1) = \lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{a(\cancel{1} + \overset{1}{\cancel{h}}) + \cancel{b} - \cancel{a} - \cancel{b}}{\underset{1}{\cancel{h}}} = a. \tag{30}$$

Thus, the condition that the two one-sided derivatives be equal requires that $a = 2$. From this and the requirement for continuity ($a + b = 1$), we see that $b = -1$. We conclude that differentiability of f at $x = 1$ requires that $a = 2$ and $b = 1$. The resulting value of f' is $f'(1) = 2$.

8. The function f given by $f(x) = x^3 + x$ can be shown to be a one-to-one function carrying the set of all real numbers onto the set of all real numbers. Hence the inverse function, f^{-1}, is also defined for all real numbers. If $u = f^{-1}(v)$, what is $\left.\frac{du}{dv}\right|_{v=10}$?

Solution If $f(x) = x^3 + x$ and $u = f^{-1}(v)$, then $v = f(u)$. But then

$$1 = \frac{dv}{dv} = \frac{d}{dv} f(u) = f'(u)\frac{du}{dv}, \tag{31}$$

by the Chain Rule. But $u = f^{-1}(v)$ exactly when $v = f(u)$, so that $u = 2$ when $v = 10$, or $2 = f^{-1}(10)$, because $10 = f(2)$, as is easily checked. Now $f'(x) = 3x^2 + 1$, so $f'(2) = 13$. Thus,

$$1 = f'(2) \cdot \left.\frac{du}{dv}\right|_{v=10}, \text{ and} \tag{32}$$

$$\left.\frac{du}{dv}\right|_{v=10} = \frac{1}{13}. \tag{33}$$

MTH 1410, Calculus I—Final Exam **Dec. 11, 2002**

Instructions: Write out your solutions for the following problems on your own paper; give your reasoning and show your supporting calculations. Your exam is due at 3:50 pm.

1. Find an equation for the line tangent to the curve
$$y = x \sin x + x \cos x \tag{1}$$
at the point where $x = \pi/2$.

2. Evaluate the limits (without using a calculator):

   (a) $\displaystyle\lim_{x\to 289} \frac{\sqrt{x}-17}{x-289}$.

   (b) $\displaystyle\lim_{x\to\infty} \left(\sqrt{3x^2+2x+2} - \sqrt{3x^2-4x+4}\right)$.

3. Use the definition of the derivative to find $f'(x)$ when
$$f(x) = \sqrt{x^2+x}\ . \tag{2}$$

4. Find the area bounded by the x-axis, the lines $x = 0$ and $x = 2\sqrt{2}$, and the curve
$$y = \frac{x}{\sqrt{1+x^2}}. \tag{3}$$

5. (a) Find an equation for the line tangent to the curve given by
$$x^3 + xy + y^3 = 11 \tag{4}$$
at the point $(1, 2)$.

   (b) Use the result of part (a) of this question to estimate the value of y near $y = 2$ that satisfies equation (4) when $x = \frac{19}{20}$. (Give your answer as a fraction whose numerator and denominator are both whole numbers.)

6. Give: f and g are functions for which $f'(x_0)$ and $g'(x_0)$ are both defined, and $g'(x_0) \neq 0$. Let F be the function defined by
$$F(x) = \frac{f(x)}{g(x)}. \tag{5}$$
Show how to derive the formula for $F'(x_0)$.

7. Let f be the function given by $f(x) = \sin^3 x$.

   (a) Explain why $f'(x) = 3\sin^2 x \cos x$.

   (b) Locate the critical numbers for f that lie in the interval $\left[-\frac{3\pi}{4}, \frac{5\pi}{4}\right]$ and determine whether each gives a local minimum, a local maximum, or neither for f.

   (c) Find the maximum value and the minimum value of f on the interval $\left[-\frac{3\pi}{4}, \frac{5\pi}{4}\right]$.

8. A Coast Guard cutter is moored offsuore at night when a motorboat speeds past its eastern side headed due north with constant speed 3.5 yards per second. The cutter's crew track the motorboat with a searchlight. If the motorboat's closest approach to the searchlight is 100 yards, how fast is the searchlight turning when the boat bears 30° north of east from the light?

MTH 1410, Calculus I—Final Exam **Solutions** **Dec. 11, 2002**

Instructions: Write out your solutions for the following problems on your own paper; give your reasoning and show your supporting calculations. Your exam is due at 3:50 pm.

1. Find an equation for the line tangent to the curve

$$y = x \sin x + x \cos x \tag{1}$$

at the point where $x = \pi/2$.

Solution: We have $y' = x \cos x - \sin x$, when $y'(\pi/2) = -1$. Because $y(\pi/2) = \pi/2$, the equation of the desired tangent line is $y = \pi/2 - (x - \pi/2)$, or $y = \pi - x$.

2. Evaluate the limits (without using a calculator):

(a) $\displaystyle\lim_{x\to 289} \frac{\sqrt{x} - 17}{x - 289}$.

(b) $\displaystyle\lim_{x\to\infty} \left(\sqrt{3x^2 + 2x + 2} - \sqrt{3x^2 - 4x + 4}\right)$.

Solution:

(a)

$$\lim_{x\to 289} \frac{\sqrt{x} - 17}{x - 289} = \lim_{x\to 289} \frac{\cancel{(\sqrt{x} - 17)}}{\cancel{(\sqrt{x} - 17)}(\sqrt{x} + 17)} = \frac{1}{34}. \tag{2}$$

(b)

$$\lim_{x\to\infty} \left(\sqrt{3x^2 + 2x + 2} - \sqrt{3x^2 - 4x + 4}\right) = \lim_{x\to\infty} \frac{(3x^2 + 2x + 2) - (3x^2 - 4x + 4)}{\sqrt{3x^2 + 2x + 2} + \sqrt{3x^2 - 4x + 4}} \tag{3}$$

$$= \lim_{x\to\infty} \frac{6x - 2}{\sqrt{3x^2 + 2x + 2} + \sqrt{3x^2 - 4x + 4}} \tag{4}$$

$$= \lim_{x\to\infty} \frac{6 - \frac{2}{x}}{\sqrt{3 - \frac{2}{x} + \frac{2}{x^2}} + \sqrt{3 - \frac{4}{x} + \frac{4}{x^2}}} \tag{5}$$

$$= \frac{6}{2\sqrt{3}} = \sqrt{3}. \tag{6}$$

3. Use the definition of the derivative to find $f'(x)$ when

$$f(x) = \sqrt{x^2 + x}\ . \tag{7}$$

Solution: Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{8}$$

$$= \lim_{h\to 0} \frac{\sqrt{(x+h)^2 + (x+h)} - \sqrt{x^2+x}}{h} \tag{9}$$

$$= \lim_{h\to 0} \frac{\left[\sqrt{(x+h)^2 + (x+h)} - \sqrt{x^2+x}\right]\left[\sqrt{(x+h)^2 + (x+h)} + \sqrt{x^2+x}\right]}{h\left[\sqrt{(x+h)^2 + (x+h)} + \sqrt{x^2+x}\right]} \tag{10}$$

$$= \lim_{h\to 0} \frac{[(x+h)^2 + (x+h)] - [x^2+x]}{h\left[\sqrt{(x+h)^2 + (x+h)} + \sqrt{x^2+x}\right]} \tag{11}$$

$$= \lim_{h\to 0} \frac{[(x+h)^2 - x^2] + [(x+h) - x]}{h\left[\sqrt{(x+h)^2 + (x+h)} + \sqrt{x^2+x}\right]} \tag{12}$$

$$= \lim_{h\to 0} \frac{[(x+h) - x]\{[(x+h) + x] + 1\}}{h\left[\sqrt{(x+h)^2 + (x+h)} + \sqrt{x^2+x}\right]} \tag{13}$$

$$= \lim_{h\to 0} \frac{\not{h}(2x+1+h)}{\not{h}\left[\sqrt{(x+h)^2 + (x+h)} + \sqrt{x^2+x}\right]} = \frac{2x+1}{2\sqrt{x^2+x}}. \tag{14}$$

4. Find the area bounded by the x-axis, the lines $x = 0$ and $x = 2\sqrt{2}$, and the curve

$$y = \frac{x}{\sqrt{1+x^2}}. \tag{15}$$

Solution: The required area is $\displaystyle\int_0^{2\sqrt{2}} \frac{x\,dx}{1+x^2}$. To evaluate this integral, we let $u = 1 + x^2$, so that $x\,dx = \frac{1}{2}du$. We also have $u = 1$ when $x = 0$ and $u = 9$ when $x = 2\sqrt{2}$. Thus,

$$\int_0^{2\sqrt{2}} \frac{x\,dx}{1+x^2} = \frac{1}{2}\int_1^9 u^{-1/2}\,du = u^{1/2}\Big|_1^9 = 3 - 1 = 2. \tag{16}$$

5. (a) Find an equation for the line tangent to the curve given by

$$x^3 + xy + y^3 = 11 \tag{17}$$

at the point $(1, 2)$.

(b) Use the result of part (a) of this question to estimate the value of y near $y = 2$ that satisfies equation (17) when $x = \frac{19}{20}$. (Give your answer as a fraction whose numerator and denominator are both whole numbers.)

Solution:

(a) By implicit differentiation, we find that

$$3x^2 + y + xy' + 3y^2y' = 0 \tag{18}$$

so that, at $(1, 2)$,

$$3 \cdot 1^2 + 2 + 1 \cdot y' + 3 \cdot 2^2 \cdot y' = 0, \tag{19}$$

or

$$y'\Big|_{(1,2)} = -\frac{5}{13}. \tag{20}$$

An equation for the tangent line at $(1, 2)$ is therefore $y = 2 - \frac{5}{13}(x - 1)$.

(b) The values of the linearization, which is the tangent line, at $(1, 2)$ give good approximations for the values of y when x is near 1, so the approximate value we seek is

$$y \sim 2 - \frac{5}{13}\left(\frac{19}{20} - 1\right) = \frac{105}{52}. \tag{21}$$

6. Give: f and g are functions for which $f'(x_0)$ and $g'(x_0)$ are both defined, and $g'(x_0) \neq 0$. Let F be the function defined by

$$F(x) = \frac{f(x)}{g(x)}. \tag{22}$$

Show how to derive the formula for $F'(x_0)$.

Solution:

$$F'(x_0) = \lim_{h \to 0} \frac{F(x_0 + h) - F(x_0)}{h} \tag{23}$$

$$= \lim_{h \to 0} \left[\frac{1}{h} \cdot \left(\frac{f(x_0 + h)}{g(x_0 + h)} - \frac{f(x_0)}{g(x_0)}\right)\right] \tag{24}$$

$$= \lim_{h \to 0} \left[\frac{1}{h} \cdot \frac{f(x_0 + h)g(x_0) - f(x_0)g(x_0 + h)}{hg(x_0)g(x_0 + h)}\right]. \tag{25}$$

But

$$\frac{f(x_0+h)g(x_0)-f(x_0)g(x_0+h)}{h} = \frac{f(x_0+h)g(x_0)-f(x_0)g(x_0)-f(x_0)g(x_0+h)+f(x_0)g(x_0)}{h} \tag{26}$$

$$= \left[\frac{f(x_0+h)-f(x_0)}{h}\right]g(x_0)-f(x_0)\left]\frac{g(x_0+h)-g(x_0)}{h}\right] \tag{27}$$

$$\to f'(x_0)g(x_0)-f(x_0)g'(x_0) \text{ as } h\to 0. \tag{28}$$

Moreover, $g'(x_0)$ exists, so g is continuous at $x=x_0$, and this guarantees that $g(x_0+h)\to g(x_0)$ as $h\to 0$. It now follows that

$$F'(x_0)=\frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{[g(x_0)]^2}. \tag{29}$$

7. Let f be the function given by $f(x)=\sin^3 x$.
   (a) Explain why $f'(x)=3\sin^2 x\cos x$.
   (b) Locate the critical numbers for f that lie in the interval $\left[-\frac{3\pi}{4},\frac{5\pi}{4}\right]$ and determine whether each gives a local minimum, a local maximum, or neither for f.
   (c) Find the maximum value and the minimum value of f on the interval $\left[-\frac{3\pi}{4},\frac{5\pi}{4}\right]$.

Solution:

(a) The Chain Rule tells us that

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}. \tag{30}$$

Taking $u=\sin x$, we have $\frac{du}{dx}=\cos x$. Thus,

$$\frac{d}{dx}\sin^3 x = \frac{d}{dx}u^3 \tag{31}$$

$$= 3u^2\frac{du}{dx} \tag{32}$$

$$= 3\sin^2 x\cos x, \tag{33}$$

as desired.

(b) If $f(x)=\sin^3 x$, then, by what we have seen, $f'(x)$ is meaningful for all values of x. The critical values for f in the interval $\left[-\frac{3\pi}{4},\frac{5\pi}{4}\right]$ are therefore the values for which $3\sin^2 x\cos x=0$ or the values $x=-\frac{\pi}{2}, 0, \frac{\pi}{2}$, and π. From our knowledge of the behaviors of the sine function and the cosine function, we see that $f'(x)$ undergoes a sign change from negative to positive at $x=-\frac{\pi}{2}$ and a sign change from positive to negative at $x=\frac{\pi}{2}$, but undergoes no sign change at either of the other two critical points. We conclude that there is a local minimum at the point $x=-\frac{\pi}{2}$, a local maximum at the point $x=\frac{\pi}{2}$, and that there is neither a local minimum nor a local maximum at the other two critical points.

(c) The absolute extremes of f are to be found either at critical points or at endpoints of the interval. We have

$$f\left(-\frac{3\pi}{4}\right) = -\frac{1}{2\sqrt{2}}, \tag{34}$$
$$f\left(-\frac{\pi}{2}\right) = -1, \tag{35}$$
$$f(0) = 0. \tag{36}$$
$$f\left(\frac{\pi}{2}\right) = 1, \tag{37}$$
$$f(\pi) = 0, \text{ and} \tag{38}$$
$$f\left(\frac{5\pi}{4}\right) = -\frac{1}{2\sqrt{2}}. \tag{39}$$

We conclude that the minimum we seek is -1 at $x = -\frac{\pi}{2}$, while or maximum is 1 at $x = \frac{\pi}{2}$.

8. A Coast Guard cutter is moored offshore at night when a motorboat speeds past its eastern side headed due north with constant speed 3.5 yards per second. The cutter's crew track the motorboat with a searchlight. If the motorboat's closest approach to the searchlight is 100 yards, how fast is the searchlight turning when the boat bears 30° north of east from the light?

Solution: Let θ denote the angle, measured counterclockwise, that the beam makes with due east. Let y denote the directed distance from the motorboat to the line passing due east through the cutter, north being the positive direction. Then $\tan\theta = \frac{y}{100}$, so, differentiating implicitly with respect to time, we obtain

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{100}\frac{dy}{dt}, \text{ or} \tag{40}$$
$$\frac{d\theta}{dt} = \frac{1}{100}\cos^2\theta \frac{dy}{dt}. \tag{41}$$

We have been given $\frac{dy}{dt} = \frac{7}{2}$, and we are interested in the moment when $\theta = \pi/6$, so that $\cos\theta = \sqrt{3}/2$. Thus, at that instant,

$$\frac{d\theta}{dt} = \frac{1}{100}\cdot\frac{3}{4}\cdot\frac{7}{2} = \frac{21}{800} \text{ radians per second.} \tag{42}$$

(This is about 1.504 degrees per second.)

MTH 1410, Calculus I—Exam I **Feb. 11, 2004**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Your exam is due at 4:50 pm.

1. A function is given by the equation $f(x) = 2x/\sqrt{25 - x^2}$.

   (a) What is the domain of f?

   (b) Evaluate

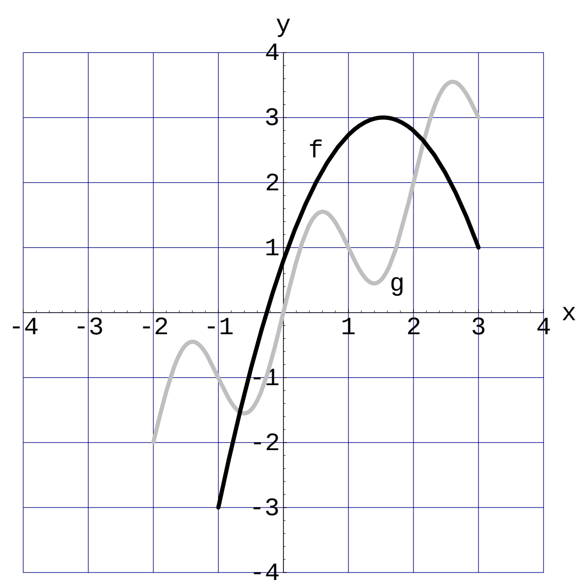
      i. $f(3)$

      ii. $f(3 + h)$

      iii. $\dfrac{f(3 + h) - f(3)}{h}$

   Simplify your answers, but do not give decimal approximations.

2. The graphs of f and g are given in the figure:



   Use the figure to estimate answers for the following:

   (a) What are $f(-1)$ and $g(2)$?

   (b) What are x-values for which $f(x) = g(x)$?

   (c) What are the solutions of the equation $f(x) = 2$?

   (d) What is the largest interval on which f is increasing.

   (e) What are the domain and range of f?

   (f) What are the domain and range of g?

3. A curve is given by the parametric equations

$$\begin{aligned} x &= \sin^2\theta; \\ y &= \cos\theta, \end{aligned}$$

for $0 \leq \theta \leq \pi$.

(a) Eliminate the parameter to find a Cartesian equation for the curve.

(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced out as the parameter increases.

4. (a) Find parametric equations to represent the line segment from $(-3, 5)$ to $(4, -2)$.

(b) Give two distinct parametrizations for the curve that consists of that part of the unit circle that lies in the first quadrant.

5. Suppose that the graph of a function f is given. Describe how the graphs of the following functions can be obtained from the graph of f:

(a) $y = f(x) + 8$

(b) $y = f(x + 8)$

(c) $y = 1 + 2f(x)$

(d) $y = f(x - 2) - 2$

(e) $y = -f(x)$

(f) $y = f(|x|)$

6. A projectile's height y (in centimeters) at time t (in seconds) is given by

$$y = 3920t - 490t^2.$$

(a) What is the projectile's height when $t = 3$?

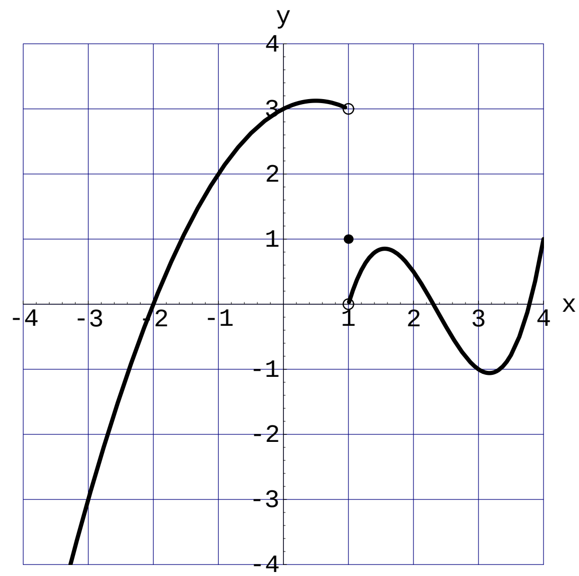
(b) What is the projectile's height when $t = 3 + h$?

(c) How far did the projectile travel during the time interval between $t = 3$ and $t = 3 + h$?

(d) what was the projectile's average velocity over the time interval between $t = 3$ and $t = 3 + h$?

(e) What was the projectile's velocity at the instant $t = 3$?

7. This figure shows the graph of a function f:



Give the value of each of the quantities below, if it exists. If the quantity in question does not exist, explain why.

(a) $\lim_{x \to 0} f(x)$

(b) $\lim_{x \to 1^-} f(x)$

(c) $\lim_{x \to 1^+} f(x)$

(d) $\lim_{x \to 1} f(x)$

(e) $f(1)$

8. Show how to find an equation for the line tangent to the curve $y = x/(1 - x)$ at the point where $x = 2$ by finding slopes of secant lines that pass through the points on the curve that correspond to $x = 2$ and to $x = 2 + h$ and then letting $h \to 0$.

MTH 1410, Calculus I—Exam I **Feb. 11, 2004**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Your exam is due at 4:50 pm.

1. A function is given by the equation $f(x) = 2x/\sqrt{25 - x^2}$.

   (a) What is the domain of f?

   (b) Evaluate

      i. $f(3)$

      ii. $f(3 + h)$

      iii. $\dfrac{f(3 + h) - f(3)}{h}$

   Simplify your answers, but do not give decimal approximations.

Solution:

(a) The fraction can be meaningful only if $25 - x^2 > 0$, which is equivalent to $-5 < x < 5$. Thus, the domain of f is $\{ x : -5 < x < 5 \}$.

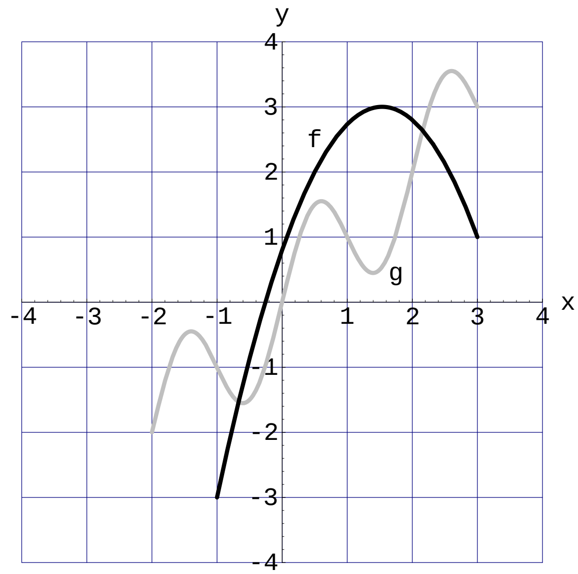
(b) i. $f(3) = \dfrac{2 \cdot 3}{\sqrt{25 - 3^2}} = \dfrac{3}{2}$.

ii. $f(3 + h) = \dfrac{2 \cdot (3 + h)}{\sqrt{25 - (3 + h)^2}} = \dfrac{6 + 2h}{\sqrt{16 - 6h - h^2}} = \dfrac{(6 + 2h)\sqrt{16 - 6h - h^2}}{16 - 6h - h^2}$.

iii.

$$\begin{aligned}
\frac{f(3 + h) - f(3)}{h} &= \frac{1}{h}\left(\frac{6 + 2h}{\sqrt{16 - 6h - h^2}} - \frac{3}{2}\right) \\
&= \frac{12 + 4h - 3\sqrt{16 - 6h - h^2}}{2h\sqrt{16 - 6h - h^2}} \quad \text{(full credit for this)} \\
&= \frac{(12 + 4h)^2 - 9(16 - 6h - h^2)}{2h\sqrt{16 - 6h - h^2}(12 + 4h + 3\sqrt{16 - 6h - h^2})} \\
&= \frac{150 + 25h}{2\sqrt{16 - 6h - h^2}(12 + 4h + 3\sqrt{16 - 6h - h^2})}
\end{aligned}$$

2. The graphs of f and g are given in the figure:



Use the figure to estimate answers for the following:

(a) What are $f(-1)$ and $g(2)$?

(b) What are x-values for which $f(x) = g(x)$?

(c) What are the solutions of the equation $f(x) = 2$?

(d) What is the largest interval on which f is increasing.

(e) What are the domain and range of f?

(f) What are the domain and range of g?

Solution:

(a) $f(-1) = -3$; $g(2) = 2$.

(b) $f(x) = g(x)$ when $x = -2/3$ and when $x = 9/4$ (approximately).

(c) The solutions of $f(x) = 2$ are about $x = 1/2$ and $x = 5/2$.

(d) The largest interval on which f is increasing is $[-1, 3/2]$.

(e) The domain of f is $[-1, 3]$, and the range of f is $[-3, 3]$.

(f) The domain of g is $[-2, 3]$, and the range of g is $[-2, 7/2]$.

3. A curve is given by the parametric equations

$$\begin{aligned} x &= \sin^2 \theta; \\ y &= \cos \theta, \end{aligned}$$

for $0 \leq \theta \leq \pi$.

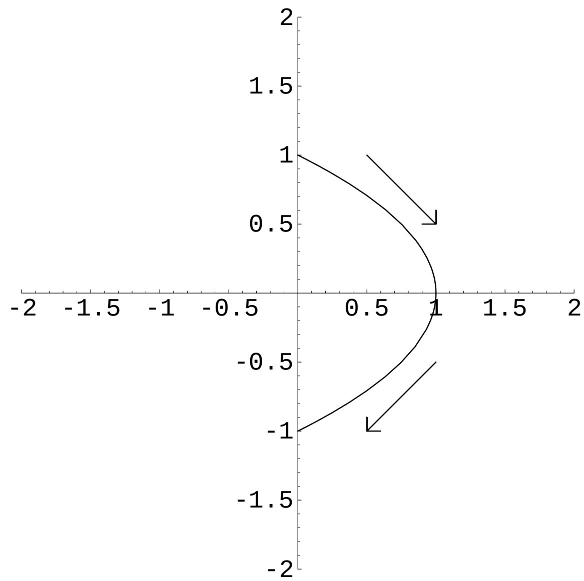
(a) Eliminate the parameter to find a Cartesian equation for the curve.

(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced out as the parameter increases.

Solution:

(a) If $x = \sin^2\theta$ and $y = \cos\theta$, then $x = 1 - y^2$. This is the required cartesian equation.

(b)



4. (a) Find parametric equations to represent the line segment from $(-3, 5)$ to $(4, -2)$.
   (b) Give two distinct parametrizations for the curve that consists of that part of the unit circle that lies in the first quadrant.

Solution: Parametric equations for the line segment connecting (x_0, y_0) to (x_1, y_1) are

$$\begin{aligned} x &= x_0 + (x_1 - x_0)t; \\ y &= y_0 + (y_1 - y_0)t; \\ 0 \leq\ & t \leq 1. \end{aligned}$$

Thus, the desired equations are:

(a)

$$\begin{aligned} x &= -3 + (4 - [-3])t \\ y &= 5 + ([-2] - 5)t, \\ 0 \leq\ & t \leq 1, \end{aligned}$$

or, equivalently,

$$\begin{aligned} x &= 7t - 3, \\ y &= -7t + 5, \\ 0 \leq\ & t \leq 1. \end{aligned}$$

(b) There are many possibilities; here are a few:

i.

$$\begin{aligned} x &= \cos t; \\ y &= \sin t; \\ 0 \leq\ & t \leq \pi/2. \end{aligned}$$

ii.

$$\begin{aligned} x &= \sin t; \\ y &= \cos t; \\ 0 \le\ & t \le \pi/2. \end{aligned}$$

iii.

$$\begin{aligned} x &= t; \\ y &= \sqrt{1-t^2}; \\ 0 \le\ & t \le 1. \end{aligned}$$

iv.

$$\begin{aligned} x &= \frac{1-t^2}{1+t^2}; \\ y &= \frac{2t}{1+t^2}; \\ 0 \le\ & t \le 1. \end{aligned}$$

5. Suppose that the graph of a function f is given. Describe how the graphs of the following functions can be obtained from the graph of f:

(a) $y = f(x) + 8$
(b) $y = f(x + 8)$
(c) $y = 1 + 2f(x)$
(d) $y = f(x - 2) - 2$
(e) $y = -f(x)$
(f) $y = f\big(|x|\big)$

Solution:

(a) Shift the graph of f 8 upward units.

(b) Shift the graph of f 8 leftward units.

(c) Stretch the graph of f vertically by a factor of 2 and then shift the result 1 unit upward.

(d) Shift the graph of f 2 units rightward and 2 units downward.

(e) Reflect the graph of f about the x axis.

(f) The right half of the graph of $y = f\big[(x|\big)$ is identical to the right half of the graph of f. The left half is the reflection of the right half about the y-axis.

6. A projectile's height y (in centimeters) at time t (in seconds) is given by

$$y = 3920t - 490t^2.$$

(a) What is the projectile's height when $t = 3$?

(b) What is the projectile's height when $t = 3 + h$?

(c) How far did the projectile travel during the time interval between $t = 3$ and $t = 3 + h$?

(d) what was the projectile's average velocity over the time interval between $t = 3$ and $t = 3 + h$?

(e) What was the projectile's velocity at the instant $t = 3$?

Solution:

(a)

$$\begin{aligned} y(3) &= 3920 \cdot 3 - 490 \cdot 3^2 \\ &= 7350 \text{ cm.} \end{aligned}$$

(b)

$$\begin{aligned} y(3+h) &= 3920(3+h) - 490(3+h)^2 \\ &= 7350 + 980h - 490h^2 \text{ cm.} \end{aligned}$$

(c) During the period between $t = 3$ sec and $t = 3+h$ sec, the projectile traveled $y(3+h)-y(3) = 980h - 490h^2$ cm.

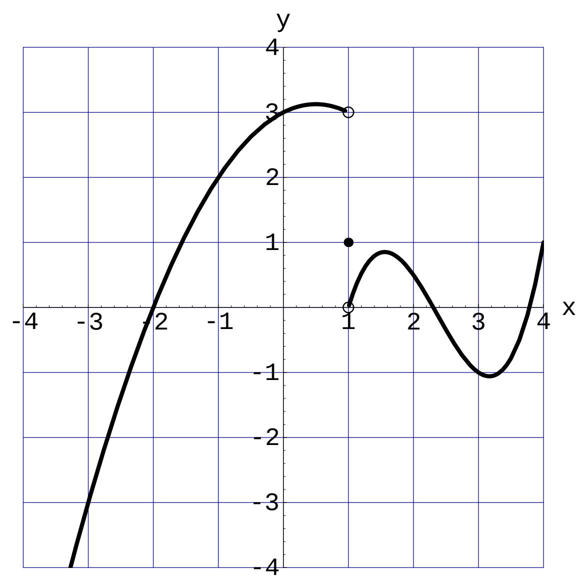
(d) The average velocity during this period was

$$\frac{y(3+h) - y(3)}{h} = 980 - 49h \text{ cm/sec.}$$

(e) At the instant $t = 3$, velocity was

$$\lim_{h\to 0}(980 - 49h) = 980 \text{ cm/sec.}$$

7. This figure shows the graph of a function f:



Give the value of each of the quantities below, if it exists. If the quantity in question does not exist, explain why.

(a) $\lim_{x\to 0} f(x)$

(b) $\lim_{x \to 1^-} f(x)$

(c) $\lim_{x \to 1^+} f(x)$

(d) $\lim_{x \to 1} f(x)$

(e) $f(1)$

Solution:

(a) $\lim_{x \to 0} f(x) = 3$ because values of y on the curve get closer and closer to 3 as x gets closer and closer to 0.

(b) $\lim_{x \to 1^-} f(x) = 3$ because values of y on the curve get closer and closer to 3 as the values of x get closer to 1 but remain less than 1.

(c) $\lim_{x \to 1^+} f(x)$ =0 because the values of y on the curve get closer and closer to 0 as the values of x get closer to 1 but remain greater than 1.

(d) $\lim_{x \to 1} f(x)$ does not exist, because

$$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x).$$

(e) $f(1) = 1$ because the point on the graph that corresponds to $x = 1$ is the point $(1, 1)$.

8. Show how to find an equation for the line tangent to the curve $y = x/(1 - x)$ at the point where $x = 2$ by finding slopes of secant lines that pass through the points on the curve that correspond to $x = 2$ and to $x = 2 + h$ and then letting $h \to 0$.

Solution: Consider the two points $(2, y(2))$ and $(2 + h, y(2 + h))$ on the graph of the curve, when $h \neq 0$. The slope of the secant line determined by these two points is

$$\begin{aligned} \frac{y(2+h) - y(2)}{(2+h) - 2} &= \frac{1}{h}\left(\frac{(2+h)}{1-(2+h)} - \frac{2}{1-2}\right) \\ &= \frac{-(\cancel{2}+h) - 2(\cancel{-1}-h)}{h(-1-h)(-1)} \\ &= \frac{\cancel{h}}{\cancel{h}(1+h)} \\ &= \frac{1}{1+h}. \end{aligned}$$

When h takes on values near 0, this quotient takes on values near 1, so the slope of the desired tangent line is 1. The tangent line is to pass through the point $(2, y[2]) = (2, -2)$, so an equation for this line is $y = -2 + 1 \cdot (x - 2)$, or $y = x - 4$.

MTH 1410, Calculus I—Exam II **Mar. 11, 2004**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Your paper is due at 4:50 pm.

1. Evaluate the limits:

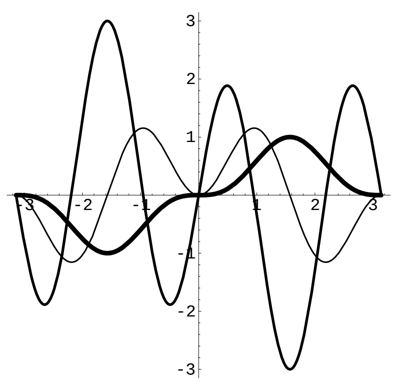
   (a) $$\lim_{x\to -3} \frac{x^2+x-6}{x^2+4x+3}$$

   (b) $$\lim_{x\to -\infty} \frac{x^3+3x^2+2}{2x^3+2x^2+2+1}$$

2. (a) Find
   $$\lim_{x\to a} \frac{\left(\dfrac{1}{x^2}-\dfrac{1}{a^2}\right)}{x-a},$$
   given that $a \neq 0$.

   (b) Find
   $$\lim_{x\to\infty} \left(\sqrt{x^2+6x}-x\right)$$

3. Here is a graph showing three functions—a skinny one, a middle-weight one, and a fat one—on the same pair of axes:



   One of the curves is f, one is f', and one is f''. Explain which is which and why.

4. (a) Where is the function f, given by
   $$f(x) = \frac{\sqrt{2+x}-2}{x},$$
   continuous? Reasons?

   (b) A function f is given by
   $$f(x) = \begin{cases} 2cx+2 & ; \quad x \leq 3 \\ 3-cx & ; \quad 3 < x. \end{cases}$$
   For what values of the constant c is f continuous on $(-\infty,\infty)$? Be sure to give your reasoning.

5. Find $f'(x)$ if

(a) $f(x) = 3x^2 - 4x + 5$

(b) $f(x) = \dfrac{a}{x^{10}} + be^x$

(c) $f(x) = \dfrac{x^2 - 2\sqrt{x}}{x^{1/3}}$

6. (a) If $f(x) = \sqrt{2x}$ for $0 \leq x$, find $f'(x)$ by directly evaluating an appropriate limit as $h \to 0$.

(b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = \sqrt{2x}$ at $x = 1$, at $x = 2$, and at $x = 8$.

MTH 1410, Calculus I—Exam II **Mar. 11, 2004**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Your paper is due at 4:50 pm.

1. Evaluate the limits:

   (a)

$$\lim_{x\to -3} \frac{x^2+x-6}{x^2+4x+3}$$

   (b)

$$\lim_{x\to -\infty} \frac{x^3+3x^2+2}{2x^3+2x^2+2x+1}$$

Solution:

(a)

$$\lim_{x\to -3} \frac{x^2+x-6}{x^2+4x+3} = \frac{3^2+3-6}{3^2+12+3} = \frac{1}{4} \tag{1}$$

(b)

$$\lim_{x\to -\infty} \frac{x^3+3x^2+2}{2x^3+2x^2+2+1} = \lim_{x\to\infty} \frac{1+\frac{3}{x}+\frac{2}{x^3}}{2+\frac{2}{x}+\frac{2}{x^2}+\frac{1}{x^3}} = \frac{1}{2}. \tag{2}$$

2. (a) Find

$$\lim_{x\to a} \frac{\left(\dfrac{1}{x^2}-\dfrac{1}{a^2}\right)}{x-a},$$

   given that $a \neq 0$.

   (b) Find

$$\lim_{x\to\infty} \left(\sqrt{x^2+6x}-x\right)$$

Solution:

(a)

$$\lim_{x\to a} \frac{\left(\dfrac{1}{x^2}-\dfrac{1}{a^2}\right)}{x-a} = \lim_{x\to a} \frac{a^2-x^2}{a^2x^2(x-a)} = -\lim_{x\to a} \frac{x^2-a^2}{a^2x^2(x-a)} \tag{3}$$

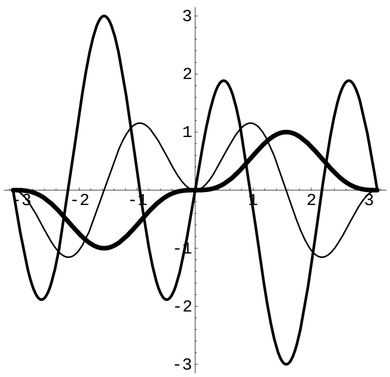
$$= -\lim_{x\to a} \frac{\cancel{(x-a)}(x+a)}{a^2x^2\cancel{(x-a)}} = -\frac{2\cancel{a}}{\cancel{a}\cdot a^3}. - \frac{2}{a^3}. \tag{4}$$

(b)

$$\lim_{x\to\infty}\left(\sqrt{x^2+6x}-x\right) = \lim_{x\to\infty}\frac{(\sqrt{x^2+6x}-x)(\sqrt{x^2+6x}+x)}{\sqrt{x^2+6x}+x} \tag{5}$$

$$= \lim_{x\to\infty}\frac{(\not{x}^{\not{2}}+6x)-\not{x}^{\not{2}}}{\sqrt{x^2+6x}+x} = \lim_{x\to\infty}\frac{6}{\sqrt{1+\frac{6}{x}}+1} = 3. \tag{6}$$

3. Here is a graph showing three functions—a skinny one, a middle-weight one, and a fat one—on the same pair of axes:



One of the curves is f, one is f', and one is f''. Explain which is which and why.

Solution: The thick curve is f, the thin curve is f', and the middle-weight curve is f''. This follows from the observations that the thin curve crosses the horizontal axis at exactly the locations where the fat curve has horizontal tangent line, is above the horizontal axis exactly where the fat curve is increasing, and is below the horizontal axis exactly where the fat curve is decreasing. Also, the middle-weight curve lies above the horizontal axis just where the fat curve is concave upward and lies below the horizontal axis just where the fat curve is concave downward.

4. (a) Where is the function f, given by

$$f(x) = \frac{\sqrt{2+x}-2}{x},$$

continuous? Reasons?

(b) A function f is given by

$$f(x) = \begin{cases} 2cx+2 & ; \quad x \le 3 \\ 3-cx & ; \quad 3 < x. \end{cases}$$

For what values of the constant c is f continuous on $(-\infty, \infty)$? Be sure to give your reasoning.

Solution:

(a) The function f is continuous on its domain, which is $[-2, 0) \cup (0, \infty)$. The number 0 isn't in the domain of f, (because division by zero is forbidden) so can't be a point of continuity. No number less than -2 is in the domain of f (because $\sqrt{2+x)}$ isn't meaningful for such numbers), and so no number smaller than -2 can be in the domain of f. The function that sends x to $x+2$ is continuous, and square roots of non-negative continuous functions are continuous. Moreover, constant functions are continuous, and the differences of continuous functions are continuous. Therefore, the numerator of f is continuous on $[-2,\infty)$. The identity function (which sends the number x to itself) is continuous everywhere, and quotients of continuous functions are continuous wherever their denominators are non-zero. Putting all of these facts together leads to the conclusion at the beginning of this paragraph.

(b) By reasoning similar to that of the previous paragraph, f is continous on $(-\infty,3)$ and on $(3,\infty)$ regardless of the value of c. Thus, the only place where f may fail to be continuous is at $x=3$. For continuity at $x=3$, we must have $\lim_{x\to 3} f(x) = f(3) = 6c+2$. But $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(2cx+2) = 6c+2$, while $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(3-cx) = 3-3c$. This means that the requirement for continuity of f at $x=3$ can be written as $6c+2 = 3-3c$, or $c=\frac{1}{9}$. We conclude that f is continuous on $(-\infty,\infty)$ precisely when $c=\frac{1}{9}$.

5. Find $f'(x)$ if

(a) $f(x) = 3x^2 - 4x + 5$

(b) $f(x) = \dfrac{a}{x^{10}} + be^x$

(c) $f(x) = \dfrac{x^2 - 2\sqrt{x}}{x^{1/3}}$

Solution:

(a)

$$f'(x) = 6x - 4.$$

(b)

$$f'(x) = -10ax^{-11} + be^x$$

(c)

$$f'(x) = \frac{(2x - x^{-1/2})x^{1/3} - \frac{1}{3}(x^2 - 2x^{1/2})x^{-2/3}}{x^{2/3}}.$$

6. (a) If $f(x) = \sqrt{2x}$ for $0 \le x$, find $f'(x)$ by directly evaluating an appropriate limit as $h \to 0$.

(b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = \sqrt{2x}$ at $x=1$, at $x=2$, and at $x=8$.

Solution:

(a)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\sqrt{2(x+h)}} - \frac{1}{\sqrt{2x}}\right] \tag{7}$$

$$= \lim_{h\to 0} \frac{\sqrt{2x} - \sqrt{2(x+h)}}{h\sqrt{2x}\sqrt{2(x+h)}} \tag{8}$$

$$= \lim_{h\to 0} \frac{\cancel{2x} - 2(\cancel{x} + \cancel{h})}{\cancel{h}\sqrt{2x}\sqrt{2(x+h)}\left(\sqrt{2x} + \sqrt{2(x+h)}\right)} = -\frac{1}{(2x)^{3/2}} \tag{9}$$

(b) The line tangent to a curve $y = g(x)$ at the point $(x_0, g(x_0))$ has equation

$$y = g(x_0) + g'(x_0)(x - x_0). \tag{10}$$

Therefore,

i. the line tangent to $y = \sqrt{2x}$ when $x = 1$ has equation $y = 2^{1/2} - \frac{1}{2^{3/2}}(x - 1)$.
ii. the line tangent to $y = \sqrt{2x}$ when $x = 2$ has equation $y = 2 - \frac{1}{8}(x - 2)$.
iii. the line tangent to $y = \sqrt{2x}$ when $x = 8$ has equation $y = 4 - \frac{1}{64}(x - 8)$.

MTH 1410, Calculus I—Exam III **April 14, 2004**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Your paper is due at 4:50 pm.

1. The position of a particle is given by $s = t^3 - \frac{9}{2}t^2 - 7t$.
   (a) When does the particle have a velocity of 5 m/sec?
   (b) When is the acceleration zero? What conclusion can you draw about velocity at this particular value for t?

2. Here is a table of values for f, g, f' and g':

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

   (a) If $h(x) = f[g(x)]$, find $h'(1)$.
   (b) If $H(x) = g[f(x)]$, find $H'(1)$.
   (c) If $G(x) = g[g(x)]$, find $G'(3)$.

3. (a) If $y = x^3 + 2x$, and $dx/dt = 5$, find dy/dt when $x = 2$.
   (b) A particle is moving along the curve $y = \sqrt{x}$. As it passes through the point $(4, 2)$, its x-coordinate increases at a rate of 3 cm/s. How fast is the distance from the particle to the origin changing at this instant?

4. Brünhilde has been looking at the curve given by $x^3 - xy^2 + y^4 = 1$. She is trying to find y' at the point where $x = -1$ and $y = 1$. Here is what she has written:

   We differentiate both sides of the equation $x^3 - xy^2 + y^4 = 1$ with respect to the variable x. This gives us

$$\begin{aligned} 3x^2 - y^2 - 2xyy' + 4y^3 &= 0; \\ -2xyy' &= y^2 - 3x^2 - 4y^3; \\ y' &= \frac{3x^2 + 4y^3 - y^2}{2xy}. \end{aligned}$$

   Cancelling an x and a y out of this fraction gives

$$\begin{aligned} y' &= \frac{3x + 4y^2 - y^2}{2} \\ &= \frac{3x + 3y^2}{2}. \end{aligned}$$

   Thus, when $x = -1$ and $y = 1$, we have

$$y' = \frac{3(-1) + 3(1)^2}{2} = 0.$$

   Critique all of her work. If her answer is wrong, show how to find the correct answer.

5. Use linearization (or the differential) to estimate the possible error in the calculated volume of a cube whose edge was measured as 30 cm with an error of up to 0.1 cm.

6. An airplane flies a level course in a straight line at 300 mph (or 440 ft/s) and passes directly over a searchlight—which tracks the plane. If the plane's altitude is one mile, what is the angular velocity of the searchlight when the airplane has traveled 1 mile (= 5280 ft) past the point directly over the searchlight? Give your answer in radians per second; do not use decimal approximations.

7. (a) Find an equation for the line tangent to the curve $x^3 - x^2y + xy^2 - 2y^3 = 4$ at the point $(2, 1)$.

   (b) Give the best estimate you can, using techniques we have studied, for the value of y for which $(41/20, y)$ is a point that lies on the curve $x^3 - x^2y + xy^2 - 2y^3 = 4$ near the point $(2, 1)$. Explain your reasoning.

MTH 1410, Calculus I—Exam III **April 14, 2004**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Your paper is due at 4:50 pm.

1. The position of a particle is given by $s = t^3 - \frac{9}{2}t^2 - 7t$.
   (a) When does the particle have a velocity of 5 m/sec?
   (b) When is the acceleration zero? What conclusion can you draw about velocity at this particular value for t?

   Solution:

   (a) Velocity is $ds/dt = 3t^2 - 9t - 7$, and this is 5 when

   $$\begin{aligned} 3t^2 - 9t - 7 &= 5; \\ 3t^2 - 9t - 12 &= 0; \\ 3(t-4)(t+1) &= 0, \end{aligned}$$

   or when $t = 4$ or $t = -1$.

   (b) Acceleration is $d^2s/dt^2 = 6t - 9$, and this is zero when $t = 3/2$. Because acceleration is negative when $t < 3/2$ but positive when $t > 3/2$, velocity is minimal at $t = 3/2$.

2. Here is a table of values for f, g, f' and g':

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

   (a) If $h(x) = f[g(x)]$, find $h'(1)$.
   (b) If $H(x) = g[f(x)]$, find $H'(1)$.
   (c) If $G(x) = g[g(x)]$, find $G'(3)$.

   Solution:

   (a) By the Chain Rule,

   $$\begin{aligned} h'(1) &= f'[g(1)]g'(1) \\ &= f'(2)g'(1) \\ &= 5 \cdot 6 \;=\; 30. \end{aligned}$$

   (b) By the Chain Rule,

   $$\begin{aligned} H'(1) &= g'[f(1)]f'(1) \\ &= g'(3)f'(1) \\ &= 9 \cdot 4 \;=\; 36. \end{aligned}$$

(c) By the Chain Rule,

$$\begin{aligned} G'(3) &= g'[g(3)]g'(3) \\ &= g'(2)g'(3) \\ &= 7 \cdot 9 \;=\; 63. \end{aligned}$$

3. (a) If $y = x^3 + 2x$, and $dx/dt = 5$, find dy/dt when $x = 2$.
   (b) A particle is moving along the curve $y = \sqrt{x}$. As it passes through the point $(4, 2)$, its x-coordinate increases at a rate of 3 cm/s. How fast is the distance from the particle to the origin changing at this instant?

   Solution:

   (a) Differentiating both sides of the equation $y = x^3 + 2x$ implicitly, we find:

$$\begin{aligned} \frac{d}{dt}(y) &= \frac{d}{dt}(x^3 + 2x); \\ \frac{dy}{dt} &= (3x^2 + 2)\frac{dx}{dt}. \end{aligned}$$

   Putting $x = 2$ and $dx/dt = 5$ in the latter equation, we find that

$$\frac{dy}{dt} = (3 \cdot 2^2 + 2) \cdot 5 \;=\; 70.$$

   (b) The square of the distance from the origin to the point $(x, \sqrt{x})$ is given by $D^2 = x^2 + x$. When $x = 4$, this gives $D = \sqrt{4^2 + 4} = 2\sqrt{5}$. Implicit differentiation gives

$$\begin{aligned} \frac{d}{dt}D^2 &= \frac{d}{dt}(x^2 + x); \\ 2D\frac{dD}{dt} &= (2x + 1)\frac{dx}{dt}. \end{aligned}$$

   Thus, if $dx/dt = 3$ cm/s at $(4, 2)$, we have

$$\begin{aligned} 2 \cdot (2\sqrt{5})\frac{dD}{dt} &= (2 \cdot 4 + 1) \cdot 3, \text{ or} \\ \frac{dD}{dt} &= \frac{27}{4\sqrt{5}} \text{ cm/s}. \end{aligned}$$

4. Brünhilde has been looking at the curve given by $x^3 - xy^2 + y^4 = 1$. She is trying to find y' at the point where $x = -1$ and $y = 1$. Here is what she has written:

   We differentiate both sides of the equation $x^3 - xy^2 + y^4 = 1$ with respect to the variable x. This gives us

$$\begin{aligned} 3x^2 - y^2 - 2xyy' + 4y^3 &= 0; \\ -2xyy' &= y^2 - 3x^2 - 4y^3; \\ y' &= \frac{3x^2 + 4y^3 - y^2}{2xy}. \end{aligned}$$

   Canceling an x and a y out of this fraction gives

$$\begin{aligned} y' &= \frac{3x + 4y^2 - y^2}{2} \\ &= \frac{3x + 3y^2}{2}. \end{aligned}$$

Thus, when $x = -1$ and $y = 1$, we have

$$y' = \frac{3(-1) + 3(1)^2}{2} = 0.$$

Critique all of her work. If her answer is wrong, show how to find the correct answer.

Solution: Brünhilde has made two mistakes. First, she wrote $(d/dx)(y^4) = 4y^3$ when the Chain Rule requires $(d/dx)(y^4) = 4y^3y'$. Second, her canceling is incorrect because neither x nor y is a factor of the numerator. Here is the correct calculation:

$$\begin{aligned} 3x^2 - y^2 - 2xyy' + 4y^3y' &= 0; \\ (2xy - 4y^3)y' &= 3x^2 - y^2; \\ y' &= \frac{3x^2 - y^2}{2xy - 4y^3}. \end{aligned}$$

Thus, when $x = -1$ and $y = 1$ we have

$$y' = \frac{3(-1)^2 - (1)^2}{2(-1)(1) - 4(1)^3} = \frac{2}{-6} = -\frac{1}{3}.$$

5. Use linearization (or the differential) to estimate the possible error in the calculated volume of a cube whose edge was measured as 30 cm with an error of up to 0.1 cm.

   Solution: Let x be the edge of the cube. Then $V = x^3$ and $V' = 3x^2$. The linearization of the volume function at $x = 30$ is thus $L(x) = 27000 + 2700(x - 30)$. The approximate magnitude of the change in V is therefore $|L(x) - 27000| = 2700|x - 30|$. When $|x - 30| \leq 0.1$, we therefore have $|L(x) - 27000| \leq 270$ cm^3, so that error in the calculated volume is, roughly, no more than 270 cm^3.

6. An airplane flies a level course in a straight line at 300 mph (or 440 ft/s) and passes directly over a searchlight—which tracks the plane. If the plane's altitude is one mile, what is the angular velocity of the searchlight when the airplane has traveled 1 mile (= 5280 ft) past the point directly over the searchlight? Give your answer in radians per second; do not use decimal approximations.

   Solution: Let x be the distance, in miles, the plane has travelled since passing over the searchlight, and let θ be the angle from the ground to the searchlight beam. Then $\tan\theta = 1/x$, so, differentiating implicitly, we find that

$$\begin{aligned} \frac{d}{dt}\tan\theta &= \frac{d}{dt}\left(\frac{1}{x}\right), \text{ or} \\ \sec^2\theta\frac{d\theta}{dt} &= -\frac{1}{x^2}\frac{dx}{dt}. \end{aligned}$$

   Thus

$$\begin{aligned} \frac{d\theta}{dt} &= -\frac{1}{x^2\sec^2\theta}\frac{dx}{dt} \\ &= -\frac{1}{x^2(1 + \tan^2\theta)}\frac{dx}{dt} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{x^2[1+(1/x)^2]}\frac{dx}{dt} \\
&= -\frac{1}{x^2+1}\frac{dx}{dt}.
\end{aligned}
$$

Putting $x = 1$ mile and $dx/dt = 300$ mph, we obtain

$$
\begin{aligned}
\frac{d\theta}{dt} &= -\frac{300}{1^2+1} \\
&= -150 \text{ radians/hour} \\
&= -\frac{150}{3600} \text{ radians/sec} = -\frac{1}{24} \text{ radians/sec}
\end{aligned}
$$

7. (a) Find an equation for the line tangent to the curve $x^3 - x^2y + xy^2 - 2y^3 = 4$ at the point $(2, 1)$.

   (b) Give the best estimate you can, using techniques we have studied, for the value of y for which $(41/20, y)$ is a point that lies on the curve $x^3 - x^2y + xy^2 - 2y^3 = 4$ near the point $(2, 1)$. Explain your reasoning.

Solution:

(a) Implicit differentiation with respect to x give

$$
\begin{aligned}
3x^2 - 2xy - x^2y' + y^2 + 2xyy' - 6y^2y' &= 0; \\
(x^2 - 2xy + 6y^2)y' &= 3x^2 - 2xy + y^2; \\
y' &= \frac{3x^2 - 2xy + y^2}{x^2 - 2xy + 6y^2}.
\end{aligned}
$$

Consequently, when $x = 2$ and $y = 1$, we have

$$
y' = \frac{3(2)^2 - 2(2)(1) + (1)^2}{(2)^2 - 2(2)(1) + 6(1)^2} = \frac{3}{2}.
$$

An equation for the tangent line is therefore $y = 1 + (3/2)(x - 2)$.

(b) In the previous part of this problem, we have calculated the linearization of y at $(2, 1)$ for this curve. Hence our best approximation for y when $x = 41/20$ is $y \sim 1 + (3/2)(41/20 - 2) = 1 + (3/2)(1/20) = 43/40 = 1.075$. Note: In fact, the correct value for y is

$$
\begin{aligned}
y &= \frac{1}{120}\left[41 + \left(1442366 + 3\sqrt{297131412009}\right)^{1/3} \right.\\
&\qquad\left. - \frac{8405}{\left(1442366 + 3\sqrt{297131412009}\right)^{1/3}}\right] \\
&\sim 1.072301579165.
\end{aligned}
$$

MTH 1410, Calculus I—Final Exam **May 10, 2004**

Instructions: Write out your solutions for the following problems on your own paper; give your reasoning and show your supporting calculations. Your paper is due at 4:50 pm.

1. Find an equation for the line tangent to the curve

$$y = x \sin x + 2 \cos x$$

at the point where $x = \pi/2$.

2. Evaluate the limits (without using a calculator):

(a) $$\lim_{x \to 289} \frac{\sqrt{x} - 17}{x - 289}$$

(b) $$\lim_{x \to \infty} \left(\sqrt{3x^2 + 2x + 2} - \sqrt{3x^2 - 4x + 4} \right)$$

3. Use the definition of the derivative to find $f'[x]$ when

$$f(x) = \sqrt{x^2 + x}.$$

4. Find the area bounded by the x-axis, the lines $x = 0$ and $x = 2\sqrt{2}$, and the curve

$$y = \frac{x}{\sqrt{1 + x^2}}.$$

5. (a) Find an equation for the line tangent to the curve given by

$$x^3 + xy + y^3 = 11 \qquad (1)$$

at the point $(1, 2)$.

(b) Use the result of part (a) of this question to estimate the value of y which satisfies equation (1) when $x = \frac{19}{20}$. (Give your answer as a fraction whose numerator and denominator are both whole numbers.)

6. A bullet strikes a target and penetrates it to a depth of 9 inches in $\frac{1}{100}$ seconds. Assume that deceleration is constant from the time of impact until the bullet comes to rest. Find the deceleration and the velocity at impact.

7. Let f be the function given by

$$f(x) \;=\; \sin^3 x.$$

   (a) Explain why

$$f'(x) \;=\; 3\sin^2 x\,\cos x.$$

   (b) Locate the critical numbers for f that lie in the interval $-3\pi/4 \le x \le 3\pi/4$ and determine whether each gives a local minimum, a local maximum, or neither for f.

   (c) Find the maximum value and the minimum value of f on the interval $-3\pi/4 \le x \le 3\pi/4$.

8. A Coast Guard cutter is moored offshore at night when a motorboat speeds past its eastern side headed due north with constant speed of 3.5 yards per second. The cutter's crew track the motorboat with a search-light. If motorboat's closest approach to the search-light is 100 yards, how fast is the searchlight turning when the boat bears 30° north of east from the light?

MTH 1410, Calculus I—Final Exam **May 10, 2004**

Instructions: Write out your solutions for the following problems on your own paper; give your reasoning and show your supporting calculations. Your paper is due at 4:50 pm.

1. Find an equation for the line tangent to the curve

$$y = x\sin x + 2\cos x$$

at the point where $x = \pi/2$.

Solution: $y' = \sin x + x\cos x - 2\sin x = x\cos x - \sin x$, so $y'(\pi/2) = -1$. When $x = \pi/2$, we have $y = \pi/2$. So an equation for the line tangent to the curve at $x = \pi/2$ is $y = \pi/2 - (x - \pi/2)$, or $y = \pi - x$.

2. Evaluate the limits (without using a calculator):

(a)

$$\lim_{x\to 289} \frac{\sqrt{x} - 17}{x - 289}$$

(b)

$$\lim_{x\to\infty} \left(\sqrt{3x^2 + 2x + 2} - \sqrt{3x^2 - 4x + 4}\right)$$

Solution:

(a)

$$\lim_{x\to 289} \frac{\sqrt{x} - 17}{x - 289} = \lim_{x\to 289} \frac{\cancel{\sqrt{x} - 17}}{\cancel{(\sqrt{x} - 17)}(\sqrt{x} + 17)} = \lim_{x\to 289} \frac{1}{\sqrt{x} + 17} = \frac{1}{34}.$$

(b)

$$\begin{aligned}
&\lim_{x\to\infty} \left(\sqrt{3x^2 + 2x + 2} - \sqrt{3x^2 - 4x + 4}\right) \\
&= \lim_{x\to\infty} \frac{(3x^2 + 2x + 2) - (3x^2 - 4x + 4)}{\sqrt{3x^2 + 2x + 2} + \sqrt{3x^2 - 4x + 4}} \\
&= \lim_{x\to\infty} \frac{6x - 2}{\sqrt{3x^2 + 2x + 2} + \sqrt{3x^2 - 4x + 4}} \\
&= \lim_{x\to\infty} \frac{6 - \frac{2}{x}}{\sqrt{3 - \frac{2}{x} + \frac{2}{x^2}} + \sqrt{3 - \frac{4}{x} + \frac{4}{x^2}}} \\
&= \frac{6}{2\sqrt{3}} = \sqrt{3}.
\end{aligned}$$

3. Use the definition of the derivative to find $f'(x)$ when

$$f(x) \quad = \quad \sqrt{x^2+x}.$$

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{(x+h)^2+(x+h)}-\sqrt{x^2+x}}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{x^2+(2h+1)x+(h^2+h)}-\sqrt{x^2+x}}{h} \\ &= \lim_{h\to 0}\frac{x^2+(2h+1)x+(h^2+h)-(x^2+x)}{h(\sqrt{x^2+(2h+1)x+(h^2+h)}+\sqrt{x^2+x})} \\ &= \lim_{h\to 0}\frac{2hx+h^2+h}{h(\sqrt{x^2+(2h+1)x+(h^2+h)}+\sqrt{x^2+x})} \\ &= \lim_{h\to 0}\frac{2x+h+1}{(\sqrt{x^2+(2h+1)x+(h^2+h)}+\sqrt{x^2+x})} \\ &= \frac{2x+1}{2\sqrt{x^2+x}}. \end{aligned}$$

4. Find the area bounded by the x-axis, the lines $x=0$ and $x=2\sqrt{2}$, and the curve

$$y \quad = \quad \frac{x}{\sqrt{1+x^2}}.$$

Solution: The required area is $\displaystyle\int_0^{2\sqrt{2}}\frac{x\,dx}{\sqrt{1+x^2}}$. Putting $u=1+x^2$, we have $du=2x\,dx$. Moreover, when $x=0$, $u=1$, and when $x=2\sqrt{2}$, $u=9$. Thus,

$$\begin{aligned} \int_0^{2\sqrt{2}}\frac{x\,dx}{\sqrt{1+x^2}} &= \frac{1}{2}\int_1^9\frac{du}{\sqrt{u}} \\ &= \frac{1}{2}\int_1^9 u^{-1/2}\,du \\ &= u^{1/2}\Big|_1^9 \;=\; \sqrt{9}-\sqrt{1} \;=\; 2. \end{aligned}$$

5. (a) Find an equation for the line tangent to the curve given by

$$x^3+xy+y^3 \quad = \quad 11 \qquad (1)$$

at the point $(1,2)$.

(b) Use the result of part (a) of this question to estimate the value of y which satisfies equation (1) when $x = \frac{19}{20}$. (Give your answer as a fraction whose numerator and denominator are both whole numbers.)

Solution: Differentiating implicitly, we find that

(a)

$$\begin{aligned} 3x^2 + y + xy' + 3y^2y' &= 0; \\ y' &= -\frac{3x^2 + y}{x + 3y^2}. \end{aligned}$$

When $x = 1$ and $y = 2$, this gives $y' = -5/13$. Consequently, an equation for the line tangent to the given curve at the point $(1, 2)$ is

$$y = 2 - \frac{5}{13}(x - 1).$$

(b) The tangent line found in part (a) approximates the curve near $(1, 2)$. Therefore, an approximate value for y when $x = \frac{19}{20}$ is

$$y \sim 2 - \frac{5}{13}\left(\frac{19}{20} - 1\right) = 2 + \frac{5}{13} \cdot \frac{1}{20} = \frac{105}{52}.$$

6. A bullet strikes a target and penetrates it to a depth of 9 inches in $\frac{1}{100}$ seconds. Assume that deceleration is constant from the time of impact until the bullet comes to rest. Find the deceleration and the velocity at impact.

Solution: Let a denote the unknown constant acceleration, and let $v(t)$ denote velocity at time t. During the hundredth of a second at issue we have $v(t) = at + v_0$, where v_0 is the velocity with which the bullet struck. Then $v(1/100) = 0$, so $a/100 + v_0 = 0$, or $a = -100v_0$. If $s(t)$ denotes depth of penetration at time t, then $s(t) = \frac{1}{2}at^2 + v_0 t = -50v_0 t^2 + v_0 t$. But $s(1/100) = 3/4$, so

$$\begin{aligned} -50\frac{v_0}{10000} + \frac{v_0}{100} &= \frac{3}{4}; \\ \frac{v_0}{200} &= \frac{3}{4} \\ v_0 &= 150 \text{ ft/sec}. \end{aligned}$$

Then $a = -100v_0 = -15000$ ft/sec^2.

7. Let f be the function given by

$$f(x) = \sin^3 x.$$

(a) Explain why

$$f'(x) = 3\sin^2 x \cos x.$$

(b) Locate the critical numbers for f that lie in the interval $-3\pi/4 \le x \le 3\pi/4$ and determine whether each gives a local minimum, a local maximum, or neither for f.

(c) Find the maximum value and the minimum value of f on the interval $-3\pi/4 \le x \le 3\pi/4$.

Solution:

(a) According to the Chain Rule, $\frac{d}{dx}u^3 = 3u^2\frac{du}{dx}$. Here, we take $u = \sin x$. Then $\frac{du}{dx} = \cos x$, so

$$\begin{aligned}\frac{d}{dx}\sin^3 x &= \frac{d}{dx}u^3 \\ &= 3u^2\frac{du}{dx} \\ &= 3\sin^2 x\cos x.\end{aligned}$$

(b) The critical numbers for f are the values of x for which $f'(x)$ either vanishes or does not exist. Because $f'(x) = 3\sin^2 x\cos x$, $f'(x)$ exists for all x. We therefore must find all solutions of the equation $3\sin^2 x\cos x = 0$ for $-3\pi/4 \le x \le 3\pi/4$. Such solutions occur where, and only where, either $\sin x = 0$ or $\cos x = 0$, so the critical numbers for f are $x = \pm\pi/2$ and $x = 0$. Now $\sin^2 x \ge 0$ for all x, while $\cos x \ge 0$ for $-\pi/2 \le x \le \pi/2$, but $\cos x \le 0$ for $\pi/2 \le |x| \le 3\pi/4$. Consequently, $f'(x)$ changes sign from negative to positive as x increases through $-\pi/2$, $f'(x)$ does not change sign as x increases through 0, and $f'(x)$ changes sign from positive to negative as x increases through $\pi/2$. The critical number $x = -\pi/2$ is therefore a local minimum for f, the critical number $x = 0$ is neither a local maximum nor a local minimum for f, and the critical number $x = \pi/2$ is a local maximum for f.

(c) The maximum value for $f(x)$ on the given interval must occur at either and end-point or at a critical number. In this case, the only critical number we need consider is $x = \pi/2$, because it is the only critical number where f has a local maximum. We have $f(-3\pi/4) < 0$, $f(\pi/2) = 1$, and $f(3\pi/4) = \sqrt{2}/4 < 1$. The maximum value attained by f on this interval is therefore $f(\pi/2) = 1$. Because the graph of f is symmetric about the origin and the interval in which we seek the minimum is symmetric, the minimum lies at $x = -\pi/2$, where $f(-\pi/2) = -1$.

8. A Coast Guard cutter is moored offshore at night when a motorboat speeds past its eastern side headed due north with constant speed of 3.5 yards per second. The cutter's crew track the motorboat with a search-light. If motorboat's closest approach to the search-light is 100 yards, how fast is the searchlight turning when the boat bears 30° north of east from the light?

Solution: Let θ be the angle at which the search-light points, measured counterclockwise from due east. If we place the origin of our coordinate system at the axis about which the search-light turns and point the x-axis due east, then we can represent the motorboat's position as $(100, y)$, where y is a function of time. The angle θ is then related to y by the equation $\tan\theta = y/100$. Consequently,

$$\begin{aligned}\sec^2\theta\frac{d\theta}{dt} &= \frac{1}{100}\frac{dy}{dt}; \\ \frac{d\theta}{dt} &= \frac{1}{100}\cos^2\theta\frac{dy}{dt}.\end{aligned}$$

We have been given $\dfrac{dy}{dt} = \dfrac{7}{2}$, and we know that $\cos(30°) = \sqrt{3}/2$. At the critical instant, we therefore have $\dfrac{d\theta}{dt} = \dfrac{1}{100} \cdot \dfrac{3}{4} \cdot \dfrac{7}{2} = \dfrac{21}{800}$ rad/sec, or about 1.504 degrees per second.

MTH 1410, Calculus I—Exam I **Feb. 7, 2007**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 2:50 pm.

1. Use the Limit Laws to find the limits:
   (a) $\lim_{t \to 2} \frac{t^2 - 4}{t^3 - 8}$
   (b) $\lim_{x \to \infty} \left(\sqrt{x^2 + 4x - 5} - \sqrt{x^2 - 2x + 8} \right)$

2. Let a be a fixed, but unspecified, positive real number. Find the limit: $\lim_{x \to a} \frac{x^{1/2} - a^{1/2}}{x - a}$.

3. We will find later that when f is given by the equation $f(x) = \tan x$, then f' is given by $f'(x) = \sec^2 x$. Use this fact to write an equation for the line tangent to the curve $y = \tan x$ at the point corresponding to $x = \pi/3$.

4. (a) Find $f'(2)$ by evaluating an appropriate limit when f is given by $f(x) = x^2 - x$.
   (b) Find $f'(x)$ by evaluating an appropriate limit when f is given by $f(x) = \frac{1}{\sqrt{x}}$.

5. A projectile's height y (in centimeters) at time t (in seconds) is given by

$$y = 3920t - 490t^2$$

   . (a) What is the projectile's height when $t = 3$?
   (b) What is the projectile's height when $t = 3 + h$?
   (c) How far did the projectile travel during the time interval between $t = 3$ and $t = 3 + h$?
   (d) what was the projectile's average velocity over the time interval between $t = 3$ and $t = 3 + h$?
   (e) What was the projectile's velocity at the instant $t = 3$?

6. (a) What criterion must f satisfy in order to be continuous at $x = a$?
   (b) What criterion must the one-sided limits $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ satisfy in order for the two-sided limit $\lim_{x \to a} f(x)$ to exist?
   (c) Find all numbers a such that the function f given by

$$f(x) = \begin{cases} x + 2 & \text{if } x \leq a \\ x^2 & \text{if } x > a \end{cases}$$

   is continuous on $\mathbb{R}$. Explain the reasoning that leads you to your conclusions.

MTH 1410, Calculus I—Exam I **Feb. 7, 2007**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 2:50 pm.

1. Use the Limit Laws to find the limits:

   (a) $\lim_{t\to 2} \frac{t^2-4}{t^3-8}$

   (b) $\lim_{x\to\infty} \left(\sqrt{x^2+4x-5} - \sqrt{x^2-2x+8}\right)$

   Solution:

   (a)

$$\lim_{t\to 2} \frac{t^2-4}{t^3-8} = \lim_{t\to 2} \frac{\cancel{(t-2)}(t+2)}{\cancel{(t-2)}(t^2+2t+4)} \tag{1}$$

$$= \lim_{t\to 2} \frac{t+2}{t^2+2t+4} \tag{2}$$

$$= \frac{\lim_{t\to 2}(t+2)}{\lim_{t\to 2}(t^2+2t+4)} \tag{3}$$

$$= \frac{4}{12} = \frac{1}{3}. \tag{4}$$

   (b)

$$\lim_{x\to\infty} \left(\sqrt{x^2+4x-5} - \sqrt{x^2-2x+8}\right) \tag{5}$$

$$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+4x-5} - \sqrt{x^2-2x+8}\right)\left(\sqrt{x^2+4x-5} + \sqrt{x^2-2x+8}\right)}{\sqrt{x^2+4x-5} + \sqrt{x^2-2x+8}} \tag{6}$$

$$= \lim_{x\to\infty} \frac{(x^2+4x-5)-(x^2-2x+8)}{\sqrt{x^2+4x-5} + \sqrt{x^2-2x+8}} \tag{7}$$

$$= \lim_{x\to\infty} \frac{6x-13}{\sqrt{x^2+4x-5} + \sqrt{x^2-2x+8}} \tag{8}$$

$$= \lim_{x\to\infty} \frac{6-13x^{-1}}{\sqrt{1+4x^{-1}-5x^{-2}} + \sqrt{1-2x^{-1}+8x^{-2}}} \tag{9}$$

$$= \frac{6-13\cdot 0}{\sqrt{1+4\cdot 0-5\cdot 0} + \sqrt{1-2\cdot 0+8\cdot 0}} = 3. \tag{10}$$

2. Let a be a fixed, but unspecified, positive real number. Find the limit: $\lim_{x\to a} \frac{x^{1/2}-a^{1/2}}{x-a}$.

   Solution:

$$\lim_{x\to a} \frac{x^{1/2}-a^{1/2}}{x-a} = \lim_{x\to a} \frac{x^{1/2}-a^{1/2}}{(x^{1/2}-a^{1/2})(x^{1/2}+a^{1/2})} \tag{11}$$

$$= \lim_{x\to a} \frac{1}{x^{1/2}+a^{1/2}} = \frac{1}{2a^{1/2}}. \tag{12}$$

3. We will find later that when f is given by the equation $f(x) = \tan x$, then f' is given by $f'(x) = \sec^2 x$. Use this fact to write an equation for the line tangent to the curve $y = f(x)$ at the point corresponding to $x = \pi/3$.

Solution: An equation for the line through the point (x_0, y_0) with slope m is $y = y_0 + m(x - x_0)$. We want an equation for the line through $(\pi/3, \tan[\pi/3])$, or $(\pi/3, \sqrt{3})$, with slope $\sec^2(\pi/3) = 2^2 = 4$. The required equation is therefore

$$y = \sqrt{3} + 4(x - \pi/3). \tag{13}$$

4. (a) Find $f'(2)$ by evaluating an appropriate limit when f is given by $f(x) = x^2 - x$.

   (b) Find $f'(x)$ by evaluating an appropriate limit when f is given by $f(x) = \dfrac{1}{\sqrt{x}}$.

Solution:

(a)

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \tag{14}$$

$$= \lim_{h\to 0} \frac{[(2+h)^2 - (2+h)] - (2^2 - 2)}{h} \tag{15}$$

$$= \lim_{h\to 0} \frac{4 + 4h + h^2 - 2 - h - 4 + 2}{h} \tag{16}$$

$$= \lim_{h\to 0} \frac{3\cancel{h} + h^{\cancel{2}}}{\cancel{h}} = \lim_{h\to 0}(3 + h) = 3. \tag{17}$$

(b)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{18}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right) \tag{19}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x(x+h)}} \tag{20}$$

$$= \lim_{h\to 0} \frac{\cancel{x} - (\cancel{x} + h)}{h\sqrt{x(x+h)}(\sqrt{x} + \sqrt{x+h})} \tag{21}$$

$$= -\lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\sqrt{x(x+h)}(\sqrt{x} + \sqrt{x+h})} \tag{22}$$

$$= -\lim_{h\to 0} \frac{1}{\sqrt{x(x+h)}(\sqrt{x} + \sqrt{x+h})} \tag{23}$$

$$= -\frac{1}{2x\sqrt{x}}. \tag{24}$$

5. A projectile's height y (in centimeters) at time t (in seconds) is given by

$$y = 3920t - 490t^2$$

.

(a) What is the projectile's height when $t = 3$?

(b) What is the projectile's height when $t = 3 + h$?

(c) How far did the projectile travel during the time interval between $t = 3$ and $t = 3 + h$?

(d) what was the projectile's average velocity over the time interval between $t = 3$ and $t = 3 + h$?

(e) What was the projectile's velocity at the instant $t = 3$?

Solution:

(a) The projectile's height when $t = 3$ is

$$y(3) = 3920 \cdot 3 - 490 \cdot 3^2 \quad (25)$$
$$= 7350 \text{ cm.} \quad (26)$$

(b) The projectile's height when $t = 3 + h$ is

$$y(3 + h) = 3920(3 + h) - 490(3 + h)^2 \quad (27)$$
$$= (7350 + 980h - 490h^2) \text{ cm.} \quad (28)$$

(c) During the interval between $t = 3$ and $t = 3 + h$, the projectile traveled

$$y(3 + h) - y(3) = (7350 + 980h - 490h^2) - 7350 \quad (29)$$
$$= (980h - 490h^2) \text{ cm.} \quad (30)$$

(d) The projectile's average velocity over the time interval between $t = 3$ and $t = 3 + h$ is

$$\frac{y(3 + h) - y(3)}{h} = \frac{980\cancel{h} - 490h^{\cancel{2}}}{\cancel{h}} \quad (31)$$
$$= (980 - 490h) \text{ cm/sec.} \quad (32)$$

(e) The projectile's velocity at the instant $t = 3$ is

$$\lim_{h \to 0} \frac{y(3 + h) - y(3)}{h} = \lim_{h \to 0} (980 - 490h) \text{ cm/sec} \quad (33)$$
$$= 980 \text{cm/sec.} \quad (34)$$

6. (a) What criterion must f satisfy in order to be continuous at $x = a$?

(b) What criterion must the one-sided limits $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ satisfy in order for the two-sided limit $\lim_{x \to a} f(x)$ to exist?

(c) Find all numbers a such that the function f given by

$$f(x) = \begin{cases} x + 2 & \text{if} \quad x \le a \\ x^2 & \text{if} \quad x > a \end{cases}$$

is continuous on $\mathbb{R}$. Explain the reasoning that leads you to your conclusions.

Solution:

(a) The function f is continuous at $x = a$ provided that $\lim_{x \to a} f(x) = f(a)$.

(b) $\lim_{x\to a} = L$ if, and only if, $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$.

(c) The function f is given by a polynomial throughout the interval $(-\infty, a)$ and by another polynomial throughout the interval (a, ∞), so it is continuous on each of those intervals. It will be continuous at $x = a$ if and only if $\lim_{x\to a} f(x) = f(a)$. Now

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} x^2 \tag{35}$$
$$= a^2, \tag{36}$$

while

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^-} x + 2 \tag{37}$$
$$= a + 2 = f(a). \tag{38}$$

But if $\lim_{x\to a} f(x)$ is to exist, we must have equality of the two-sided limits at $x = a$, so the condition that a must satisfy for $\lim_{x\to a} f(x)$ to be equal to $f(a)$ is $a^2 = a + 2$ or $a^2 - a - 2 = 0$. This is a quadratic equation, and the solutions are given by the quadratic formula:

$$a = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-2)}}{2 \cdot 1} \tag{39}$$
$$= \frac{1 \pm \sqrt{9}}{2} = \frac{1 \pm 3}{2}. \tag{40}$$

The function f is therefore continuous when $a = 2$ and when $a = -1$.

MTH 1410, Calculus I—Exam II **March 14, 2007**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 2:50 pm.

1. Find $f'(x)$ if

   (a) $f(x) = 4x^8 - 31x^5 + 22x^3 - x^2 + 12$.

   (b) $f(x) = \dfrac{x^3 - x^2}{4x^2 + 1}$.

2. Find $f'(x)$ if

   (a) $f(x) = \sin^2 x \cos 3x$.

   (b) $f(x) = \tan^{-1}\big(\ln x\big)$.

3. Use the Limit Laws, in conjunction with the fact that you know the value of $\lim_{x \to 0} \dfrac{\sin x}{x}$, to evaluate

$$\lim_{x \to 0} \frac{\sin x}{2x + \tan x}.$$

4. Suppose that $f(2) = 5$, $f(4) = 3$, $f'(2) = 4$, $f'(4) = -2$, $g(2) = 4$, $g(4) = 2$, $g'(2) = -6$, and $g'(4) = -8$.

   (a) Find $F(2)$ and $F'(2)$, where $F(x) = \dfrac{f(x)}{g(x)}$.

   (b) Find $G(2)$ and $G'(2)$, where $G(x) = f[g(x)]$.

   (c) Find $H(2)$ and $H'(2)$, where $H(x) = f[g(2x)]$.

5. (a) If x and y are related by the equation $x^3 - 4xy + 2y^3 = 2$, find the value of y' at the point $(2, 1)$.

   (b) Find an equation for the line tangent to the curve $x^3 - 4xy + 2y^3 = 2$ at the point $(2, 1)$.

   (c) Show how to use the result in part (b) to estimate the value of y for the point on the curve near $(2, 1)$ where $x = 1.997$.

6. When a resistor whose resistance is r ohms is placed in a circuit in parallel with another resistance of 100 ohms, the effective resistance, R, in ohms, in that circuit because of the presence of the two resistors is given by

$$R = \frac{100r}{100 + r}.$$

   Suppose that r is measured as 50 ohms with an error no worse than ± 3 ohms. Use linearization (or differentials) to estimate the maximum error in the computed value for R.

MTH 1410, Calculus I—Exam II **March 14, 2007**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 2:50 pm.

1. Find $f'(x)$ if

   (a) $f(x) = 4x^8 - 31x^5 + 22x^3 - x^2 + 12$.

   (b) $f(x) = \dfrac{x^3 - x^2}{4x^2 + 1}$.

   Solution:

   (a) $f'(x) = 32x^3 - 155x^4 + 66x^2 - 2x$

   (b) $f'(x) = \dfrac{(3x^2 - 2x)(4x^2 + 1) - (x^3 - x^2)(8x)}{(4x^2 + 1)^2}$

2. Find $f'(x)$ if

   (a) $f(x) = \sin^2 x \cos 3x$.

   (b) $f(x) = \tan^{-1}\big(\ln x\big)$.

   Solution:

   (a) $f'(x) = (2 \sin x \cos x) \cos 3x + \sin^2 x(-3 \sin 3x)$

   (b) $f'(x) = \dfrac{1}{1 + (\ln x)^2} \cdot \dfrac{1}{x}$

3. Use the Limit Laws, in conjunction with the fact that you know the value of $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$, to evaluate

$$\lim_{x \to 0} \frac{\sin x}{2x + \tan x}.$$

   Solution:

$$\lim_{x \to 0} \frac{\sin x}{2x + \tan x} = \lim_{x \to 0} \frac{\left(\dfrac{\sin x}{x}\right)}{\left(2 + \dfrac{\tan x}{x}\right)} \tag{1}$$

$$= \frac{\lim\limits_{x \to 0} \left(\dfrac{\sin x}{x}\right)}{\lim\limits_{x \to 0} \left(2 + \dfrac{\sin x}{x \cos x}\right)} \tag{2}$$

$$= \frac{1}{2 + \lim\limits_{x \to 0} \left(\dfrac{\sin x}{x}\right) \cdot \lim\limits_{x \to 0} \left(\dfrac{1}{\cos x}\right)} = \frac{1}{2 + 1 \cdot 1} = \frac{1}{3}. \tag{3}$$

4. Suppose that $f(2) = 5$, $f(4) = 3$, $f'(2) = 4$, $f'(4) = -2$, $g(2) = 4$, $g(4) = 2$, $g'(2) = -6$, and $g'(4) = -8$. Give fully simplified answers for the following:

   (a) Find $F(2)$ and $F'(2)$, where $F(x) = \dfrac{f(x)}{g(x)}$.

   (b) Find $G(2)$ and $G'(2)$, where $G(x) = f[g(x)]$.

   (c) Find $H(2)$ and $H'(2)$, where $H(x) = f[g(2x)]$.

   Solution:

   (a) $F(2) = f(2)/g(2) = 5/4$. By the Quotient Rule,

$$F'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}. \tag{4}$$

   Thus,

$$F'(2) = \frac{f'(2)g(2) - f(2)g'(2)}{[g(2)]^2} = \frac{23}{8}. \tag{5}$$

   (b) $G(2) = f[g(2)] = f(4) = 3$. By the Chain Rule,

$$G'(x) = f'[g(x)]g'(x). \tag{6}$$

   Thus,

$$G'(2) = f'[g(2)]g'(2) = f'(4)g'(2) = 12. \tag{7}$$

   (c) By the Chain Rule,

$$\begin{aligned} H'(x) &= f'[g(2x)]\frac{d}{dx}[g(2x)] \qquad (8)\\ &= f'[g(2x)]g'(2x)\frac{d}{dx}[2x] \qquad (9)\\ &= 2f'[g(2x)]g'(2x). \qquad (10) \end{aligned}$$

   Thus

$$\begin{aligned} H'(2) &= 2f'[g(2 \cdot 2)]g'(2 \cdot 2) = 2f'[g(4)]g'(4) \qquad (11)\\ &= 2 \cdot f'(2) \cdot (-8) = -64. \qquad (12) \end{aligned}$$

5. (a) If x and y are related by the equation $x^3 - 4xy + 2y^3 = 2$, find the value of y' at the point $(2, 1)$.

   (b) Find an equation for the line tangent to the curve $x^3 - 4xy + 2y^3 = 2$ at the point $(2, 1)$.

   (c) Show how to use the result in part (b) to estimate the value of y for the point on the curve near $(2, 1)$ where $x = 1.997$.

   Solution:

(a) Differentiating the equation $x^3 - 4xy + 2y^3 = 2$ implicitly we find that

$$3x^2 - 4y - 4xy' + 6y^2y' = 0, \tag{13}$$

or

$$y' = \frac{3x^2 - 4y}{4x - 6y^2}. \tag{14}$$

Hence

$$y'\Big|_{(2,1)} = \frac{3(2)^2 - 4(1)}{4(2) - 6(1)} = \frac{12 - 4}{8 - 6} = 4. \tag{15}$$

(b) From the previous calculation, we know that the slope of the line tangent to this curve at $(2,1)$ is 4. Hence, the required equation is $y = 1 + 4(x - 2)$, or $y = 4x - 7$.

(c) From the previous part of the problem, we know that if $y = f(x)$ for values of x and y near $(2,1)$, the linearization of f is given by $L(x) = 1 + 4(x - 2)$. For values of x near 2, $L(x)$ is a good approximation to $f(x)$. Hence $f(1.997)$ is approximately

$$\begin{aligned} L(1.997) &= 1 + 4(1.997 - 2) && (16)\\ &= 1 + 4(-0.003) && (17)\\ &= 0.988. && (18) \end{aligned}$$

6. When a resistor whose resistance is r ohms is placed in a circuit in parallel with another resistance of 100 ohms, the effective resistance, R, in ohms, in that circuit because of the presence of the two resistors is given by

$$R = \frac{100r}{100 + r}.$$

Suppose that r is measured as 50 ohms with an error no worse than ± 3 ohms. Use linearization (or differentials) to estimate the maximum error in the computed value for R.

Solution: When $R = f(r)$, the linearization $L(r)$ of f at r_0, which is given by

$$L(r) = f(r_0) + f'(r_0)(r - r_0), \tag{19}$$

gives good approximations for $f(r)$ when r is near r_0. In this case, we have

$$\begin{aligned} f'(r) &= \frac{100(100 + r) - 100r(1)}{(100 + r)^2} && (20)\\ &= \frac{10000}{(100 + r)^2}. && (21) \end{aligned}$$

Thus,

$$\begin{aligned} L(r) &= \frac{100 \cdot 50}{100 + 50} + \frac{10000}{22500}(r - 50) && (22)\\ &= \frac{100}{3} + \frac{4}{9}(r - 50). && (23) \end{aligned}$$

We are interested in error, which is $|R(r) - R(50)|$. Replacing $R(r)$ with $L(r)$ to obtain an approximation, we have

$$|L(r) - R(50)| = \left|\frac{4}{9}(r - 50)\right|. \tag{24}$$

Because $|r - 50| \leq 3$, this gives

$$|L(r) - R(50)| \leq \frac{4}{9} \cdot 3 = \frac{4}{3}. \tag{25}$$

The maximum error in the computed value (which is $100/3$ ohms) for R when $r = 50$ to within ± 3 is about $4/3$ ohm.

MTH 1410, Calculus I—Exam III **April 17, 2007**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 2:50 pm.

1. Locate the critical points of the function f which is given by

$$f(x) = (2x^2 + 3x)e^{-x},$$

and determine whether each is a local maximum, a local minimum, or neither. Give your reasoning.

2. Find the absolute maximum and the absolute minimum values of the function g which is given by

$$g(x) = (x^2 - 1)^3$$

on the interval $[-2, 3]$. Give your reasoning.

3. Let F be the function given by

$$F(x) = x^5(x + 2)^8.$$

Then in fully factored form

$$F'(x) = x^4(x + 2)^7(13x + 10)$$

and

$$F''(x) = 4x^3(x + 2)^6(39x^2 + 60x + 20).$$

Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. Give your reasoning.

4. Show how to evaluate the following limits. Be sure to give your reasoning.

   (a) $\displaystyle\lim_{x\to 0} \frac{12\cos x - 12 + 6x^2}{7x^4}$

   (b) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x - 1}$

5. A spherical snowball is melting in such a way that its radius is decreasing at a rate of 1/50 cm/min. At what rate is the volume changing when the diameter is 14 cm? Give your reasoning

6. A Front Range rancher wants to create a fenced 40-acre rectangular plot whose sides run east-west and north-south. Owing to frequent strong winds from the west, it costs four times as much per foot to fence the east and west boundaries of such a plot as it does to fence the north and south boundaries. What are the dimensions of his most economical plot? Give your reasoning. (One acre contains 43,560 ft^2.)

MTH 1410, Calculus I—Exam III **April 17, 2007**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 2:50 pm.

1. Locate the critical points of the function f which is given by

$$f(x) = (2x^2 + 3x)e^{-x},$$

and determine whether each is a local maximum, a local minimum, or neither. Give your reasoning.

Solution: We have

$$f'(x) = (4x + 3)e^{-x} - (2x^2 + 3x)e^{-x} \tag{1}$$
$$= -(2x^2 - x - 3)e^{-x}, \tag{2}$$

which is zero when

$$2x^2 - x - 3 = 0 \tag{3}$$

or

$$(2x - 3)(x + 1) = 0, \tag{4}$$

Thus, $x = -1$ and $x = 3/2$ give the critical points for f. The quantity $-(2x^2 - x - 3)e^{-x}$ is negative when $x < -1$, positive when $-1 < x < 3/2$, and negative when $3/2 < x$. Thus, $f'(x)$ changes sign from negative to positive at $x = -1$, meaning that f has a local minimum at $x = -1$. On the other hand, $f'(x)$ changes sign from positive to negative at $x = 3/2$, so f has a local maximum at $x = 3/2$.

2. Find the absolute maximum and the absolute minimum values of the function g which is given by

$$g(x) = (x^2 - 1)^3$$

on the interval $[-2, 3]$. Give your reasoning.

Solution: The derivative is given by

$$g'(x) = 6x(x^2 - 1)^2. \tag{5}$$

This is zero when $x = 0$ and when $x = \pm 1$. Because absolute extrema must occur at either critical points or end-points, it now suffices to examine $g(-2) = 27$, $g(-1) = 0$, $g(0) = -1$, $g(1) = 0$, and $g(3) = 512$. The largest of these is the maximum and the smallest the minimum. We conclude that on the interval $[-2, 3]$, the function g has an absolute minimum of -1 at $x = 0$ and an absolute maximum of 512 at $x = 3$.

3. Let F be the function given by

$$F(x) = x^5(x+2)^8.$$

Then in fully factored form

$$F'(x) = x^4(x+2)^7(13x+10)$$

and

$$F''(x) = 4x^3(x+2)^6(39x^2+60x+20).$$

Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. Give your reasoning.

Solution: F' is positive when $x < -2$, negative when $-2 < x < -10/13$, positive when $-10/13 < x < 0$, and positive when $0 < x$. Consequently, F is increasing on $(-\infty, -2]$, decreasing on $[-2, -10/13]$, and increasing on $[-10/13, \infty)$. (The *isolated* critical point at $x = 0$ doesn't affect the growth behavior of F.) The quadratic equation $39x^2 + 60x + 20 = 0$ has roots $x = 2(-15 \pm \sqrt{30})/39$, and the quadratic factor $39x^2 + 60x + 20$ is negative only when x lies between the roots. Thus, $F''(x)$ is negative when $x < -2$, negative when $-2 < x < 2(-15 - \sqrt{30})/39$, positive when $2(-15-\sqrt{30})/39 < x < 2(-15+\sqrt{30})/39$, negative when $2(-15+\sqrt{30})/39 < x < 0$, and positive when $0 < x$. Thus F' is decreasing on $(-\infty, 2(-15 - \sqrt{30})/39]$ and on $[2(-15+\sqrt{30})/39, 0]$. On the other hand, F' is increasing on $[2(-15 - \sqrt{30})/39, 2(-15+\sqrt{30})/39]$ and increasing on $[0, \infty)$. This means that F is concave downward on $(-\infty, 2(-15-\sqrt{30})/39]$ and on $[2(-15+\sqrt{30})/39, 0]$ and concave upward on $[2(-15 - \sqrt{30})/39, 2(-15+\sqrt{30})/39]$ as well as on $[0, \infty)$.

4. Show how to evaluate the following limits. Be sure to give your reasoning.

   (a) $\displaystyle \lim_{x\to 0} \frac{12\cos x - 12 + 6x^2}{7x^4}$

   (b) $\displaystyle \lim_{x\to 0} \frac{\sin x}{x-1}$

   Solution:

   (a) As $x \to 0$, both $12\cos x - 12 + 6x^2 \to 0$ and $7x^4 \to 0$, so we may attempt L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{12\cos x - 12 + 6x^2}{7x^4} = \lim_{x\to 0} \frac{-12\sin x + 12x}{28x^3}, \tag{6}$$

   provided the latter limit exists. But in this latter expression, both numerator and denominator go to zero as $x \to 0$, so

$$\lim_{x\to 0} \frac{-12\sin x + 12x}{28x^3} = \lim_{x\to 0} \frac{-12\cos x + 12}{84x^2} \tag{7}$$

   if this last limit exists. L'Hôpital's Rule is applicable to this one, too, so

$$\lim_{x\to 0} \frac{-12\cos x + 12}{84x^2} = \lim_{x\to 0} \frac{12\sin x}{168x} \tag{8}$$

$$= \frac{1}{14} \lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{14}. \tag{9}$$

(b)

$$\lim_{x\to 0}\frac{\sin x}{x-1} = \frac{\lim_{x\to 0}\sin x}{\lim_{x\to 0}(x-1)} \tag{10}$$
$$= \frac{0}{-1} = 0. \tag{11}$$

5. A spherical snowball is melting in such a way that its radius is decreasing at a rate of 1/5 cm/min. At what rate is the volume changing when the diameter is 14 cm? Give your reasoning

Solution: The volume V of a sphere of radius r is given by $V = 4\pi r^3/3$. Treating V and r both as functions of time and taking derivatives implicitly, we find that

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}. \tag{12}$$

We are given that $dr/dt = -1/50$ cm/min, and we want to know dV/dt when the diameter is 14 cm, or, equivalently, when $r = 7$. From (12), we have at that moment

$$\frac{dV}{dt} = 4\pi \cdot 7^2 \cdot \left(-\frac{1}{50}\right) = -\frac{98\pi}{25}\text{cm}^3/\text{min}. \tag{13}$$

6. A Front Range rancher wants to create a fenced 40-acre rectangular plot whose sides run east-west and north-south. Owing to frequent strong winds from the west, it costs four times as much per foot to fence the east and west boundaries of such a plot as it does to fence the north and south boundaries. What are the dimensions of his most economical plot? Give your reasoning. (One acre contains 43,560 ft^2.)

Solution: Let k denote the cost per foot for fence on the north and south boundaries of the plot. Then per-foot cost for the east and west boundaries is $4k$. Letting x denote the common length of the north and south boundary fences, and y the common length of the east and west boundary fences, we must have $xy = 40 \cdot 43560 = 1742400$. Thus, $y = 1742400/x$. The total cost, C, is

$$C = k\cdot(2x) + (4k)\cdot(2y) \tag{14}$$
$$= 2kx + 8ky \tag{15}$$
$$= 2kx + 8k\cdot\frac{1742400}{x} \tag{16}$$
$$= 2kx + \frac{13939200k}{x}, \tag{17}$$

from which we find that

$$\frac{dC}{dx} = 2k - \frac{13939200k}{x^2}. \tag{18}$$

This is zero when $x^2 = 6969600$, or when $x = 2640$. (There is a negative solution, but it is of no interest to us because x is the length of a fence.) Then $y = 1742400/2640 = 660$. The north and south boundaries of the plot should be 2640 feet (half a mile) long, and the east and west boundaries should be 660 feet long (one eighth of a mile).

MTH 1410, Calculus I—Final Exam **May 7, 2007**

Instructions: Write out your solutions for the following problems on your own paper; give your reasoning and show your supporting calculations. Your paper is due at 2:50 pm.

1. Find an equation for the line tangent to the curve

$$y = x\sin x + 2\cos x$$

at the point where $x = \pi/2$.

2. Show how to use either the Limit Laws or L'Hôpital's Rule to evaluate the limits:

(a) $$\lim_{x\to 289} \frac{\sqrt{x}-17}{x-289}$$

(b) $$\lim_{x\to\infty} \left(\sqrt{3x^2+2x+2} - \sqrt{3x^2-4x+4}\right)$$

3. Show how to use the definition of the derivative as the limiting value of a certain quotient to find $f'(x)$ when

$$f(x) = \sqrt{x^2+x}.$$

4. Use the Midpoint Rule (that is, a midpoint Riemann sum) with four equal subdivisions to find the approximate value of

$$\int_0^8 \frac{dx}{1+x^3}.$$

Show your calculations and give your answer correct to five decimal places.

5. Find the area bounded by the x-axis, the lines $x = 0$ and $x = 2\sqrt{2}$, and the curve

$$y = \frac{x}{\sqrt{1+x^2}}.$$

6. (a) Find an equation for the line tangent to the curve given by

$$x^3 + xy + y^3 = 11 \qquad (1)$$

at the point $(1, 2)$.

(b) Use the result of part (a) of this question to estimate the value of y which satisfies equation (1) when $x = \frac{19}{20}$. (Give your answer as a fraction whose numerator and denominator are both whole numbers.)

7. Let f be the function given by

$$f(x) = \sin^3 x.$$

(a) Explain why

$$f'(x) = 3\sin^2 x\,\cos x.$$

(b) Locate the critical numbers for f that lie in the interval $-3\pi/4 \le x \le 3\pi/4$ and determine whether each gives a local minimum, a local maximum, or neither for f.

(c) Find the maximum value and the minimum value of f on the interval $-3\pi/4 \le x \le 3\pi/4$.

8. A Coast Guard cutter is moored offshore at night when a motorboat speeds past its eastern side headed due north at constant speed of 3.5 yards per second. The cutter's crew track the motorboat with a search-light. If the motorboat's closest approach to the search-light is 100 yards, how fast is the searchlight turning when the boat bears 30° north of east from the light?

MTH 1410, Calculus I—Final Exam **May 7, 2007**

Instructions: Write out your solutions for the following problems on your own paper; give your reasoning and show your supporting calculations. Your paper is due at 2:50 pm.

1. Find an equation for the line tangent to the curve

$$y = x\sin x + 2\cos x$$

at the point where $x = \pi/2$.

Solution: If

$$f(x) = x\sin x + 2\cos x,$$

then

$$f'(x) = \sin x + x\cos x - 2\sin x = x\cos x - \sin x,$$

so

$$\begin{aligned} f'(\pi/2) &= (\pi/2)\cos(\pi/2) - \sin(\pi/2) \\ &= -1. \end{aligned}$$

The equation of the tangent line to the curve $y = f(x)$ at $x = \pi/2$ is therefore

$$y = f(\pi/2) + f'(\pi/2)(x - \pi/2),$$

or

$$y = 2 - \left(x - \frac{\pi}{2}\right).$$

2. Show how to use either the Limit Laws or L'Hôpital's Rule to evaluate the limits:

(a)

$$\lim_{x\to 289} \frac{\sqrt{x} - 17}{x - 289}$$

(b)

$$\lim_{x\to\infty} \left(\sqrt{3x^2 + 2x + 2} - \sqrt{3x^2 - 4x + 4}\right)$$

Solution:

(a)

$$\begin{aligned} \lim_{x\to 289} \frac{\sqrt{x} - 17}{x - 289} &= \lim_{x\to 289} \frac{(\sqrt{x} - 17)}{(\sqrt{x} - 17)(\sqrt{x} + 17)} \\ &= \lim_{x\to 289} \frac{1}{(\sqrt{x} + 17)} = \frac{1}{34}. \end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{3x^2+2x+2}-\sqrt{3x^2-4x+4}\right) &= \lim_{x\to\infty}\left[\frac{(3x^2+2x+2)-(3x^2-4x+4)}{\sqrt{3x^2+2x+2}+\sqrt{3x^2-4x+4}}\right]\\
&= \lim_{x\to\infty}\left[\frac{6x-2}{\sqrt{3x^2+2x+2}+\sqrt{3x^2-4x+4}}\right]\\
&= \lim_{x\to\infty}\left[\frac{6-\frac{2}{x}}{\sqrt{3+\frac{2}{x}+\frac{2}{x^2}}+\sqrt{3-\frac{4}{x}+\frac{4}{x}}}\right]\\
&= \frac{6}{2\sqrt{3}} = \sqrt{3}.
\end{aligned}$$

3. Show how to use the definition of the derivative as the limiting value of a certain quotient to find $f'(x)$ when

$$f(x) = \sqrt{x^2+x}.$$

Solution: We have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{(x+h)^2+(x+h)}-\sqrt{x^2+x}}{h}\\
&= \lim_{h\to 0}\frac{[(x+h)^2+(x+h)]-[x^2+x]}{h(\sqrt{(x+h)^2+(x+h)}+\sqrt{x^2+x})}\\
&= \lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2+\cancel{x}+h-\cancel{x^2}-\cancel{x}}{h(\sqrt{(x+h)^2+(x+h)}+\sqrt{x^2+x})}\\
&= \lim_{h\to 0}\frac{\cancel{h}(2x+h+1)}{\cancel{h}(\sqrt{(x+h)^2+(x+h)}+\sqrt{x^2+x})}\\
&= \lim_{h\to 0}\frac{2x+h+1}{\sqrt{(x+h)^2+(x+h)}+\sqrt{x^2+x}}\\
&= \frac{2x+1}{2\sqrt{x^2+x}}.
\end{aligned}$$

4. Use the Midpoint Rule (that is, a midpoint Riemann sum) with four equal subdivisions to find the approximate value of

$$\int_0^8 \frac{dx}{1+x^3}.$$

Show your calculations and give your answer correct to five decimal places.

Solution: For the Midpoint Rule approxmation to $\int_0^8 f(x)\,dx$ with $n=4$, we need four intervals of equal length: $[0,2]$, $[2,4]$, $[4,6]$, and $[6,8]$. The midpoints of these intervals are, respectively, $x_1^*=1$, $x_2^*=3$, $x_3^*=5$, and $x_4^*=7$. Then

$$\begin{aligned}
\int_0^8 \frac{dx}{1+x^3} &\sim \sum_{k=1}^{4} f(x_k^*)(x_k-x_{k-1})\\
&= \frac{1}{1+1^3}\cdot(2-0)+\frac{1}{1+3^3}\cdot(4-2)+\frac{1}{1+5^3}\cdot(6-4)+\frac{1}{1+7^3}\cdot(8-6)\\
&= \frac{2}{2}+\frac{2}{28}+\frac{2}{126}+\frac{2}{344} = \frac{11845}{10836} \sim 1.09312
\end{aligned}$$

Note: Advanced techniques of integration give the result

$$\int_0^8 \frac{dx}{1+x^3} = \frac{1}{18}\left[\sqrt{3}\pi + 3\log\left(\frac{27}{19}\right) + 6\sqrt{3}\arctan\left(5\sqrt{3}\right)\right] \sim 1.20139. \qquad (1)$$

5. Find the area bounded by the x-axis, the lines $x = 0$ and $x = 2\sqrt{2}$, and the curve

$$y = \frac{x}{\sqrt{1+x^2}}.$$

Solution: The required area is

$$A = \int_0^{2\sqrt{2}} \frac{x\,dx}{\sqrt{1+x^2}}.$$

In order to evaluate this definite integral, we let $u = 1 + x^2$. Then $du = 2x\,dx$, or $x\,dx = (u/2)\,du$. Moreover, when $x = 0$, $u = 1$, and when $x = 2\sqrt{2}$, $u = 9$. Therefore,

$$\int_0^{2\sqrt{2}} \frac{x\,dx}{\sqrt{1+x^2}} = \frac{1}{2}\int_1^9 \frac{u\,du}{\sqrt{u}} = \frac{1}{2}\int_1^9 u^{-1/2}\,du = \frac{1}{2}\cdot 2u^{1/2}\Big|_1^9 = 2.$$

6. (a) Find an equation for the line tangent to the curve given by

$$x^3 + xy + y^3 = 11 \qquad (2)$$

at the point $(1, 2)$.

(b) Use the result of part (a) of this question to estimate the value of y which satisfies equation (2) when $x = \frac{19}{20}$. (Give your answer as a fraction whose numerator and denominator are both whole numbers.)

Solution:

(a) Treating y as a function of x and differentiating equation 2 implicitly, we obtain

$$3x^2 + y + xy' + 3y^2y' = 0;$$
$$(x + 3y^2)y' = -(3x^2 + y);$$
$$y' = -\frac{3x^2 + y}{x + 3y^2}.$$

Putting $x = 1$ and $y = 2$ in the latter equation, we obtain

$$y'\Big|_{(1,2)} = -\frac{5}{13}.$$

An equation for the line tangent to the curve at $(1, 2)$ is therefore

$$y = 2 - \frac{5}{13}(x - 1).$$

(b) Replacing the curve with its linearization at $(1, 2)$, we find that when $x = 19/20$, we have

$$y \sim 2 - \frac{5}{13}\left(\frac{19}{20} - 1\right) = 2 + \frac{5}{13}\cdot\frac{1}{20} = \frac{105}{52}.$$

7. Let f be the function given by

$$f(x) = \sin^3 x.$$

(a) Explain why

$$f'(x) \quad = \quad 3\sin^2 x \, \cos x.$$

(b) Locate the critical numbers for f that lie in the interval $-3\pi/4 \le x \le 3\pi/4$ and determine whether each gives a local minimum, a local maximum, or neither for f.

(c) Find the maximum value and the minimum value of f on the interval $-3\pi/4 \le x \le 3\pi/4$.

Solution:

(a) By the Chain Rule, we have, when u is any functions of x,

$$\frac{d}{dx}u^3 = 3u^2\frac{d}{dx}u.$$

Thus,

$$\begin{aligned}\frac{d}{dx}\sin^3 x &= \frac{d}{dx}\big(\sin x\big)^3\\ &= 3\big(\sin x\big)^2\frac{d}{dx}\sin x\\ &= 3\sin^2 x\cos x\end{aligned}$$

(b) The critical numbers for f are the values of x in $(-3\pi/4, 3\pi/3)$ for which $f'(x) = 0$, or, in this case, for which $3\sin^2 x\cos x = 0$. This latter can be zero only when $\sin x = 0$ or $\cos x = 0$, so the critical numbers are $x = 0$ and $x = \pm\pi/2$. Because the squared sine function is non-negative everywhere on $[-3\pi/4, 3\pi/4]$ while the cosine function is negative only on $[-3\pi/4, -\pi/2)$ and on $(\pi/2, 3\pi/4]$, the derivative undergoes a sign change from negative to positive at $x = -\pi/2$ and another sign change from positive to negative at $x = \pi/2$. The derivative is positive to both sides of $x = 0$. Hence f has a local minimum at $x = -\pi/2$, a local maximum at $x = \pi/2$, and neither at $x = 0$.

(c) In order to find the absolute maximum and the absolute minimum for $f(x)$ on $[-3\pi/4, 3\pi/4]$, we calculate $f(-3\pi/4) = -\sqrt{2}/4$, $f(-\pi/2) = -1$, $f(0) = 0$, $f(\pi/2) = 1$, $f(3\pi/4) = \sqrt{2}/4$. The absolute maximum for f on the interval is the largest of these, which is $1 = f(\pi/2)$. The absolute minimum is the smallest, or $-1 = f(-\pi/2)$.

8. A Coast Guard cutter is moored offshore at night when a motorboat speeds past its eastern side headed due north at constant speed of 3.5 yards per second. The cutter's crew track the motorboat with a search-light. If motorboat's closest approach to the search-light is 100 yards, how fast is the searchlight turning when the boat bears 30° north of east from the light?

Solution: Let y denote the distance, measured in the northward direction, from the east-west line passing through the Coast Guard cutter to the motorboat. We are given $dy/dt = 3.5$ y/s. If θ denotes the angle, measured in the counter-clockwise direction, that the searchlight beam makes with the east, then

$$\theta = \arctan\left(\frac{y}{100}\right),$$

whence

$$\frac{d\theta}{dt} = \frac{1}{1+(y/100)^2}\frac{d}{dt}\frac{y}{100} = \frac{100}{10000+y^2}\frac{dy}{dt}.$$

At the critical instant when the motorboat bears 30° north of east from the light, we have

$$y = 100\tan\frac{\pi}{6} = \frac{100}{\sqrt{3}}.$$

At that moment, we then have

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{100}{10000 + (100/\sqrt{3})^2} \cdot \frac{7}{2} \\ &= \frac{21}{800} \text{ radians/sec.}\end{aligned}$$

This is about 1.504 deg/sec.

MTH 1410, Calculus I—Exam I **Sept. 19, 2007**

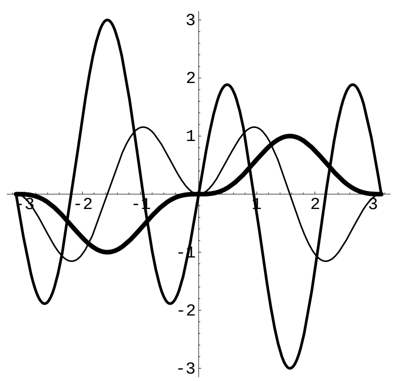
Instructions: Work the following problems; give your reasoning and show your supporting calculations. Your paper is due at 4:50 pm.

1. Evaluate the following limits. Use the Limit Laws. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $$\lim_{x\to -3} \frac{x^2+x-6}{x^2+4x+3}$$

   (b) $$\lim_{x\to -\infty} \frac{x^3+3x^2+2}{2x^3+2x^2+2+1}$$

2. Here is a graph showing three functions—a skinny one, a middle-weight one, and a fat one—on the same pair of axes:



   One of the curves is f, one is f', and one is f''. Explain which is which and how you know.

3. Find $f'(x)$ if

   (a) $f(x) = 3x^2 - 4x + 5$

   (b) $f(x) = \dfrac{a}{x^{10}} + be^x$, where a and b are fixed but unspecified constants.

   (c) $f(x) = \dfrac{x^2 - 2\sqrt{x}}{x^{1/3}}$

4. A function f is given by

$$f(x) = \begin{cases} 2cx + 2 & ; \quad x \le 3 \\ 3 - cx & ; \quad 3 < x. \end{cases}$$

   For what values of the constant c is f continuous on $(-\infty, \infty)$? Be sure to give your reasoning.

5. (a) If $f(x) = \sqrt{2x}$ for $0 \leq x$, find $f'(x)$ by setting up and evaluating (via the Limit Laws) an appropriate limit.
   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = \sqrt{2x}$ at $x = 1$, at $x = 2$, and at $x = 8$.

6. Let F be the function given by

$$F(x) = f(x)g(x),$$

where f and g are functions for which $f'(x)$ and $g'(x)$ are both defined for all real values of x.

   (a) What is $F'(x)$?
   (b) Explain why the formula you have given in part 6a is correct.

MTH 1410, Calculus I—Exam I **Sept. 19, 2007**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Your paper is due at 4:50 pm.

1. Evaluate the following limits. Use the Limit Laws. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a)
$$\lim_{x\to -3} \frac{x^2+x-6}{x^2+4x+3}$$

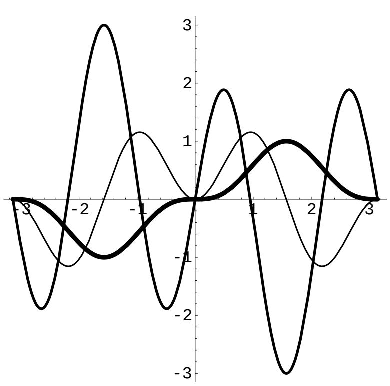
   (b)
$$\lim_{x\to -\infty} \frac{x^3+3x^2+2}{2x^3+2x^2+2+1}$$

   Solution:

   (a)
$$\lim_{x\to -3} \frac{x^2+x-6}{x^2+4x+3} = \lim_{x\to -3} \frac{\cancel{(x+3)}(x-2)}{\cancel{(x+3)}(x+1)} = \lim_{x\to -3} \frac{x-2}{x+1} = \frac{-3-2}{-3+1} = \frac{5}{2}.$$

   (b)
$$\lim_{x\to -\infty} \frac{x^3+3x^2+2}{2x^3+2x^2+2+1} = \lim_{x\to -\infty} \frac{1+3/x+2/x^3}{2+2/x+2/x^3+1/x^3} = \frac{1}{2}.$$

2. Here is a graph showing three functions—a skinny one, a middle-weight one, and a fat one—on the same pair of axes:



   One of the curves is f, one is f', and one is f''. Explain which is which and how you know.

   Solution: The skinny curve lies above the x-axis exactly where the fat curve is increasing, touches the x-axis exactly where the fat curve has a horizontal tangent, and lies below the x-axis exactly where the fat curve is decreasing. The middle-weight curve lies above the x-axis exactly where the skinny one is increasing, touches the x-axis exactly where the skinny one has a horizontal tangent, and lies below the x-axis exactly where the skinny one is decreasing. Thus, the fat curve is f, the skinny curve is f', and the middle-weight curve is f''.

3. Find $f'(x)$ if

   (a) $f(x) = 3x^2 - 4x + 5$

   (b) $f(x) = \dfrac{a}{x^{10}} + be^x$, where a and b are fixed but unspecified constants.

   (c) $f(x) = \dfrac{x^2 - 2\sqrt{x}}{x^{1/3}}$

   Solution:

   (a)

$$\begin{aligned} f'(x) &= \frac{d}{dx}(3x^2 - 4x + 5) \\ &= 6x - 4. \end{aligned}$$

   (b)

$$\begin{aligned} f'(x) &= \frac{d}{dx}(ax^{-10} + be^x) \\ &= -10ax^{-11} + be^x. \end{aligned}$$

   (c)

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^{5/3} - 2x^{1/6}) \\ &= \frac{5}{3}x^{2/3} - \frac{1}{3}x^{-5/6}. \end{aligned}$$

4. A function f is given by

$$f(x) = \begin{cases} 2cx + 2 & ; \quad x \le 3 \\ 3 - cx & ; \quad 3 < x. \end{cases}$$

   For what values of the constant c is f continuous on $(-\infty, \infty)$? Be sure to give your reasoning.

   Solution: If $a \neq 3$ the values of the function $f(x)$ are given by a polynomial function in some open interval centered at a, so f is a continuous function everywhere except possibly at $x = 3$. In order for f to be continuous at $x = 3$, we need to have $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) = 6c + 2$. Now $\lim_{x\to 3^-} f(x) = 6c + 2$, but $\lim_{x\to 3^+} = 3 - 3c$. Consequently, f is continuous at $x = 3$ precisely when c is chosen so that $6c + 2 = 3 - 3c$, or when $c = 1/9$.

5. (a) If $f(x) = \sqrt{2x}$ for $0 \leq x$, find $f'(x)$ by setting up and evaluating (via the Limit Laws) an appropriate limit.

(b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = \sqrt{2x}$ at $x = 1$, at $x = 2$, and at $x = 8$.

Solution:

(a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{2(x+h)} - \sqrt{2x}}{h} \\ &= \lim_{h\to 0} \frac{2(x+h) - 2x}{h\left[\sqrt{2(x+h)} + \sqrt{2x}\right]} \\ &= \lim_{h\to 0} \frac{\cancel{2x} + 2\cancel{h} - \cancel{2x}}{\cancel{h}\left[\sqrt{2(x+h)} + \sqrt{2x}\right]} \\ &= \frac{2}{\sqrt{2x} + \sqrt{2x}} = \frac{1}{\sqrt{2x}}. \end{aligned}$$

(b)

i. Tangent line at $x = 1$:

$$y = f(1) + f'(1)(x - 1),$$

which is

$$y = \sqrt{2} + \frac{1}{\sqrt{2}}(x - 1).$$

ii. Tangent line at $x = 2$:

$$y = f(2) + f'(2)(x - 2),$$

which is

$$y = 2 + \frac{1}{2}(x - 2).$$

iii. Tangent line at $x = 8$:

$$y = f(8) + f'(8)(x - 8),$$

which is

$$y = 4 + \frac{1}{4}(x - 8).$$

6. Let F be the function given by

$$F(x) = f(x)g(x),$$

where f and g are functions for which $f'(x)$ and $g'(x)$ are both defined for all real values of x.

(a) What is $F'(x)$?

(b) Explain why the formula you have given in part 6a is correct.

Solution:

(a) $F'(x) = f'(x)g(x) + f(x)g'(x)$.

(b)

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} g(x+h)\right] + \lim_{h\to 0} \left[f(x)\frac{g(x+h) - g(x)}{h}\right] \\ &= \left[\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right] \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \left[\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right], \end{aligned}$$

provided all of the limits in the latter expression exist. But

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= f'(x), \\ \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} &= g'(x), \text{ and} \\ \lim_{h\to 0} f(x) &= f(x). \end{aligned}$$

We are given that $g'(x)$ exists for all real x, and this means that the function g is continuous everywhere. Consequently,

$$\lim_{h\to 0} g(x+h) = g(x).$$

It now follows that

$$\begin{aligned} F'(x) &= \left[\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right] \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \left[\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right] \\ &= f'(x)g(x) + f(x)g'(x). \end{aligned}$$

MTH 1410, Calculus I—Exam II **October 17, 2007**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 4:50 pm.

1. Find $f'(x)$ if
   (a) $f(x) = 3x^6 - 14x^5 + 12x^3 + 7x^2 - 8x$.
   (b) $f(x) = \dfrac{3x^2 - 5x}{x^2 + x + 1}$.

2. Find $f'(x)$ if
   (a) $f(x) = \cos^3 x \sin 2x$.
   (b) $f(x) = \ln\left(e^x \sin^3 x\right)$.

3. If f is the function given by

$$f(x) = \frac{3\sin x}{2\sin x + 4\cos x},$$

   find an equation for the line tangent to the curve $y = f(x)$ at $x = \pi/4$.

4. Use the Limit Laws, in conjunction with the fact that you know the value of $\lim_{x\to 0} \dfrac{\sin x}{x}$, to evaluate

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2}.$$

5. Suppose that $f(2) = 2$, $f(4) = 4$, $f'(2) = 4$, $f'(4) = -2$, $g(2) = 4$, $g(4) = 2$, $g'(2) = -6$, and $g'(4) = -8$.
   (a) Find $F(4)$ and $F'(4)$, where $F(x) = \dfrac{f(x)}{g(x)}$.
   (b) Find $G(2)$ and $G'(2)$, where $G(x) = g[2f(x)]$.
   (c) Find $H(4)$ and $H'(4)$, where $H(x) = g[f(x^2)]$.

6. (a) Show that the point $(2, 1)$ lies on the curve whose equation is $x^3 - 4x^2y + 2xy^3 + 4 = 0$.
   (b) If x and y are related by the equation $x^3 - 4x^2y + 2xy^3 + 4 = 0$, find the value of y' at the point $(2, 1)$.
   (c) Find an equation for the line tangent to the curve $x^3 - 4x^2y + 2xy^3 + 4 = 0$ at the point $(2, 1)$.

MTH 1410, Calculus I—Exam II, Solutions **October 17, 2007**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 4:50 pm.

1. Find $f'(x)$ if

   (a) $f(x) = 3x^6 - 14x^5 + 12x^3 + 7x^2 - 8x$.

   (b) $f(x) = \dfrac{3x^2 - 5x}{x^2 + x + 1}$.

   Solution:

   (a) $f'(x) = 18x^5 - 70x^4 + 36x^2 + 14x - 8$.

   (b) $f'(x) = \dfrac{(6x-5)(x^2+x+1) - (3x^2-5x)(2x+1)}{(x^2+x+1)^2}$.

2. Find $f'(x)$ if

   (a) $f(x) = \cos^3 x \sin 2x$.

   (b) $f(x) = \ln\left(e^x \sin^3 x\right)$. .

   Solution:

   (a) $f'(x) = -3\cos^2 x \sin x \sin 2x + 2\cos^3 x \cos 2x$.

   (b)

$$f'(x) = D_x\left[\ln\left(e^x \sin^3 x\right)\right] = \frac{D_x\left[e^x \sin^3 x\right]}{e^x \sin^3 x} \quad (1)$$

$$= \frac{e^x \sin^3 x + 3e^x \sin^2 x \cos x}{e^x \sin^3 x}. \quad (2)$$

3. If f is the function given by

$$f(x) = \frac{3\sin x}{2\sin x + 4\cos x},$$

   find an equation for the line tangent to the curve $y = f(x)$ at $x = \pi/4$.

   Solution: We have

$$f\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}/2}{\sqrt{2} + 2\sqrt{2}} = \frac{1}{2},$$

   and

$$f'(x) = \frac{3\cos x(2\sin x + 4\cos x) - 3\sin x(2\cos x - 4\sin x)}{(2\sin x + 4\cos x)^2}$$

$$= \frac{3}{(\sin x + 2\cos x)^2}$$

so that

$$f'\left(\frac{\pi}{4}\right) = \frac{3}{(\sqrt{2}/2 + \sqrt{2})^2} = \frac{2}{3}.$$

The tangent line thus passes through the point with coordinates $(\pi/4, 1/2)$ and has slope $2/3$. The required equation is therefore

$$y = \frac{1}{2} + \frac{2}{3}\left(x - \frac{\pi}{4}\right).$$

4. Use the Limit Laws, in conjunction with the fact that you know the value of $\lim_{x\to 0} \frac{\sin x}{x}$, to evaluate

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2}.$$

Solution: Multiplying the top and bottom by $1 + \cos x$, we find that

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{x^2} &= \lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)} \\
&= \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)} \\
&= \left(\lim_{x\to 0} \frac{\sin^2 x}{x^2}\right) \cdot \left(\lim_{x\to 0} \frac{1}{1+\cos x}\right) \\
&= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 \cdot \left(\lim_{x\to 0} \frac{1}{1+\cos x}\right) = 1 \cdot \frac{1}{2} = \frac{1}{2}.
\end{aligned}$$

5. Suppose that $f(2) = 2$, $f(4) = 4$, $f'(2) = 4$, $f'(4) = -2$, $g(2) = 4$, $g(4) = 2$, $g'(2) = -6$, and $g'(4) = -8$.

   (a) Find $F(4)$ and $F'(4)$, where $F(x) = \dfrac{f(x)}{g(x)}$.
   (b) Find $G(2)$ and $G'(2)$, where $G(x) = g[2f(x)]$.
   (c) Find $H(2)$ and $H'(2)$, where $H(x) = g[f(x^2)]$.

Solutions:

(a)

$$\begin{aligned}
F(4) &= \frac{f(4)}{g(4)} = 2; \\
F'(4) &= \frac{f'(4)g(4) - f(4)g'(4)}{[g(4)]^2} = 7.
\end{aligned}$$

(b)

$$\begin{aligned}
G(2) &= g[2f(2)] = 2; \\
G'(x) &= g'[2f(x)]D_x[2f(x)] = 2g'[2f(x)]f'(x),
\end{aligned}$$

so

$$G'(2) = 2g'[2f(2)]f'(2) = -64.$$

(c)

$$\begin{aligned} H(2) &= g[f(2)] = 2; \\ H'(x) &= g'[f(x^2)] \cdot D_x f(x^2) = g'[f(x^2)]f'(x^2)D_x x^2 = 2xg'[f(x^2)]f'(x^2), \end{aligned}$$

so

$$H'(2) = 4g'[f(4)]f'(4) = 64.$$

6. (a) Show that the point $(2,1)$ lies on the curve whose equation is $x^3 - 4x^2y + 2xy^3 + 4 = 0$.
   (b) If x and y are related by the equation $x^3 - 4x^2y + 2xy^3 + 4 = 0$, find the value of y' at the point $(2,1)$.
   (c) Find an equation for the line tangent to the curve $x^3 - 4x^2y + 2xy^3 + 4 = 0$ at the point $(2,1)$.

Solution:

(a) When $x = 2$ and $y = 1$,

$$\begin{aligned} x^3 - 4x^2y + 2xy^3 + 4 &= 2^3 - 4 \cdot 2^2 \cdot 1 + 2 \cdot 2 \cdot 1^3 + 4 \\ &= 8 - 16 + 4 + 4 = 0, \end{aligned}$$

so the point $(2,1)$ lies on the curve $x^3 - 4x^2y + 2xy^3 + 4 = 0$.

(b) Treating y as a function of x and taking the derivative with respect to x on both sides of the equation, we find that

$$\begin{aligned} D_x(x^3 - 4x^2y + 2xy^3 + 4) &= D_x(0) \\ 3x^2 - (8xy + 4x^2y') + (2y^3 + 6xy^2y') + 0 &= 0 \\ (6xy^2 - 4x^2)y' &= 8xy - 2y^3 - 3x^2 \\ y' &= \frac{8xy - 2y^3 - 3x^2}{6xy^2 - 4x^2}. \end{aligned}$$

At $x = 2$, $y = 1$, this gives

$$y'\Big|_{(2,1)} = \frac{8 \cdot 2 \cdot 1 - 2 \cdot 1^3 - 3 \cdot 2^2}{6 \cdot 2 \cdot 1^2 - 4 \cdot 2^2} = \frac{2}{-4} = -\frac{1}{2}.$$

(c) The tangent line is the line that passes through the point $(2,1)$ with slope $-1/2$, so an equation for the tangent line is

$$y = 1 - \frac{1}{2}(x - 2).$$

MTH 1410, Calculus I—Exam III **November 14, 2007**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 4:50 pm.

1. Find the limits:

   (a) $\lim_{x \to -1} \dfrac{x + x^2}{\ln(2 + x)}$

   (b) $\lim_{x \to 0} \dfrac{xe^{-2x}}{\pi e^{2x} - \pi}$

2. Find the absolute maximum and the absolute minimum for the function

$$f(x) = 2x^3 - 3x^2 - 12x + 20$$

   on the interval $[-3, 3]$.

3. Take $x_0 = 2$ as your initial guess in the Newton's method approximation of a root of the equation $x^3 - 11 = 0$ and find x_1 and x_2. Give your answers as fractions of integers.

4. Let F be the function given by

$$F(x) = (x - 1)^2(x + 1)^3.$$

   Then, in fully factored form,

$$F'(x) = (x - 1)(x + 1)^2(5x - 1)$$

   and, also in fully factored form,

$$F''(x) = 20(x + 1)\left[x - \frac{1}{5}\left(1 - \sqrt{6}\right)\right]\left[x - \frac{1}{5}\left(1 + \sqrt{6}\right)\right].$$

   Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward.

5. Find the points on the ellipse $4x^2 + y^2 = 4$ whose distance from the point $(1, 0)$ is maximal.

6. Murgatroyd was driving his car toward an intersection at 60 miles per hour. A police cruiser was approaching the same intersection but on the cross-street (which is at right angles to the road that Muratroyd is on), at 50 miles per hour. When both cars were a quarter of a mile from the intersection, a police officer in the cruiser pointed a radar gun at Murgatroyd and measured the speed at which the two cars were approaching each other. What did she get?

MTH 1410, Calculus I—Exam III, Solutions **November 14, 2007**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 4:50 pm.

1. Find the limits:

   (a) $\lim_{x\to -1} \dfrac{x+x^2}{\ln(2+x)}$

   (b) $\lim_{x\to 0} \dfrac{xe^{-2x}}{\pi e^{2x}-\pi}$

   Solution:

   (a) $\lim_{x\to-1}(x+x^2) = 0 = \lim_{x\to-1}\ln(2+x)$, so we may attempt l'Hôpital's rule:

$$\begin{aligned}\lim_{x\to -1}\frac{x+x^2}{\ln(2+x)} &= \lim_{x\to-1}\frac{1+2x}{[1/(2+x)]}\\ &= \lim_{x\to-1}(1+2x)(2+x)\\ &= -1.\end{aligned}$$

   (b) $\lim_{x\to 0} xe^{-2x} = 0 = \lim_{x\to 0}(\pi e^{2x}-\pi)$, so we can attempt l'Hôpital's rule again:

$$\begin{aligned}\lim_{x\to 0}\frac{xe^{-2x}}{\pi e^{2x}-\pi} &= \lim_{x\to 0}\frac{e^{-2x}-2xe^{-2x}}{2\pi e^{2x}}\\ &= \frac{1}{2\pi}.\end{aligned}$$

2. Find the absolute maximum and the absolute minimum for the function

$$f(x) = 2x^3 - 3x^2 - 12x + 20$$

   on the interval $[-3, 3]$.

   Solution: Absolute extrema are to be found only at endpoints and critical numbers. We have $f'(x) = 6x^2 - 6x - 12 = 6(x+1)(x-2)$, which is defined everywhere and is zero only when $x = -1$ or $x = 2$. Thus, the extrema are among the numbers $f(-3)$, $f(-1)$, $f(2)$, and $f(3)$. We find that $f(-3) = -25$, $f(-1) = 27$, $f(2) = 0$, and $f(3) = 11$. The absolute minimum is $f(-3) = -25$, and the absolute maximum is $f(-1) = 27$.

3. Take $x_0 = 2$ as your initial guess in the Newton's method approximation of a root of the equation $x^3 - 11 = 0$ and find x_1 and x_2. Give your answers as fractions of integers.

   Solution: The Newton's method relation is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

   We have $f(x) = x^3 - 11$, so $f'(x) = 3x^2$. Thus,

$$x_1 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-3}{12} = \frac{9}{4},$$

and

$$x_2 = \frac{9}{4} - \frac{f(9/4)}{f'(9/4)} = \frac{9}{4} - \frac{25/64}{243/16} = \frac{1081}{486}.$$

4. Let F be the function given by

$$F(x) = (x-1)^2(x+1)^3.$$

Then, in fully factored form,

$$F'(x) = (x-1)(x+1)^2(5x-1)$$

and, also in fully factored form,

$$F''(x) = 20(x+1)\left[x - \frac{1}{5}\left(1-\sqrt{6}\right)\right]\left[x - \frac{1}{5}\left(1+\sqrt{6}\right)\right].$$

Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward.

Solution: The quantity $(x-1)$ is positive when $x > 1$ and negative when $x < 1$; $(x+1)^2$ is positive unless $x = -1$; and $(5x-1)$ is positive when $x > 1/5$, negative when $x < 1/5$. Thus, $F'(x) > 0$ when $-\infty < x < -1$, when $-1 < x < 1/5$ and when $1 < x < \infty$. It follows that F is increasing on the intervals $(-\infty, 1/5]$ and $[1, \infty)$, but decreasing on $[1/5, 1]$.

The quantity $(x+1)$ is negative when $x < -1$ and positive when $x > -1$; $[x - (1-\sqrt{6})/5]$ is negative when $x < (1-\sqrt{6})/5$ and positive when $x > (1-\sqrt{6})/5$; $[x-(1+\sqrt{6})/5]$ is negative when $x < (1+\sqrt{6})/5$ and positive when $x > (1+\sqrt{6})/5$. Consequently $F''(x) < 0$ when $-\infty < x < -1$ and when $(1-\sqrt{6})/5 < x < (1+\sqrt{6})/5$, but $F''(x) > 0$ when $-1 < x < (1-\sqrt{6})/5$ and when $(1+\sqrt{6})/5 < x < \infty$. So F is concave upward on $[-1, (1-\sqrt{6})/5]$ and on $[(1+\sqrt{6})/5, \infty)$, but concave downward on $(-\infty, -1]$ and on $[(1-\sqrt{6})/5, (1+\sqrt{6})/5]$.

5. Find the points on the ellipse $4x^2 + y^2 = 4$ whose distance from the point $(1, 0)$ is maximal.

Solution: The distance D from a point (x, y) to the point $(1, 0)$ is given by

$$D^2 = (x-1)^2 + y^2.$$

If the point (x, y) lies on the given ellipse, then $4x^2 + y^2 = 4$, so that $y^2 = 4 - 4x^2$. Substituting this latter equation into the relation for D, we find that

$$D^2 = (x-1)^2 + 4 - 4x^2 = 5 - 2x - 3x^2.$$

Because $y^2 = 4 - 4x^2$ must not be negative, we are interested only in those values of x for which $-1 \le x \le 1$. We have

$$2D\frac{dD}{dx} = -2 - 6x.$$

Thus, $dD/dx = 0$ when $x = -1/3$. We must check the value of D when $x = -1$, when $x = -1/3$, and when $x = 1$. The corresponding values of D^2 are 4, $16/3$, and 0. Thus, D takes on the maximal value $4/\sqrt{3}$ when $x = -1/3$.

6. Murgatroyd was driving his car toward an intersection at 60 miles per hour. A police cruiser was approaching the same intersection but on the cross-street (which is at right angles to the road that Muratroyd is on), at 50 miles per hour. When both cars were a quarter of a mile from the intersection, a police officer in the cruiser pointed a radar gun at Murgatroyd and measured the speed at which the two cars were approaching each other. What did she get?

Solution: Let x denote the distance from Murgatroyd to the intersection, and let y denote the distance from the police car to the interesection. The distance D between the two cars satisfies

$$D^2 = x^2 + y^2,$$

so

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt},$$

or

$$\frac{dD}{dt} = \frac{1}{D}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

At the critical instant, we have $x = y = 1/4$, $D = 1/(2\sqrt{2})$, $dx/dt = -60$, and $dy/dt = -50$. Thus, rate of change of the distance between the two cars was

$$\begin{aligned}\frac{dD}{dt} &= 2\sqrt{2}\left[\frac{1}{4}(-60) + \frac{1}{4}(-50)\right] \\ &= -\frac{110}{\sqrt{2}} \text{ mph.}\end{aligned}$$

The reading on the radar gun was about 78 mph.

MTH 1410, Calculus I—Final Exam **Dec. 10, 2007**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problme requires you to do so. Your exam is due at 4:50 pm.

1. Evaluate the following definite integrals. Give all of your reasoning.
   (a) $\int_3^5 (3x^2 - 24x + 54)\,dx$
   (b) $\int_\pi^{2\pi} \sin x\,dx$
2. Evaluate the following definite integrals. Give all of your reasoning.
   (a) $\int_0^3 3t\sqrt{9-t^2}\,dt$
   (b) $\int_1^4 \frac{(1-\sqrt{x})^4}{\sqrt{x}}\,dx$
3. Suppose that $f(2) = 2$, $f(4) = 4$, $f'(2) = 4$, $f'(4) = -2$, $g(2) = 4$, $g'(2) = -6$, and $g'(4) = -8$.
   (a) Find $F(4)$ and $F'(4)$, where $F(x) = \frac{f(x)}{g(x)}$.
   (b) Find $G(2)$ and $G'(2)$, where $G(x) = g[2f(x)]$.
   (c) Find $H(2)$ and $H'(2)$, where $H(x) = g]f(x^2)]$.
4. (a) Show that the point $(3, 2)$ lies on the curve given by the equation
   $$x^3 - 5x^3y^3 + 8y^4 + 205 = 0. \tag{1}$$
   (b) If x and y are related by equation (1), find the value of y' at $(3, 2)$.
   (c) Show how to use the results of parts (a) and (b) of this problem to find an approximate value for y near 2 when $x = \frac{74}{25}$.
5. Find the points on the ellipse $4x^2 + y^2 = 4$ whose distance from the point $(1, 0)$ is maximal.
6. Let F be the function given by
   $$F(x) = f(x)g(x),$$
   where f and g are functions for which $f'(x)$ and $g'(x)$ are both defined for all real values of x.
   (a) What is $F'(x)$?
   (b) Explain why the formula you have given in part 6(a) is correct.

MTH 1410, Calculus I—Final Exam Solutions **Dec. 10, 2007**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problme requires you to do so. Your exam is due at 4:50 pm.

1. Evaluate the following definite integrals. Give all of your reasoning.

   (a) $\displaystyle\int_3^5 (3x^2 - 24x + 54)\,dx$

   (b) $\displaystyle\int_\pi^{2\pi} \sin x\,dx$

   Solution:

   (a)

$$\begin{aligned}\int_3^5 (3x^2 - 24x + 54)\,dx &= \left(x^3 - 12x^2 + 54x\right)\Big|_3^5 \\ &= (125 - 300 + 270) - (27 - 108 + 162) = 14.\end{aligned}$$

   (b)

$$\int_\pi^{2\pi} \sin x\,dx = -\cos x\Big|_\pi^{2\pi} = (-\cos 2\pi) - (-\cos \pi) = -2.$$

2. Evaluate the following definite integrals. Give all of your reasoning.

   (a) $\displaystyle\int_0^3 3t\sqrt{9 - t^2}\,dt$

   (b) $\displaystyle\int_1^4 \frac{(1 - \sqrt{x})^4}{\sqrt{x}}\,dx$

   Solution:

   (a) Let $u = 9 - t^2$. Then $du = -t\,dt$, or $t\,dt = -\dfrac{1}{2}\,du$. Moreover, $u = 9$ when $t = 0$, and $u = 0$ when $t = 3$. Thus,

$$\begin{aligned}\int_0^3 3t\sqrt{9 - t^2}\,dt &= -\frac{3}{2}\int_9^0 u^{1/2}\,du = \frac{3}{2}\int_0^9 u^{1/2}\,du \\ &= u^{3/2}\Big|_0^9 = 9^{3/2} - 0^{3/2} = 27,\end{aligned}$$

   (b) Let $u = 1 - \sqrt{x}$. Then $du = -\dfrac{dx}{2\sqrt{x}}$, or $\dfrac{dx}{\sqrt{x}} = -2\,du$. So

$$\begin{aligned}\int \frac{(1 - \sqrt{x})^4}{\sqrt{x}}\,dx &= -2\int u^4\,du \\ &= -\frac{2}{5}u^5 = -\frac{2}{5}\left(1 - \sqrt{x}\right)^5.\end{aligned}$$

It follows that

$$\int_1^4 \frac{(1-\sqrt{x})^4}{\sqrt{x}}\,dx = -\frac{2}{5}\left(1-\sqrt{x}\right)^5\Big|_1^4$$
$$= -\frac{2}{5}(1-\sqrt{4})^5 + \frac{2}{5}(1-\sqrt{1})^5 = \frac{2}{5}.$$

3. Suppose that $f(2) = 2$, $f(4) = 4$, $f'(2) = 4$, $f'(4) = -2$, $g(2) = 4$, $g(4) = 2$, $g'(2) = -6$, and $g'(4) = -8$.

   (a) Find $F(4)$ and $F'(4)$, where $F(x) = \dfrac{f(x)}{g(x)}$.

   (b) Find $G(2)$ and $G'(2)$, where $G(x) = g[2f(x)]$.

   (c) Find $H(2)$ and $H'(2)$, where $H(x) = g]f(x^2)]$.

Solution:

(a)

$$F(4) = \frac{f(4)}{g(4)} = 2;$$
$$F'(4) = \frac{f'(4)g(4) - f(4)g'(4)}{[g(4)]^2} = 7.$$

(b)

$$G(2) = g[2f(2)] = 2;$$
$$G'(2) = g'[2f(2)] \cdot 2f'(2) = -64.$$

(c)

$$H'(2) = g'[f(x^2)] \cdot f'(x^2) \cdot 2x, \text{ so}$$
$$H'(2) = 64.$$

4. (a) Show that the point $(3, 2)$ lies on the curve given by the equation

$$x^3 - 5x^2y^3 + 8y^4 + 205 = 0. \qquad (1)$$

   (b) If x and y are related by equation (1), find the value of y' at $(3, 2)$.

   (c) Show how to use the results of parts (a) and (b) of this problem to find an approximate value for y near 2 when $x = \dfrac{74}{25}$.

Solution:

(a) When $x = 3$ and $y = 2$, we have

$$3^3 - 5 \cdot 3^2 \cdot 2^3 + 8 \cdot 2^4 + 205 = 27 - 360 + 128 + 205 = 0, \qquad (2)$$

so the point with coordinates $(3, 2)$ lies in the curve given by (2).

(b) Treating y as a function of x and differentiating implicitly gives

$$3x^2 - 10xy^3 - 15x^2y^2y' + 32y^3y' = 0 \tag{3}$$

Setting $x = 3$ and $y = 2$ in (3) and solving gives

$$y'\Big|_{(3,2)} = -\frac{3}{4}. \tag{4}$$

(c) From the previous part of this problem, we knwo that we can write the linearization of y at $(3, 2)$ as

$$L(x) = 2 - \frac{3}{4}(x - 3). \tag{5}$$

This linearization gives a good approximation for values of y near 2 when x is near 3, so the approximate value we seek is

$$L\left(\frac{74}{25}\right) = 2 - \frac{3}{4}\left(\frac{74}{25} - 3\right) \tag{6}$$

$$= 2 + \frac{3}{4} \cdot \frac{1}{25} = \frac{203}{100}. \tag{7}$$

5. Find the points on the ellipse $4x^2 + y^2 = 4$ whose distance from the point $(1, 0)$ is maximal.

Solution: The distancd D from a point (x, y) to the point $(1, 0)$ satisfies

$$D^2 = (x - 1)^2 + y^2. \tag{8}$$

If the point (x, y) lies on the given ellipse, then $4x^2 + y^2 = 4$, so that $y^2 = 4 - 4x^2$. Substituting this for y^2 in (8), we obtain

$$D^2 = (x - 1)^2 + 4 - 4x^2 = 5 - 2x - 3x^2. \tag{9}$$

Because $y^2 = 4 - 4x^2$ must not be negative, we require that $-1 \le x \le 1$. Differentiating (9) with respect to x while treating D as a function of x, we find that

$$2D\frac{dD}{dx} = -2 - 6x. \tag{10}$$

Thus, $\dfrac{dD}{dx} = 0$ when $x = -\dfrac{1}{3}$. We therefore check for the maximum value of D when $x = -1$, when $x = -\dfrac{1}{3}$, and when $x = 1$. The corresponding values of D^2 are 4, $\dfrac{16}{3}$, and 0. Thus, D takes on the maximal value $\dfrac{4}{\sqrt{3}}$ when $x = -\dfrac{1}{3}$.

6. Let F be the function given by

$$F(x) = f(x)g(x),$$

where f and g are functions for which $f'(x)$ and $g'(x)$ are both defined for all real values of x.

(a) What is $F'(x)$?

(b) Explain why the formula you have given in part 6(a) is correct.

Soluion:

(a) $F'(x) = f'(x)g(x) + f(x)g'(x)$.

(b)

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} g(x+h)\right] + \lim_{h\to 0} \left[f(x)\frac{g(x+h) - g(x)}{h}\right] \\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h}\right] \lim_{h\to 0} g(x+h) + f(x) \lim_{h\to 0} \left[\frac{g(x+h) - g(x)}{h}\right],
\end{aligned}$$

provided all of the limits in the latter expression exist. But

$$\lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h}\right] = f'(x) \text{ and}$$
$$\lim_{h\to 0} \left[\frac{g(x+h) - g(x)}{h}\right] = g'(x)$$

for all real x, and this guarantees that g is continuous at every real x. This in turn guarantees that

$$\lim_{h\to 0} g(x+h) = g(x)$$

for all real values of x. It now follows that

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h}\right] \lim_{h\to 0} g(x+h) + f(x) \lim_{h\to 0} \left[\frac{g(x+h) - g(x)}{h}\right] \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

MTH 1410, Calculus I—Exam I **Sept. 17, 2008**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Your paper is due at 4:50 pm.

1. Evaluate the following limits. Use the Limit Laws. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $$\lim_{x \to -3} \frac{x^2 + x - 6}{x^2 + 4x + 3}$$

   (b) $$\lim_{x \to \infty} \frac{(5 - x)(10 + 8x)}{(3 - 3x)(3 + 10x)}$$

2. A rock is thrown off of a 100-meter cliff with an upward velocity of 50 m/s. As a result, its height (above the ground at the base of the cliff) after t seconds is given by the formula $h(t) = 100 + 50t - 5t^2$.

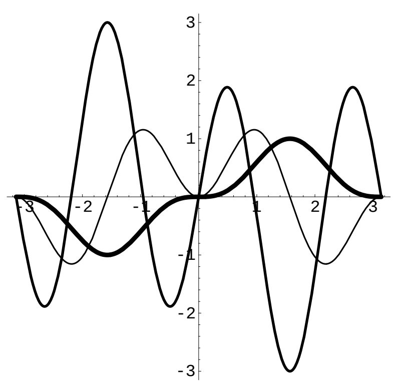
   (a) What is its height after 7 seconds?

   (b) What is its average velocity over the first seven seconds?

   (c) What is its velocity after 7 seconds?

   (d) What is its velocity when it hits the ground at the base of the cliff?

3. Here is a graph showing three functions—a skinny one, a middle-weight one, and a fat one—on the same pair of axes:



   One of the curves is f, one is f', and one is f''. Explain which is which and how you know.

4. Find $f'(x)$ if

   (a) $f(x) = 3x^2 - 4x + 5$

   (b) $f(x) = \dfrac{x^2 - 2\sqrt{x}}{x^{1/3}}$

5. Find $f'(x)$ if

   (a) $f(x) = (3x^2 - x + 1)^2(5x + 4)^{12}$

   (b) $f(x) = \dfrac{a}{x^{10}} + \sin^3 bx$, where a and b are fixed but unspecified constants.

6. A function f is given by

$$f(x) = \begin{cases} 2cx + 2 & ; \quad x \leq 3 \\ 3 - cx & ; \quad 3 < x. \end{cases}$$

   For what values of the constant c is f continuous on $(-\infty, \infty)$? Be sure to give your reasoning.

7. (a) Use the definition of the derivative to find $f'(x)$ if $f(x) = \sqrt{2x}$.

   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = \sqrt{2x}$ at $x = 1$, at $x = 2$, and at $x = 8$.

8. Let F be the function given by

$$F(x) = f(x)g(x),$$

   where f and g are functions for which $f'(x)$ and $g'(x)$ are both defined for all real values of x.

   (a) What is $F'(x)$?

   (b) Derive the formula you have given in part 8a.

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 1410, Calculus I—Exam I **Sept. 17, 2008**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Your paper is due at 4:50 pm.

1. Evaluate the following limits. Use the Limit Laws. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a)

$$\lim_{x\to -3} \frac{x^2+x-6}{x^2+4x+3}$$

   (b)

$$\lim_{x\to \infty} \frac{(5-x)(10+8x)}{(3-3x)(3+10x)}$$

   Solution:

   (a)

$$\lim_{x\to -3} \frac{x^2+x-6}{x^2+4x+3} = \lim_{x\to -3} \frac{\cancel{(x+3)}(x-2)}{\cancel{(x+3)}(x+1)} = \frac{-5}{-2} = \frac{5}{2}.$$

   (b)

$$\begin{aligned}\lim_{x\to \infty} \frac{(5-x)(10+8x)}{(3-3x)(3+10x)} &= \lim_{x\to \infty} \frac{[(5-x)(10+8x)]/x^2}{[(3-3x)(3+10x)]/x^2} \\ &= \lim_{x\to \infty} \frac{(5/x-1)(10/x+8)}{(3/x-3)(3/x+10)} = \frac{(-1)\cdot(8)}{(-3)\cdot(10)} = \frac{4}{15}.\end{aligned}$$

2. A rock is thrown off of a 100-meter cliff with an upward velocity of 50 m/s. As a result, its height (above the ground at the base of the cliff) after t seconds is given by the formula $h(t) = 100 + 50t - 5t^2$.

   (a) What is its height after 7 seconds?
   (b) What is its average velocity over the first seven seconds?
   (c) What is its velocity after 7 seconds?
   (d) What is its velocity when it hits the ground at the base of the cliff?

   Solution:

   (a) $h(7) = 100 + 50\cdot 7 - 5\cdot 7^2 = 205$ meters.

   (b) Average velocity over the first seven seconds is

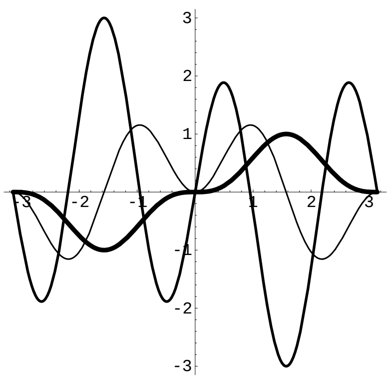
$$\frac{h(7)-h(0)}{7-0} = \frac{205-100}{7} = 15 \text{ meters per second.}$$

(c) Velocity is the derivative $h'(t) = 50 - 10t$, so when $t = 7$, velocity is $h'(7) = 50 - 10 \cdot 7 = -20$ meters per second.

(d) The rock hits the ground when $t > 0$ satisfies $h(t) = 0$. But the only positive solution of the equation $100 + 50t - 5t^2 = 0$ is $t = 5 + 3\sqrt{5}$. So the rock hits the ground at the base of the cliff with velocity

$$h'(5 + 3\sqrt{5}) = 50 - 10(5 + 3\sqrt{5}) = -30\sqrt{5} \text{ meters per second.}$$

3. Here is a graph showing three functions—a skinny one, a middle-weight one, and a fat one—on the same pair of axes:



One of the curves is f, one is f', and one is f''. Explain which is which and how you know.

Solution: The skinny curve lies above the x-axis exactly where the fat curve is increasing, touches the x-axis exactly where the fat curve has a horizontal tangent, and lies below the x-axis exactly where the fat curve is decreasing. The middle-weight curve lies above the x-axis exactly where the skinny one is increasing, touches the x-axis exactly where the skinny one has a horizontal tangent, and lies below the x-axis exactly where the skinny one is decreasing. Thus, the fat curve is f, the skinny curve is f', and the middle-weight curve is f''.

4. Find $f'(x)$ if

(a) $f(x) = 3x^2 - 4x + 5$

(b) $f(x) = \dfrac{x^2 - 2\sqrt{x}}{x^{1/3}}$

Solution:

(a) $f'(x) = 6x - 4$.

(b) Carrying out the division, we find that $f(x) = x^{2-1/3} - 2x^{1/2-1/3} = x^{5/3} - 2x^{1/6}$. Consequently, $f'(x) = \dfrac{5}{3}x^{2/3} - \dfrac{1}{3}x^{-5/6}$.

5. Find $f'(x)$ if

   (a) $f(x) = (3x^2 - x + 1)^2(5x + 4)^{12}$

   (b) $f(x) = \dfrac{a}{x^{10}} + \sin^3 bx$, where a and b are fixed but unspecified constants.

   Solution:

   (a) $f'(x) = 2(3x^2 - x + 1)(6x - 1)(5x + 4)^{12} + 60(3x^2 - x + 1)^2(5x + 4)^{11}$

   (b) $f(x) = ax^{-10} + [\sin(bx)]^3$, so $f'(x) = -10ax^{-11} + 3b\sin^2(bx)\cos(bx)$.

6. A function f is given by

$$f(x) = \begin{cases} 2cx + 2 & ; \quad x \le 3 \\ 3 - cx & ; \quad 3 < x. \end{cases}$$

   For what values of the constant c is f continuous on $(-\infty, \infty)$? Be sure to give your reasoning.

   Solution: If $a \neq 3$ the values of the function $f(x)$ are given by a polynomial function in some open interval centered at a, so f is a continuous function everywhere except possibly at $x = 3$. In order for f to be continuous at $x = 3$, we need to have $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) = 6c + 2$. Now $\lim_{x\to 3^-} f(x) = 6c + 2$, but $\lim_{x\to 3^+} = 3 - 3c$. Consequently, f is continuous at $x = 3$ precisely when c is chosen so that $6c + 2 = 3 - 3c$, or when $c = 1/9$.

7. (a) Use the definition of the derivative to find $f'(x)$ if $f(x) = \sqrt{2x}$.

   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = \sqrt{2x}$ at $x = 1$, at $x = 2$, and at $x = 8$.

   Solution:

   (a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{2(x+h)} - \sqrt{2x}}{h} \\ &= \lim_{h\to 0} \frac{2(x+h) - 2x}{h\left[\sqrt{2(x+h)} + \sqrt{2x}\right]} \\ &= \lim_{h\to 0} \frac{\cancel{2x} + 2\cancel{h} - \cancel{2x}}{\cancel{h}\left[\sqrt{2(x+h)} + \sqrt{2x}\right]} \\ &= \frac{2}{\sqrt{2x} + \sqrt{2x}} = \frac{1}{\sqrt{2x}}. \end{aligned}$$

   (b) i. Tangent line at $x = 1$:

$$y = f(1) + f'(1)(x - 1),$$

which is

$$y = \sqrt{2} + \frac{1}{\sqrt{2}}(x-1).$$

ii. Tangent line at $x = 2$:

$$y = f(2) + f'(2)(x-2),$$

which is

$$y = 2 + \frac{1}{2}(x-2).$$

iii. Tangent line at $x = 8$:

$$y = f(8) + f'(8)(x-8),$$

which is

$$y = 4 + \frac{1}{4}(x-8).$$

8. Let F be the function given by

$$F(x) = f(x)g(x),$$

where f and g are functions for which $f'(x)$ and $g'(x)$ are both defined for all real values of x.

(a) What is $F'(x)$?

(b) Derive the formula you have given in part 8a.

Solution:

(a) $F'(x) = f'(x)g(x) + f(x)g'(x)$.

(b)

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h} g(x+h)\right] + \lim_{h\to 0} \left[f(x)\frac{g(x+h)-g(x)}{h}\right] \\
&= \left[\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right] \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \left[\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right],
\end{aligned}$$

provided all of the limits in the latter expression exist. But

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x),$$

$$\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = g'(x), \text{ and}$$

$$\lim_{h\to 0} f(x) = f(x).$$

We are given that $g'(x)$ exists for all real x, and this means that the function g is continuous everywhere. Consequently,

$$\lim_{h\to 0} g(x+h) = g(x).$$

It now follows that

$$\begin{aligned} F'(x) &= \left[\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right] \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \left[\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right] \\ &= f'(x)g(x) + f(x)g'(x). \end{aligned}$$

MTH 1410, Calculus I—Exam II **October 15, 2008**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 4:50 pm.

1. Find $f'(x)$ if
   (a) $f(x) = 3x^6 - 14x^5 + 12x^3 + 7x^2 - 8x$.
   (b) $f(x) = \dfrac{3x^2 - 5x}{x^2 + x + 1}$.
2. Find $f'(x)$ if
   (a) $f(x) = \cos^3 x \sin 2x$.
   (b) $f(x) = \ln\left[e^x \sin^3 x\right]$.
3. (a) *Use the definition of the derivative* to find $f'(x)$ if $f(x) = 1/\sqrt{x}$.
   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = 1/\sqrt{x}$ at $x = 1$, at $x = 4$, and at $x = 9$.
4. Let f be the function given by

$$f(x) = \begin{cases} x^2 + 2x, & x \le 2 \\ ax^2 + b, & x > 2. \end{cases}$$

   (a) What condition must the constants a and b satisfy if f is to be a continuous function?
   (b) Find all pairs of values for a and b which make the function f a differentiable function.
5. Let g be the function given by $g(x) = x^3 + x$. Then g is a one-to-one function from $(-\infty, \infty)$ onto $(-\infty, \infty)$, and so g is invertible. Let $G = g^{-1}$. Find an equation for the line tangent to the curve $y = G(x)$ at the point on the curve where $x = 10$.
6. A man started walking north at 4 feet per second from a point P. Five minutes later, a woman started walking south at 5 feet per second from a point 500 feet due east of P. At what rate were the people moving apart 15 minutes after the woman started walking.
7. (a) Find an equation for the line tangent to the curve $x^3 - 4x^2y + 2xy^3 + 4 = 0$ at the point whose coordinates are $(2, 1)$.
   (b) Use the result of part (a) of this problem to find an approximate value for the y-coordinate of the point $(2.04, y)$ that lies on the curve $x^3 - 4x^2y + 2xy^3 + 4 = 0$ near the point $(2, 1)$.

MTH 1410, Calculus I—Exam II **October 15, 2008**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 4:50 pm.

1. Find $f'(x)$ if

   (a) $f(x) = 3x^6 - 14x^5 + 12x^3 + 7x^2 - 8x$.

   (b) $f(x) = \dfrac{3x^2 - 5x}{x^2 + x + 1}$.

   Solution:

   (a) $f'(x) = 18x^5 - 70x^4 + 36x^2 + 14x - 8$.

   (b) $f'(x) = \dfrac{(6x-5)(x^2+x+1) - (3x^2-5x)(2x+1)}{(x^2+x+1)^2}$.

2. Find $f'(x)$ if

   (a) $f(x) = \cos^3 x \sin 2x$.

   (b) $f(x) = \ln\left[e^x \sin^3 x\right]$.

   Solution:

   (a) $f'(x) = -3\cos^2 x \sin x \sin 2x + 2\cos^3 x \cos 2x$.

   (b) $f'(x) = D_x(x + 3\ln\sin x) = 1 + 3\dfrac{\cos x}{\sin x} = 1 + 3\cot x$.

3. (a) *Use the definition of the derivative* to find $f'(x)$ if $f(x) = 1/\sqrt{x}$.

   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = 1/\sqrt{x}$ at $x = 1$, at $x = 4$, and at $x = 9$.

   Solution:

   (a)

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right] \\
&= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \\
&= \lim_{h\to 0} \frac{x - (x+h)}{h\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} \\
&= -\lim_{h\to 0} \frac{1}{\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} = -\frac{1}{2x\sqrt{x}}.
\end{aligned}
$$

(b) $f'(1) = -1/2$, so the equation of the tangent line at $x = 1$ is $y = 1-(x-1)/2$. $f'(4) = -1/16$, so the equation of the tangent line at $x = 4$ is $y = (1/2) - (1/16)(x-4)$. $f'(9) = -1/54$; the equation of the tangent line at $x = 9$ is $y = (1/3)-(1/54)(x-9)$.

4. Let f be the function given by

$$f(x) = \begin{cases} x^2 + 2x, & x \le 2 \\ ax^2 + b, & x > 2. \end{cases}$$

(a) What condition must the constants a and b satisfy if f is to be a continuous function?

(b) Find all pairs of values for a and b which make the function f a differentiable function.

Solution:

(a) If f is to be continuous, we must have $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$. But

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} ax^2 + b = 4a + b,$$

while

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 + 2x = 8.$$

If f is to be continuous, the constants a and b must therefore satisfy the equation $4a + b = 8$.

(b) If f is to be differentiable, f must be continuous at $x = 2$ and the derivatives from the left and from the right must match at $x = 2$. Thus, we must have

$$\begin{aligned} f'(2) &= \lim_{h\to 0^+} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0^+} \frac{(2+h)^2 + 2(2+h) - 8}{h} \\ &= \lim_{h\to 0^+} \frac{4 + 4h + h^2 + 4 + 2h - 8}{h} \\ &= \lim_{h\to 0^+} (6 + h) = 6, \end{aligned}$$

and

$$\begin{aligned} f'(2) &= \lim_{h\to 0^-} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0^-} \frac{[a(2+h)^2 + b] - (4a + b)}{h} \\ &= \lim_{h\to 0^-} (4a + ah) = 4a \end{aligned}$$

together with the equation we derived in part (a): $4a + b = 8$. Thus, $4a = 6$ and $4a + b = 8$, so that $a = 3/2$ and $b = 2$.

5. Let g be the function given by $g(x) = x^3 + x$. Then g is a one-to-one function from $(-\infty, \infty)$ onto $(-\infty, \infty)$, and so g is invertible. Let $G = g^{-1}$. Find an equation for the line tangent to the curve $y = G(x)$ at the point on the curve where $x = 10$.

   Solution: We note that $G(10) = 2$ because $g(2) = 2^3 + 2 = 10$. Thus,

$$G'(10) = \frac{1}{g'[G(10)]} = \frac{1}{g'(2)}.$$

   But $g'(x) = 3x^2 + 1$, so $g'(2) = 13$ and it follows that $G'(10) = 1/13$. The equation for the required tangent line is therefore

$$y = 2 + \frac{1}{13}(x - 10).$$

   Alternate Solution: We know that $y = G(x)$ iff $x = g(y) = y^3 + y$. Differentiating this latter equation implicitly with respect to x gives $1 = 3y^2y' + y'$, whence $y' = 1/(3y^2 + 1)$. But, as above, $G(10) = 2$, so $y'(10) = 1/13$.

6. A man started walking north at 4 feet per second from a point P. Five minutes later, a woman started walking south at 5 feet per second from a point 500 feet due east of P. At what rate were the people moving apart 15 minutes after the woman started walking.

   Solution: Let x denote the distance the man has walked north of the point P, and let y denote the distance the woman has walked south of her starting point. Let D be the distance between the two. By the Pythagorean Theorem, $D^2 = (x+y)^2 + 500^2 = (x+y)^2 + 250000$. We differentiate this latter equation implicitly with respect to time to obtain:

$$2D\frac{dD}{dt} = 2(x + y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right), \text{ or}$$

$$\frac{dD}{dt} = \frac{1}{D}(x + y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right).$$

   We are given that $dx/dt = 4$ and $dy/dt = 5$, and this means that

$$\frac{dD}{dt} = \frac{9(x + y)}{D}.$$

   At the critical instant, the woman has been walking for 15 minutes, or 900 seconds, so $y = 4500$. At that instant, the man has been walking for 20 minutes, so $x = 4800$. Thus, $D = \sqrt{(4800 + 4500)^2 + 250000} = 100\sqrt{8674}$, and this means that

$$\frac{dD}{dt} = \frac{9(4800 + 4500)}{100\sqrt{8674}} = \frac{837}{\sqrt{8674}} \text{ ft/sec.}$$

7. (a) Find an equation for the line tangent to the curve $x^3 - 4x^2y + 2xy^3 + 4 = 0$ at the point whose coordinates are $(2, 1)$.

(b) Use the result of part (a) of this problem to find an approximate value for the y-coordinate of the point $(2.04, y)$ that lies on the curve $x^3 - 4x^2y + 2xy^3 + 4 = 0$ near the point $(2, 1)$.

Solution:

(a) Differentiating the equation $x^3 - 4x^2y + 2xy^3 + 4 = 0$ implicitly with respect to x, we find that

$$3x^2 - 8xy - 4x^2y' + 2y^3 + 6xy^2y' = 0;$$

$$y' = \frac{8xy - 3x^2 - 2y^3}{6xy^2 - 4x^2};$$

$$y'\bigg|_{(2,1)} = -\frac{1}{2},$$

and the equation of the required tangent line is $y = 1 - \frac{1}{2}(x - 2)$.

(b) The equation $x^3 - 4x^2y + 2xy^3 + 4 = 0$ defines y implicitly as a function φ of x near the point $(2, 1)$; we have just computed the linearization of φ at $x = 2$. When x is near 2, we may estimate $\varphi(x)$ by using the linearization in its place. Consequently,

$$\varphi(2.04) \sim 1 - \frac{1}{2}(2.04 - 2) = 1 - 0.02 = 0.98.$$

MTH 1410, Calculus I—Exam III **November 12, 2008**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 4:50 pm.

1. Find the limits:

   (a) $\displaystyle\lim_{x\to -1} \frac{x+x^2}{\ln(2+x)}$

   (b) $\displaystyle\lim_{x\to 0} \frac{xe^{-2x}}{\pi e^{2x}-\pi}$

2. Find the absolute maximum and the absolute minimum for the function

$$f(x) = 2x^3 - 3x^2 - 12x + 20$$

   on the interval $[-3, 3]$.

3. Let F be the function given by

$$F(x) = (x-1)^2(x+1)^3.$$

   Then, in fully factored form,

$$F'(x) = (x-1)(x+1)^2(5x-1)$$

   and, also in fully factored form,

$$F''(x) = 20(x+1)\left[x - \frac{1}{5}\left(1-\sqrt{6}\right)\right]\left[x - \frac{1}{5}\left(1+\sqrt{6}\right)\right].$$

   Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

4. Let f be the function given by

$$f(x) = \begin{cases} x^2 + 2x, & x \le 2 \\ ax^2 + b, & x > 2. \end{cases}$$

   (a) What condition must the constants a and b satisfy if f is to be a continuous function?

   (b) Find all pairs of values for a and b which make the function f a differentiable function.

5. Find the points on the ellipse $x^2 + 4y^2 = 4$ whose distance from the point $(1, 0)$ is minimal.

6. Murgatroyd was driving his car toward an intersection at 60 miles per hour. A police cruiser was approaching the same intersection but on the cross-street (which is at right angles to the road that Muratroyd is on), at 50 miles per hour. When both cars were a quarter of a mile from the intersection, a police officer in the cruiser pointed a radar gun at Murgatroyd and measured the speed at which the two cars were approaching each other. What did she get?

MTH 1410, Calculus I—Exam III **November 12, 2008**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 4:50 pm.

1. Find the limits:

   (a) $\lim_{x\to -1} \dfrac{x+x^2}{\ln(2+x)}$

   (b) $\lim_{x\to 0} \dfrac{xe^{-2x}}{\pi e^{2x}-\pi}$

   Solution:

   (a) $\lim_{x\to -1}(x+x^2) = 0 = \lim_{x\to -1}\ln(2+x)$, so we may attempt l'Hôpital's rule:

$$\begin{aligned}\lim_{x\to -1}\frac{x+x^2}{\ln(2+x)} &= \lim_{x\to -1}\frac{1+2x}{[1/(2+x)]}\\ &= \lim_{x\to -1}(1+2x)(2+x)\\ &= -1.\end{aligned}$$

   (b) $\lim_{x\to 0} xe^{-2x} = 0 = \lim_{x\to 0}(\pi e^{2x}-\pi)$, so we can attempt l'Hôpital's rule again:

$$\begin{aligned}\lim_{x\to 0}\frac{xe^{-2x}}{\pi e^{2x}-\pi} &= \lim_{x\to 0}\frac{e^{-2x}-2xe^{-2x}}{2\pi e^{2x}}\\ &= \frac{1}{2\pi}.\end{aligned}$$

2. Find the absolute maximum and the absolute minimum for the function

$$f(x) = 2x^3 - 3x^2 - 12x + 20$$

   on the interval $[-3, 3]$.

   Solution: Absolute extrema are to be found only at endpoints and critical numbers. We have $f'(x) = 6x^2 - 6x - 12 = 6(x+1)(x-2)$, which is defined everywhere and is zero only when $x = -1$ or $x = 2$. Thus, the extrema are among the numbers $f(-3)$, $f(-1)$, $f(2)$, and $f(3)$. We find that $f(-3) = -25$, $f(-1) = 27$, $f(2) = 0$, and $f(3) = 11$. The absolute minimum is $f(-3) = -25$, and the absolute maximum is $f(-1) = 27$.

3. Let F be the function given by

$$F(x) = (x-1)^2(x+1)^3.$$

   Then, in fully factored form,

$$F'(x) = (x-1)(x+1)^2(5x-1)$$

and, also in fully factored form,

$$F''(x) = 20(x+1)\left[x - \frac{1}{5}\left(1-\sqrt{6}\right)\right]\left[x - \frac{1}{5}\left(1+\sqrt{6}\right)\right].$$

Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

Solution: The quantity $(x-1)$ is positive when $x > 1$ and negative when $x < 1$; $(x+1)^2$ is positive unless $x = -1$; and $(5x-1)$ is positive when $x > 1/5$, negative when $x < 1/5$. Thus, $F'(x) > 0$ when $-\infty < x < -1$, when $-1 < x < 1/5$ and when $1 < x < \infty$. It follows that F is increasing on the intervals $(-\infty, 1/5]$ and $[1, \infty)$, but decreasing on $[1/5, 1]$.

The quantity $(x+1)$ is negative when $x < -1$ and positive when $x > -1$; $[x - (1-\sqrt{6})/5]$ is negative when $x < (1-\sqrt{6})/5$ and positive when $x > (1-\sqrt{6})/5$; $[x-(1+\sqrt{6})/5]$ is negative when $x < (1+\sqrt{6})/5$ and positive when $x > (1+\sqrt{6})/5$. Consequently $F''(x) < 0$ when $-\infty < x < -1$ and when $(1-\sqrt{6})/5 < x < (1+\sqrt{6})/5$, but $F''(x) > 0$ when $-1 < x < (1-\sqrt{6})/5$ and when $(1+\sqrt{6})/5 < x < \infty$. So F is concave upward on $[-1, (1-\sqrt{6})/5]$ and on $[(1+\sqrt{6})/5, \infty)$, but concave downward on $(-\infty, -1]$ and on $[(1-\sqrt{6})/5, (1+\sqrt{6})/5]$.

From the expression for $F'(x)$, we see that the critical numbers for F are $x = 1$, $x = -1$, and $x = 1/5$. Our analysis of the increasing/decreasing behavior of F above, shows that $x = 1$ gives a local minimum for F because F is decreasing just to the left of $x = 1$ but increasing just to the right of $x = 1$. Similarly, $x = 1/5$ gives a local maximum for F, and $x = -1$ gives neither a local maximum nor a local minimum.

4. Let f be the function given by

$$f(x) = \begin{cases} x^2 + 2x, & x \le 2 \\ ax^2 + b, & x > 2. \end{cases}$$

(a) What condition must the constants a and b satisfy if f is to be a continuous function?

(b) Find all pairs of values for a and b which make the function f a differentiable function.

Solution:

(a) If f is to be continuous, we must have $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$. But

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} ax^2 + b = 4a + b,$$

while

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 + 2x = 8.$$

If f is to be continuous, the constants a and b must therefore satisfy the equation $4a+b = 8$.

(b) If f is to be differentiable, f must be continuous at $x = 2$ and the derivatives from the left and from the right must match at $x = 2$. Thus, we must have

$$\begin{aligned} f'(2) &= \lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0^+} \frac{(2+h)^2 + 2(2+h) - 8}{h} \\ &= \lim_{h \to 0^+} \frac{4 + 4h + h^2 + 4 + 2h - 8}{h} \\ &= \lim_{h \to 0^+} (6 + h) = 6, \end{aligned}$$

and

$$\begin{aligned} f'(2) &= \lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0^-} \frac{[a(2+h)^2 + b] - (4a + b)}{h} \\ &= \lim_{h \to 0^-} (4a + ah) = 4a \end{aligned}$$

together with the equation we derived in part (a): $4a + b = 8$. Thus, $4a = 6$ and $4a + b = 8$, so that $a = 3/2$ and $b = 2$.

5. Find the points on the ellipse $x^2 + 4y^2 = 4$ whose distance from the point $(1, 0)$ is minimal.

Solution: Let (x, y) be any point on the curve. Then $x^2 + 4y^2 = 4$ and the square S of the distance from (x, y) to $(1, 0)$ is $S = (x - 1)^2 + y^2$. We can minimize distance by minimizing S. From $x^2 + 4y^2 = 4$, we see that $y^2 = 1 - x^2/4$, where $|x| \leq 2$. Thus

$$\begin{aligned} S(x) &= (x - 1)^2 + y^2 \\ &= (x^2 - 2x + 1) + \left(1 - \frac{x^2}{4}\right) \\ &= x^2 - 2x + 1 + 1 - \frac{x^2}{4} \\ &= \frac{3}{4}x^2 - 2x + 2, \end{aligned}$$

so that

$$\frac{dS}{dx} = \frac{3}{2}x - 2.$$

The only critical number for $S(x)$ is thus at $x = 4/3$. The minimum for $S(x)$ must occur either at $x = 4/3$ or at an endpoint $x = \pm 2$. We have $S(-2) = 9$, $S(4/3) = 2/3$, and $S(2) = 1$. The minimal distance therefore occurs when $x = 4/3$ and $y = \pm\sqrt{5}/3$—that is, at the points $(4/3, \sqrt{5}/3)$ and $(4/3, -\sqrt{5}/3)$.

6. Murgatroyd was driving his car toward an intersection at 60 miles per hour. A police cruiser was approaching the same intersection but on the cross-street (which is at right angles to the road that Muratroyd is on), at 50 miles per hour. When both cars were a quarter of a mile from the intersection, a police officer in the cruiser pointed a radar gun at Murgatroyd and measured the speed at which the two cars were approaching each other. What did she get?

Solution: Let x denote the distance from Murgatroyd to the intersection, and let y denote the distance from the police car to the interesection. The distance D between the two cars satisfies

$$D^2 = x^2 + y^2,$$

so

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt},$$

or

$$\frac{dD}{dt} = \frac{1}{D}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

At the critical instant, we have $x = y = 1/4$, $D = 1/(2\sqrt{2})$, $dx/dt = -60$, and $dy/dt = -50$. Thus, rate of change of the distance between the two cars was

$$\begin{aligned}\frac{dD}{dt} &= 2\sqrt{2}\left[\frac{1}{4}(-60) + \frac{1}{4}(-50)\right] \\ &= -\frac{110}{\sqrt{2}} \text{ mph.}\end{aligned}$$

The reading on the radar gun was $110/\sqrt{2}$ mph, or about 78 mph.

MTH 1410, Calculus I—Final Exam **December 8, 2008**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 4:50 pm.

1. (a) *Use the definition of the derivative* to find $f'(x)$ if $f(x) = 1/\sqrt{x}$.

   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = 1/\sqrt{x}$ at $x = 1$, at $x = 4$, and at $x = 9$.

2. Evaluate the following definite integrals. Give all of your reasoning.

   (a) $\displaystyle\int_3^5 (3x^2 - 24x + 54)\,dx$

   (b) $\displaystyle\int_0^3 3t\sqrt{9 - t^2}\,dt$

3. Let F be the function given by

$$F(x) = (x-1)^2(x+1)^3.$$

   Then, in fully factored form,

$$F'(x) = (x-1)(x+1)^2(5x-1)$$

   and, also in fully factored form,

$$F''(x) = 20(x+1)\left[x - \frac{1}{5}\left(1 - \sqrt{6}\right)\right]\left[x - \frac{1}{5}\left(1 + \sqrt{6}\right)\right].$$

   Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

4. Suppose that $f(2) = 2$, $f(4) = 4$, $f'(2) = 4$, $f'(4) = -2$, $g(2) = 4$, $g(4) = 2$, $g'(2) = -6$, and $g'(4) = -8$.

   (a) Find $F(4)$ and $F'(4)$, where $F(x) = \dfrac{f(x)}{g(x)}$.

   (b) Find $G(2)$ and $G'(2)$, where $G(x) = g[2f(x)]$.

   (c) Find $H(2)$ and $H'(2)$, where $H(x) = g[f(x^2)]$.

5. (a) Show that the point $(3, 2)$ lies on the curve given by the equation

$$x^3 - 5x^2y^3 + 8y^4 + 205 = 0.$$

   (b) If x and y are related by the equation, $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$, find the value of y' at $(3, 2)$.

   (c) Show how to use the results of parts (a) and (b) of this problem to find an approximate value for y near 2 when $x = 74/25$.

6. Find the points on the ellipse $4x^2 + y^2 = 4$ whose distance from the point $(1, 0)$ is maximal.

MTH 1410, Calculus I—Final Exam **December 8, 2008**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 4:50 pm.

1. (a) *Use the definition of the derivative* to find $f'(x)$ if $f(x) = 1/\sqrt{x}$.

   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = 1/\sqrt{x}$ at $x = 1$, at $x = 4$, and at $x = 9$.

   Solution:

   (a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right] \\ &= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \\ &= \lim_{h\to 0} \frac{x - (x+h)}{h\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} \\ &= -\lim_{h\to 0} \frac{1}{\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} = -\frac{1}{2x\sqrt{x}}. \end{aligned}$$

   (b) From the immediately preceding equation, $f'(1) = -1/2$, so the equation of the tangent line at $x = 1$ is $y = 1 - (x-1)/2$, or $x + 2y = 3$. $f'(4) = -1/16$, so the equation of the tangent line at $x = 4$ is $y = (1/2) - (1/16)(x-4)$, or $x + 16y = 12$. $f'(9) = -1/54$; the equation of the tangent line at $x = 9$ is $y = (1/3) - (1/54)(x-9)$, or $x + 54y = 27$.

2. Evaluate the following definite integrals. Give all of your reasoning.

   (a) $\displaystyle\int_3^5 (3x^2 - 24x + 54)\,dx$

   (b) $\displaystyle\int_0^3 3t\sqrt{9-t^2}\,dt$

   Solution:

   (a)

$$\begin{aligned} \int_3^5 (3x^2 - 24x + 54)\,dx &= (x^3 - 12x^2 + 54x)\Big|_3^5 \\ &= (125 - 300 + 270) - (27 - 108 + 162) = 14. \end{aligned}$$

   (b) Let $u = 9 - t^2$. Then $du = -2t\,dt$, or $t\,dt = -(1/2)\,du$. Moreover, $t = 0 \Rightarrow u = 9$ and $t = 3 \Rightarrow u = 0$. Thus

$$\begin{aligned} \int_0^3 3t\sqrt{9-t^2}\,dt &= -\frac{3}{2}\int_9^0 u^{1/2}\,du = \frac{3}{2}\int_0^9 u^{1/2}\,du \\ &= u^{3/2}\Big|_0^9 = 9^{3/2} - 0^{3/2} = 27. \end{aligned}$$

3. Let F be the function given by

$$F(x) = (x-1)^2(x+1)^3.$$

Then, in fully factored form,

$$F'(x) = (x-1)(x+1)^2(5x-1)$$

and, also in fully factored form,

$$F''(x) = 20(x+1)\left[x - \frac{1}{5}\left(1-\sqrt{6}\right)\right]\left[x - \frac{1}{5}\left(1+\sqrt{6}\right)\right].$$

Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

Solution: The quantity $(x-1)$ is positive when $x > 1$ and negative when $x < 1$; $(x+1)^2$ is positive unless $x = -1$; and $(5x-1)$ is positive when $x > 1/5$, negative when $x < 1/5$. Thus, $F'(x) > 0$ when $-\infty < x < -1$, when $-1 < x < 1/5$ and when $1 < x < \infty$. It follows that F is increasing on the intervals $(-\infty, 1/5]$ and $[1, \infty)$, but decreasing on $[1/5, 1]$.

The quantity $(x+1)$ is negative when $x < -1$ and positive when $x > -1$; $[x-(1-\sqrt{6})/5]$ is negative when $x < (1-\sqrt{6})/5$ and positive when $x > (1-\sqrt{6})/5$; $[x-(1+\sqrt{6})/5]$ is negative when $x < (1+\sqrt{6})/5$ and positive when $x > (1+\sqrt{6})/5$. Consequently $F''(x) < 0$ when $-\infty < x < -1$ and when $(1-\sqrt{6})/5 < x < (1+\sqrt{6})/5$, but $F''(x) > 0$ when $-1 < x < (1-\sqrt{6})/5$ and when $(1+\sqrt{6})/5 < x < \infty$. So F is concave upward on $[-1, (1-\sqrt{6})/5]$ and on $[(1+\sqrt{6})/5, \infty)$, but concave downward on $(-\infty, -1]$ and on $[(1-\sqrt{6})/5, (1+\sqrt{6})/5]$.

From the expression for $F'(x)$, we see that the critical numbers for F are $x = 1$, $x = -1$, and $x = 1/5$. Our analysis of the increasing/decreasing behavior of F above, shows that $x = 1$ gives a local minimum for F because F is decreasing just to the left of $x = 1$ but increasing just to the right of $x = 1$. Similarly, $x = 1/5$ gives a local maximum for F, and $x = -1$ gives neither a local maximum nor a local minimum.

4. Suppose that $f(2) = 2$, $f(4) = 4$, $f'(2) = 4$, $f'(4) = -2$, $g(2) = 4$, $g(4) = 2$, $g'(2) = -6$, and $g'(4) = -8$.

   (a) Find $F(4)$ and $F'(4)$, where $F(x) = \dfrac{f(x)}{g(x)}$.

   (b) Find $G(2)$ and $G'(2)$, where $G(x) = g[2f(x)]$.

   (c) Find $H(2)$ and $H'(2)$, where $H(x) = g[f(x^2)]$.

Solution:

(a)

$$F(4) = \frac{f(4)}{g(4)} = 2;$$
$$F'(4) = \frac{f'(4)g(4) - f(4)g'(4)}{[g(4)]^2} = 7.$$

(b)

$$G(2) = g[2f(2)] = 2;$$
$$G'(x) = g'[2f(x)]D_x[2f(x)] = 2g'[2f(x)]f'(x),$$

so

$$G'(2) = 2g'[2f(2)]f'(2) = -64.$$

(c)

$$H(2) = g[f(2)] = 2;$$
$$H'(x) = g'[f(x^2)] \cdot D_x f(x^2) = g'[f(x^2)]f'(x^2)D_x x^2 = 2xg'[f(x^2)]f'(x^2),$$

so

$$H'(2) = 4g'[f(4)]f'(4) = 64.$$

5. (a) Show that the point $(3, 2)$ lies on the curve given by the equation

$$x^3 - 5x^2y^3 + 8y^4 + 205 = 0.$$

(b) If x and y are related by the equation, $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$, find the value of y' at $(3, 2)$.

(c) Show how to use the results of parts (a) and (b) of this problem to find an approximate value for y near 2 when $x = 74/25$.

Solution:

(a) When $x = 3$ and $y = 2$, we have

$$3^3 - 5 \cdot 3^2 \cdot 2^3 + 8 \cdot 2^4 + 205 = 27 - 360 + 128 + 205 = 0,$$

so the point with coordinates $(3, 2)$ lies on the curve whose equation is $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$.

(b) Treating y as a function of x and differentiating implicitly gives

$$3x^2 - 10xy^3 - 15x^2y^2y' + 32y^3y' = 0;$$
$$(32y^3 - 15x^2y^2)y' = 10xy^3 - 3x^2;$$
$$y' = \frac{10xy^3 - 3x^2}{32y^3 - 15x^2y^2}.$$

Thus,

$$y'\Big|_{(3,2)} = \frac{10 \cdot 3 \cdot 2^3 - 3 \cdot 3^2}{32 \cdot 2^3 - 15 \cdot 3^2 \cdot 2^2} = \frac{240 - 27}{256 - 540} = \frac{213}{-284} = -\frac{3}{4}.$$

(c) From the previous part of this problem, we know that the equation of the line tangent to the curve at $(3, 2)$ is

$$y = 2 - \frac{3}{4}(x - 3).$$

When a point (x_0, y_0) lies near $(3, 2)$ on the curve $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$, it lies near the line tangent to the curve at $(3, 2)$. Thus, we can approximate the value of y near 2 that satisfies the equation

$$\left(\frac{74}{25}\right)^3 - 5\left(\frac{74}{25}\right)^2 y^3 + 8y^4 + 205 = 0$$

as

$$y \sim 2 - \frac{3}{4}\left(\frac{74}{25} - 3\right) = 2 + \frac{3}{4} \cdot \frac{1}{25} = \frac{203}{100}.$$

6. Find all points on the ellipse $4x^2 + y^2 = 4$ whose distance from the point $(1, 0)$ is maximal.

Solution: The distance D from a point (x, y) to the point $(1, 0)$ is given by

$$D^2 = (x - 1)^2 + y^2.$$

If the point (x, y) lies on the given ellipse, then $4x^2 + y^2 = 4$, so that $y^2 = 4 - 4x^2$. Substituting this latter equation into the relation for D, we find that

$$D^2 = (x-1)^2 + 4 - 4x^2 = 5 - 2x - 3x^2.$$

Because $y^2 = 4 - 4x^2$ must not be negative, we are interested only in those values of x for which $-1 \leq x \leq 1$. We have

$$2D\frac{dD}{dx} = -2 - 6x.$$

Thus, $dD/dx = 0$ when $x = -1/3$. We must check the value of D when $x = -1$, when $x = -1/3$, and when $x = 1$. The corresponding values of D^2 are 4, $16/3$, and 0. Thus, D takes on the maximal value $4/\sqrt{3}$ when $x = -1/3$. The corresponding points on the ellipse are $(-1/3, 4\sqrt{2}/9)$ and $(-1/3, -4\sqrt{2}/9)$.

MTH 1410, Calculus I—Exam I **Feb. 18, 2009**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Your paper is due at 2:50 pm.

1. Evaluate the following limits. Use the Limit Laws. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $$\lim_{x\to -3} \frac{x^2+x-6}{x^2+4x+3}$$

   (b) $$\lim_{x\to\infty} \frac{(5-x)(10+8x)}{(3-3x)(3+10x)}$$

2. A rock is thrown off of a 100-meter cliff with an upward velocity of 50 m/s. As a result, its height (above the ground at the base of the cliff) after t seconds is given by the formula $h(t) = 100 + 50t - 5t^2$.

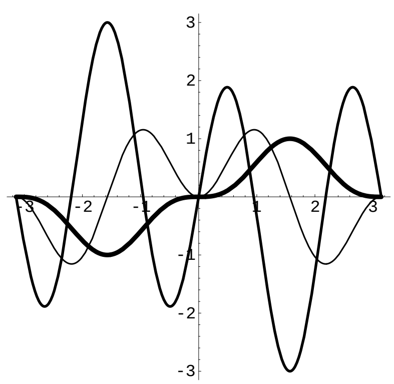
   (a) What is its height after 7 seconds?

   (b) What is its average velocity over the first seven seconds?

   (c) What is its velocity after 7 seconds?

   (d) What is its velocity when it hits the ground at the base of the cliff?

3. Here is a graph showing three functions—a skinny one, a middle-weight one, and a fat one—on the same pair of axes:



   One of the curves is f, one is f', and one is f''. Explain which is which and how you know.

4. Find $f'(x)$ if

   (a) $f(x) = 3x^2 - 4x + 5$

   (b) $f(x) = \dfrac{x^2 - 2\sqrt{x}}{x^{1/3}}$

5. Find $f'(x)$ if

   (a) $f(x) = (3x^2 - x + 1)^2(5x + 4)^{12}$

   (b) $f(x) = \dfrac{a}{x^{10}} + \sin^3 bx$, where a and b are fixed but unspecified constants.

6. A function f is given by

$$f(x) = \begin{cases} 2cx + 2 & ; \quad x \leq 3 \\ 3 - cx & ; \quad 3 < x. \end{cases}$$

   For what values of the constant c is f continuous on $(-\infty, \infty)$? Be sure to give your reasoning.

7. (a) Use the definition of the derivative to find $f'(x)$ if $f(x) = \sqrt{2x}$.

   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = \sqrt{2x}$ at $x = 1$, at $x = 2$, and at $x = 8$.

8. Let F be the function given by

$$F(x) = f(x)g(x),$$

   where f and g are functions for which $f'(x)$ and $g'(x)$ are both defined for all real values of x.

   (a) What is $F'(x)$?

   (b) Derive the formula you have given in part 8a.

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 1410, Calculus I—Exam I **Feb. 18, 2009**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Your paper is due at 2:50 pm.

1. Evaluate the following limits. Use the Limit Laws. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a)

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x^2 + 4x + 3}$$

   (b)

$$\lim_{x \to \infty} \frac{(5 - x)(10 + 8x)}{(3 - 3x)(3 + 10x)}$$

   Solution:

   (a)

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x^2 + 4x + 3} = \lim_{x \to -3} \frac{\cancel{(x + 3)}(x - 2)}{\cancel{(x + 3)}(x + 1)} = \frac{-5}{-2} = \frac{5}{2}.$$

   (b)

$$\begin{aligned} \lim_{x \to \infty} \frac{(5 - x)(10 + 8x)}{(3 - 3x)(3 + 10x)} &= \lim_{x \to \infty} \frac{[(5 - x)(10 + 8x)]/x^2}{[(3 - 3x)(3 + 10x)]/x^2} \\ &= \lim_{x \to \infty} \frac{(5/x - 1)(10/x + 8)}{(3/x - 3)(3/x + 10)} = \frac{(-1) \cdot (8)}{(-3) \cdot (10)} = \frac{4}{15}. \end{aligned}$$

2. A rock is thrown off of a 100-meter cliff with an upward velocity of 50 m/s. As a result, its height (above the ground at the base of the cliff) after t seconds is given by the formula $h(t) = 100 + 50t - 5t^2$.

   (a) What is its height after 7 seconds?

   (b) What is its average velocity over the first seven seconds?

   (c) What is its velocity after 7 seconds?

   (d) What is its velocity when it hits the ground at the base of the cliff?

   Solution:

   (a) $h(7) = 100 + 50 \cdot 7 - 5 \cdot 7^2 = 205$ meters.

   (b) Average velocity over the first seven seconds is

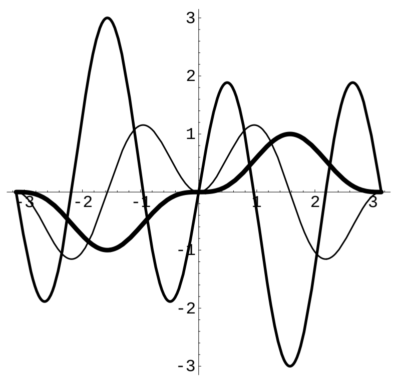
$$\frac{h(7) - h(0)}{7 - 0} = \frac{205 - 100}{7} = 15 \text{ meters per second.}$$

(c) Velocity is the derivative $h'(t) = 50 - 10t$, so when $t = 7$, velocity is $h'(7) = 50 - 10 \cdot 7 = -20$ meters per second.

(d) The rock hits the ground when $t > 0$ satisfies $h(t) = 0$. But the only positive solution of the equation $100 + 50t - 5t^2 = 0$ is $t = 5 + 3\sqrt{5}$. So the rock hits the ground at the base of the cliff with velocity

$$h'(5 + 3\sqrt{5}) = 50 - 10(5 + 3\sqrt{5}) = -30\sqrt{5} \text{ meters per second.}$$

3. Here is a graph showing three functions—a skinny one, a middle-weight one, and a fat one—on the same pair of axes:



One of the curves is f, one is f', and one is f''. Explain which is which and how you know.

Solution: The skinny curve lies above the x-axis exactly where the fat curve is increasing, touches the x-axis exactly where the fat curve has a horizontal tangent, and lies below the x-axis exactly where the fat curve is decreasing. The middle-weight curve lies above the x-axis exactly where the skinny one is increasing, touches the x-axis exactly where the skinny one has a horizontal tangent, and lies below the x-axis exactly where the skinny one is decreasing. Thus, the fat curve is f, the skinny curve is f', and the middle-weight curve is f''.

4. Find $f'(x)$ if

(a) $f(x) = 3x^2 - 4x + 5$

(b) $f(x) = \dfrac{x^2 - 2\sqrt{x}}{x^{1/3}}$

Solution:

(a) $f'(x) = 6x - 4$.

(b) Carrying out the division, we find that $f(x) = x^{2-1/3} - 2x^{1/2-1/3} = x^{5/3} - 2x^{1/6}$. Consequently, $f'(x) = \dfrac{5}{3}x^{2/3} - \dfrac{1}{3}x^{-5/6}$.

5. Find $f'(x)$ if

(a) $f(x) = (3x^2 - x + 1)^2(5x+4)^{12}$

(b) $f(x) = \dfrac{a}{x^{10}} + \sin^3 bx$, where a and b are fixed but unspecified constants.

Solution:

(a) $f'(x) = 2(3x^2 - x + 1)(6x-1)(5x+4)^{12} + 60(3x^2 - x + 1)^2(5x+4)^{11}$

(b) $f(x) = ax^{-10} + [\sin(bx)]^3$, so $f'(x) = -10ax^{-11} + 3b\sin^2(bx)\cos(bx)$.

6. A function f is given by

$$f(x) = \begin{cases} 2cx + 2 & ; \quad x \le 3 \\ 3 - cx & ; \quad 3 < x. \end{cases}$$

For what values of the constant c is f continuous on $(-\infty, \infty)$? Be sure to give your reasoning.

Solution: If $a \neq 3$ the values of the function $f(x)$ are given by a polynomial function in some open interval centered at a, so f is a continuous function everywhere except possibly at $x = 3$. In order for f to be continuous at $x = 3$, we need to have $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) = 6c + 2$. Now $\lim_{x\to 3^-} f(x) = 6c + 2$, but $\lim_{x\to 3^+} = 3 - 3c$. Consequently, f is continuous at $x = 3$ precisely when c is chosen so that $6c + 2 = 3 - 3c$, or when $c = 1/9$.

7. (a) Use the definition of the derivative to find $f'(x)$ if $f(x) = \sqrt{2x}$.

(b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = \sqrt{2x}$ at $x = 1$, at $x = 2$, and at $x = 8$.

Solution:

(a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{2(x+h)} - \sqrt{2x}}{h} \\ &= \lim_{h\to 0} \frac{2(x+h) - 2x}{h\left[\sqrt{2(x+h)} + \sqrt{2x}\right]} \\ &= \lim_{h\to 0} \frac{\cancel{2x} + 2\cancel{h} - \cancel{2x}}{\cancel{h}\left[\sqrt{2(x+h)} + \sqrt{2x}\right]} \\ &= \frac{2}{\sqrt{2x} + \sqrt{2x}} = \frac{1}{\sqrt{2x}}. \end{aligned}$$

(b) i. Tangent line at $x = 1$:

$$y = f(1) + f'(1)(x-1),$$

which is

$$y = \sqrt{2} + \frac{1}{\sqrt{2}}(x-1).$$

ii. Tangent line at $x = 2$:

$$y = f(2) + f'(2)(x-2),$$

which is

$$y = 2 + \frac{1}{2}(x-2).$$

iii. Tangent line at $x = 8$:

$$y = f(8) + f'(8)(x-8),$$

which is

$$y = 4 + \frac{1}{4}(x-8).$$

8. Let F be the function given by

$$F(x) = f(x)g(x),$$

where f and g are functions for which $f'(x)$ and $g'(x)$ are both defined for all real values of x.

(a) What is $F'(x)$?

(b) Derive the formula you have given in part **??**.

Solution:

(a) $F'(x) = f'(x)g(x) + f(x)g'(x)$.

(b)

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} g(x+h)\right] + \lim_{h\to 0} \left[f(x)\frac{g(x+h) - g(x)}{h}\right] \\
&= \left[\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right] \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \left[\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right],
\end{aligned}$$

provided all of the limits in the latter expression exist. But

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x),$$

$$\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = g'(x), \text{ and}$$

$$\lim_{h\to 0} f(x) = f(x).$$

We are given that $g'(x)$ exists for all real x, and this means that the function g is continuous everywhere. Consequently,

$$\lim_{h\to 0} g(x+h) = g(x).$$

It now follows that

$$\begin{aligned} F'(x) &= \left[\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right] \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \left[\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right] \\ &= f'(x)g(x) + f(x)g'(x). \end{aligned}$$

MTH 1410, Calculus I—Exam I **Feb. 18, 2010**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 8:50 pm. Complete solutions to the exam problems will be available from the course web-site later this evening.

1. Show how to evaluate the following limits by using the Limit Laws:

   (a) $\lim_{x\to 4} \dfrac{x^2 - x - 12}{4x^2 - 18x + 8}$

   (b) $\lim_{x\to\infty} \left[\sqrt{x^2 + 8x - 7} - \sqrt{x^2 + x - 3}\right]$

2. Let f be the function given by the equation $f(x) = \sqrt{x+3}$.

   (a) Show how to use the definition of the derivative to find $f'(a)$, where it is given that a is some real number greater than -3.

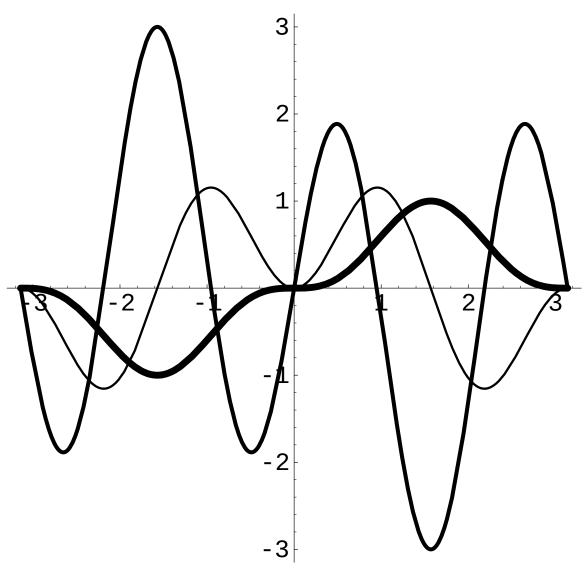
   (b) Use the value you have just found for $f'(a)$ of this problem to write an equation for the line tangent to the curve $y = \sqrt{x+3}$ at the point where $x = 13$.

3. Find the derivatives; you may use any (correct) method you know.

   (a) $f(x) = 3x^5 - 5x^4 + x^3 + 7x^2 - 12x + 4 - \dfrac{8}{x^2}$

   (b) $f(x) = \dfrac{(x-3)\sqrt{x}}{x^2+4}$.

4. Here is a graph showing three functions—a skinny one, a middle-weight one, and a fat one—on the same pair of axes:



   One of the curves is f, one is f', and one is f''. Explain which is which and how you know.

5. Find $f'(x)$, using any means at your disposal, if

   (a) $f(x) = (3x^2 - x + 1)^2(5x+4)^{12}$

   (b) $f(x) = \dfrac{a}{x^{10}} + \sin^3 bx$, where a and b are fixed, but unspecified, constants.

6. Let g be the function given by

$$g(x) = \begin{cases} x^2 - 4; & \text{if } x \leq 3, \\ mx + b; & \text{if } x > 3, \end{cases}$$

where m and b both stand for fixed, but unspecified, constants.

(a) What relationship must m and b satisfy if it is given that the function g is continuous on $(-\infty, \infty)$? Be sure to give your reasoning.

(b) Find all values of m and b for which the function g is differentiable on $(-\infty, \infty)$. Be sure to give your reasoning.

7. A conical tank, whose vertex points downward, is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of water in the tank at the instant when the tank is 8 feet deep.

8. Consider the curve given by the equation $x^2y + xy^2 = 3x + 3$.

(a) Verify that the point $(1, 2)$ lies on this curve.

(b) Write an equation for the line tangent to the curve at $(1, 2)$.

(c) Show how to use the equation you have written for the tangent line to estimate the value y must have if the point $(1.03, y)$ is to lie on the curve near $(1, 2)$.

MTH 1410, Calculus I—Exam I **Feb. 18, 2010**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 8:50 pm. Complete solutions to the exam problems will be available from the course web-site later this evening.

1. Show how to evaluate the following limits by using the Limit Laws:

   (a) $\lim_{x\to 4} \dfrac{x^2 - x - 12}{4x^2 - 18x + 8}$

   (b) $\lim_{x\to\infty} \left[\sqrt{x^2+8x-7} - \sqrt{x^2+x-3}\right]$

   Solution:

   (a)

$$\lim_{x\to 4} \frac{x^2 - x - 12}{4x^2 - 18x + 8} = \lim_{x\to 4} \frac{\cancel{(x-4)}(x+3)}{2\cancel{(x-4)}(2x-1)} = \lim_{x\to 4} \frac{x+3}{2(2x-1)} = \frac{1}{2}.$$

   (b)

$$\begin{aligned}\lim_{x\to\infty} \left[\sqrt{x^2+8x-7} - \sqrt{x^2+x-3}\right] &= \lim_{x\to\infty} \frac{(x^2+8x-7)-(x^2+x-3)}{\sqrt{x^2+8x-7}+\sqrt{x^2+x-3}} \\ &= \lim_{x\to\infty} \frac{7x-4}{\sqrt{x^2+8x-7}+\sqrt{x^2+x-3}} \\ &= \lim_{x\to\infty} \frac{7-(4/x)}{\sqrt{1+(8/x)-(7/x^2)}+\sqrt{1+(1/x)-(3/x^2)}} = \frac{7}{2}.\end{aligned}$$

2. Let f be the function given by the equation $f(x) = \sqrt{x+3}$.

   (a) Show how to use the definition of the derivative to find $f'(a)$, where it is given that a is some real number greater than -3.

   (b) Use the value you have just found for $f'(a)$ of this problem to write an equation for the line tangent to the curve $y = \sqrt{x+3}$ at the point where $x = 13$.

   Solution:

   (a)

$$\begin{aligned}f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\sqrt{(a+h)+3}-\sqrt{a+3}}{h} \\ &= \lim_{h\to 0} \frac{[(\cancel{a}+h)+\cancel{3}]-[\cancel{a}+\cancel{3}]}{\cancel{h}[\sqrt{(a+h)+3}+\sqrt{a+3}]} = \lim_{h\to 0} \frac{1}{\sqrt{(a+h)+3}+\sqrt{a+3}} = \frac{1}{2\sqrt{a+3}}.\end{aligned}$$

   (b) We have $f(13) = \sqrt{13+3} = 4$. By what we have seen above, $f'(13) = 1/(2\sqrt{13+3}) = 1/8$. Hence, the required equation is $y = 4 + (x-13)/8$.

3. Find the derivatives; you may use any (correct) method you know.

   (a) $f(x) = 3x^5 - 5x^4 + x^3 + 7x^2 - 12x + 4 - \dfrac{8}{x^2}$

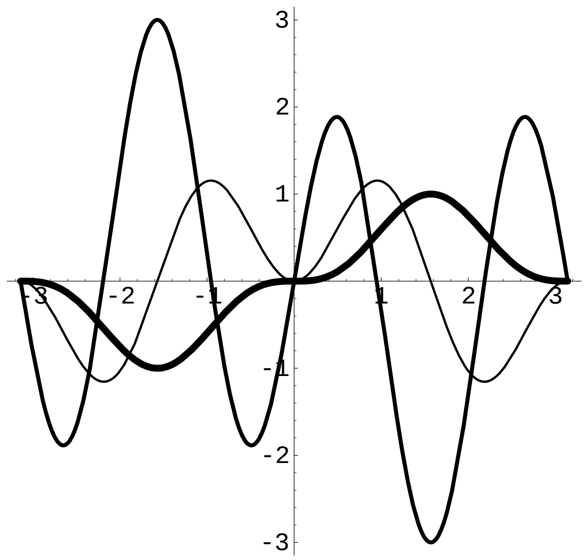
   (b) $f(x) = \dfrac{(x-3)\sqrt{x}}{x^2+4}$.

Solution:

(a) $f'(x) = 15x^4 - 20x^3 + 3x^2 + 14x - 12 + 16x^{-3}$.

(b) $f'(x) = \dfrac{[\sqrt{x} + (x-3)x^{-1/2}/2](x^2+4) - 2x(x-3)\sqrt{x}}{(x^2+4)^2}$.

4. Here is a graph showing three functions—a skinny one, a middle-weight one, and a fat one—on the same pair of axes:



One of the curves is f, one is f', and one is f''. Explain which is which and how you know.

Solution: The skinny curve, and only the skinny curve, touches the x-axis at precisely the places where the fat curve has a horizontal tangent. It also lies above the x-axis where the fat curve is increasing, and it lies below the x-axis where the fat curve is decreasing—so the skinny curve is the derivative of the fat curve. The middle-weight curve, and only the middle-weight curve, touches the x-axis at precisely the places where the skinny curve has a horizontal tangent. The middle-weight curve also lies above the x-axis where the skinny one is increasing, and it lies below the x-axis where the skinny curve is decreasing—so the middle-weight curve is the derivative of the skinny curve. Hence the fat curve is the graph of f, the skinny curve is the graph of f', and the middle-weight curve is the graph of f''.

5. Find $f'(x)$, using any means at your disposal, if

(a) $f(x) = (3x^2 - x + 1)^2(5x+4)^{12}$

(b) $f(x) = \dfrac{a}{x^{10}} + \sin^3 bx$, where a and b are fixed, but unspecified, constants.

Solution:

(a) $f'(x) = 2(3x^2 - x + 1)(6x-1)(5x+4)^{12} + 60(3x^2 - x + 1)^2(5x+4)^{11}$.

(b) $f'(x) = -10ax^{-11} + 3b\sin^2 bx \cos bx$.

6. Let g be the function given by

$$g(x) = \begin{cases} x^2 - 4; & \text{if } x \le 3, \\ mx + b; & \text{if } x > 3, \end{cases}$$

where m and b both stand for fixed, but unspecified, constants.

(a) What relationship must m and b satisfy if it is given that the function g is continuous on $(-\infty, \infty)$? Be sure to give your reasoning.

(b) Find all values of m and b for which the function g is differentiable on $(-\infty, \infty)$. Be sure to give your reasoning.

Solution:

(a) If g is continuous everywhere, it must be continuous at $x = 3$, so $\lim_{x\to 3} g(x)$ must exist and be equal to $g(3) = 5$. Thus, we will need to have $5 = \lim_{x\to 3^+}(mx + b) = 3m + b$. We conclude that if g is continuous on $(-\infty, \infty)$, then $3m + b = 5$.

(b) If g is to be differentiable everywhere, then $g'(3)$ must exist. But we must have

$$g'(5) = \lim_{h\to 3^-} \frac{g(3+h) - g(3)}{h} = \lim_{h\to 3^-} \frac{[(3+h)^2 - 4] - 5}{h}$$
$$= \lim_{h\to 3^-} \frac{6h + h^2}{h} = \lim_{h\to 3^-} (6 + h) = 6$$

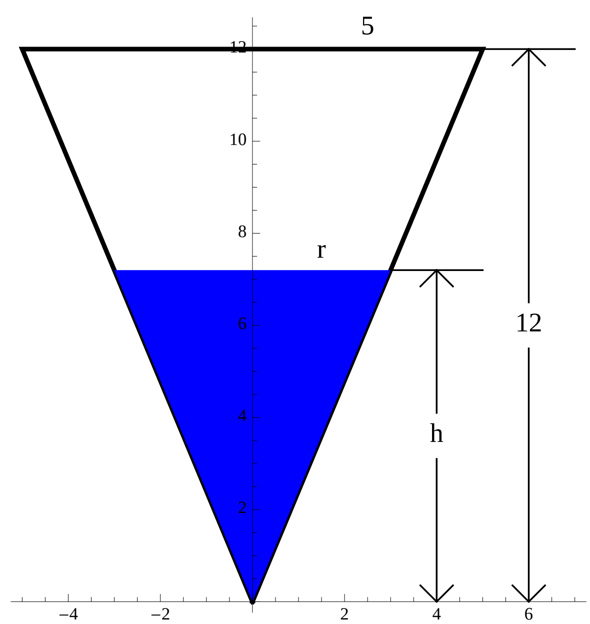
on the one hand, while we must also have

$$g'(3) = \lim_{h\to 3^+} \frac{g(3+h) - g(3)}{h} = \lim_{h\to 3^+} \frac{[m(3+h) + b] - [m3 + b]}{h}$$
$$= \lim_{h\to 3^+} \frac{mh}{h} = \lim_{h\to 3^+} m = m$$

on the other hand. Thus, $g'(3)$ can exist only if $m = 6$. Because g must be continuous at $x = 3$ if $g'(3)$ exists, we will also need [by part (a), above] to have $3m + b = 5$, and from $m = 6$ it then follows that we must also have $b = -13$. We conclude that g is differentiable everywhere only when $m = 6$ and $b = -13$.

7. A conical tank, whose vertex points downward, is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of water in the tank at the instant when the tank is 8 feet deep.

Solution: Here is a figure, showing the tank partially full:



By similar triangles, $12r = 5h$. When $h = 8$, this gives $r = 10/3$. Differentiating the relation $12r = 5h$ with respect to t gives us $12\dot{r} = 5\dot{h}$. On the other hand, the volume V of water in the tank is given by $V = \pi r^2 h/3$, from which we see that $\dot{V} = 2\pi rh\dot{r}/3 + \pi r^2\dot{h}/3$, or, making use of the relation $12\dot{r} = 5\dot{h}$, that $\dot{V} = [5\pi rh/18 + \pi r^2/3]\dot{h}$. Finally, we make use of the information pertaining to the critical instant, $h = 8$, $r = 10/3$, $\dot{V} = 10$, to find that

$$\dot{V} = \pi\left(\frac{5}{18}rh + \frac{1}{3}r^2\right)\dot{h};$$

$$10 = \pi\left[\frac{5}{18}\cdot\frac{10}{3}\cdot 8 + \frac{1}{3}\left(\frac{10}{3}\right)^2\right]\cdot\dot{h} = \frac{100\pi}{9}\dot{h}$$

so that, at the critical instant we have

$$\dot{h} = \frac{9}{10\pi}\text{ ft/min.}$$

8. Consider the curve given by the equation $x^2y + xy^2 = 3x + 3$.
   (a) Verify that the point $(1, 2)$ lies on this curve.
   (b) Write an equation for the line tangent to the curve at $(1, 2)$.
   (c) Show how to use the equation you have written for the tangent line to estimate the value y must have if the point $(1.03, y)$ is to lie on the curve near $(1, 2)$.

Solution:

(a) Substituting $x = 1$ and $y = 2$ into the equation for the curve gives

$$(1)^2 \cdot 2 + 1 \cdot (2)^2 = 3 \cdot 1 + 3$$

or

$$2 + 4 = 3 + 3,$$

which is true. The point $(1, 2)$ therefore lies on the curve whose equation is $x^2y + xy^2 = 3x + 3$.

(b) Differentiating the equation $x^2y + xy^2 = 3x + 3$ implicitly with respect to x while treating y as a function of x gives

$$\begin{aligned}[2xy + x^2y'] + [y^2 + 2xyy'] &= 3;\\ (x^2 + 2xy)y' &= 3 - 2xy - y^2;\\ y' &= \frac{3 - 2xy - y^2}{x^2 + 2xy},\end{aligned}$$

whence

$$y'\Big|_{(1,2)} = \frac{3 - 2\cdot 1\cdot 2 - 2^2}{1^2 + 2\cdot 1\cdot 2} = -1.$$

This is the slope of the tangent line at $(1, 2)$, so an equation for the tangent line is $y = 2 - (x - 1)$, or just $y = 3 - x$.

(c) The curve $x^2y + xy^2 = 3x + 3$ defines y implicitly as a function of x near the point $(1, 2)$. Writing $y = f(x)$ for this function, the linearization of f at $x = 1$ is given by the equation $y = 2 - (x - 1)$ that we have just calculated. The linearization approximation for $f(1.03)$ is therefore $2 - (1.03 - 1) = 1.97$. [Note: The equation $x^2y + xy^2 = 3x + 3$ is quadratic in y and we can actually solve for y in terms of x. Doing so gives $f\left(\frac{103}{100}\right) = \frac{50}{100}\left(\frac{\sqrt{2621630881} - 10609}{10000}\right)$, which is 1.97052738050163128169 to twenty digits past the decimal.]

MTH 1410, Calculus I—Exam II **March 18, 2010**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 8:50 pm. Complete solutions to the exam problems will be available from the course web-site later this evening.

1. Show how to use the Limit Laws to evaluate the following limits. *Do not use l'Hôpital's Rule.*

   (a) $\displaystyle\lim_{x\to 3} \frac{x^2-4x+3}{x^3-6x^2+12x-9}$

   (b) $\displaystyle\lim_{x\to\infty} \left[\sqrt{x^2-4x+4}-x\right]$

2. Let f and g be differentiable functions for which $f(3)=3$, $g(3)=5$, $f'(3)=4$, and $g'(3)=-2$. Find the following values, being sure to show your reasoning.

   (a) $S'(3)$, where $S(x)=f(x)+g(x)$.

   (b) $P'(3)$, where $P(x)=f(x)g(x)$.

   (c) $Q'(3)$, where $Q(x)=\dfrac{g(x)}{f(x)}$.

   (d) $F'(3)$, where $F(x)=f(x)e^{g(x)}$.

   (e) $G'(3)$, where $G(x)=g[f(x)]$.

3. Let f be the function given by the equation $f(x)=\dfrac{1}{x+2}$.

   (a) Show how to use the definition of the derivative to find $f'(a)$, where it is given only that a is some real number other than -2.

   (b) Use the value you have just found for $f'(a)$ of this problem to write an equation for the line tangent to the curve $y=\dfrac{1}{x+2}$ at the point where $x=-1$.

4. Evaluate the following limits. You may use l'Hôpital's Rule if it is applicable. Be sure to give your reasoning.

   (a) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x+\cos x}$

   (b) $\displaystyle\lim_{x\to 0} \frac{2x-\arctan 2x}{x^3}$

5. Find the absolute maximum and the absolute minimum of the function $f(x)=3x^4-4x^3$ when $-1\le x\le 2$. Where do these extremes occur?

6. A culture of bacteria starts at time $t=0$ with 760 bacteria and grows at a rate that is proportional to its size. After 5 hours (that is, when $t=5$) there are 3800 bacteria in the culture.

   (a) Express the population as a function of t.

   (b) What will the population be when $t=6$?

   (c) How long will it take for the population to reach 2170?

7. (a) Use the definition of the hyperbolic sine function to show that

   $$\sinh^{-1} x = \ln\left(x+\sqrt{1+x^2}\right).$$

   (b) Show how to use the formula from part (a) of this problem to derive the fact that

   $$\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{1+x^2}}.$$

MTH 1410, Calculus I—Exam II **March 18, 2010**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 8:50 pm. Complete solutions to the exam problems will be available from the course web-site later this evening.

1. Show how to use the Limit Laws to evaluate the following limits. *Do not use l'Hôpital's Rule.*

   (a) $\displaystyle\lim_{x\to 3} \frac{x^2-4x+3}{x^3-6x^2+12x-9}$

   (b) $\displaystyle\lim_{x\to\infty} \left[\sqrt{x^2-4x+4}-x\right]$

   Solution:

   (a)

$$\lim_{x\to 3} \frac{x^2-4x+3}{x^3-6x^2+12x-9} = \lim_{x\to 3} \frac{\cancel{(x-3)}(x-1)}{\cancel{(x-3)}(x^2-3x+3)} = \frac{\lim_{x\to 3}(x-1)}{\lim_{x\to 3}(x^2-3x+3)} = \frac{2}{3}.$$

   (b)

$$\begin{aligned}\lim_{x\to\infty}\left[\sqrt{x^2-4x+4}-x\right] &= \lim_{x\to\infty}\left[\frac{(x^2-4x+4)-x^2}{\sqrt{x^2-4x+4}+x}\right]\\ &= \lim_{x\to\infty}\left[\frac{-4x+4}{\sqrt{x^2-4x+4}+x}\right]\\ &= \lim_{x\to\infty}\left[\frac{-4+(4/x)}{\sqrt{1-(4/x)+(4/x^2)}+1}\right]\\ &= \frac{\lim_{x\to\infty}\left[-4+(4/x)\right]}{\lim_{x\to\infty}\left[\sqrt{1-(4/x)+(4/x^2)}+1\right]}\\ &= \frac{-4}{1+1} = -2.\end{aligned}$$

2. Let f and g be differentiable functions for which $f(3)=3$, $g(3)=5$, $f'(3)=4$, and $g'(3)=-2$. Find the following values, being sure to show your reasoning.

   (a) $S'(3)$, where $S(x)=f(x)+g(x)$.

   (b) $P'(3)$, where $P(x)=f(x)g(x)$.

   (c) $Q'(3)$, where $Q(x)=\dfrac{g(x)}{f(x)}$.

   (d) $F'(3)$, where $F(x)=f(x)e^{g(x)}$.

   (e) $G'(3)$, where $G(x)=g[f(x)]$.

   Solution:

   (a) $S'(3)=f'(3)+g'(3)=4+(-2)=2.$

   (b) $P'(3)=f'(3)g(3)+f(3)g'(3)=4\cdot 5+3\cdot(-2)=14.$

   (c) $Q'(3)=\dfrac{g'(3)f(3)-g(3)f'(3)}{[f(3)]^2}=\dfrac{(-2)\cdot 3-5\cdot 4}{(3)^2}=-\dfrac{26}{9}.$

(d) $F'(3) = f'(3)e^{g(3)} + f(3)g'(3)e^{g(3)} = [f'(3) + f(3)g'(3)]e^{g(3)} = [4 + 3 \cdot (-2)]e^5 = -2e^5$.

(e) $G'(3) = g'[f(3)]f'(3) = g'(3)f'(3) = (-2) \cdot (4) = -8$.

3. Let f be the function given by the equation $f(x) = \dfrac{1}{x+2}$.

   (a) Show how to use the definition of the derivative to find $f'(a)$, where it is given only that a is some real number other than -2.

   (b) Use the value you have just found for $f'(a)$ of this problem to write an equation for the line tangent to the curve $y = \dfrac{1}{x+2}$ at the point where $x = -1$.

Solution:

(a)

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(a+h)+2} - \frac{1}{a+2}\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{(a+2) - (a+h+2)}{(a+h+2)(a+2)}\right] = -\lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(a+h+2)(a+2)} \\ &= -\frac{1}{(a+2)^2}. \end{aligned}$$

(b) From the previous part of this problem, $f'(-1) = -1$. Consequently, an equation for the line tangent to the curve at the point $(-1, 1)$ where $x = -1$ is $y = 1 - (x+1)$, or just $y = -x$.

4. Evaluate the following limits. You may use l'Hôpital's Rule if it is applicable. Be sure to give your reasoning.

   (a) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x + \cos x}$

   (b) $\displaystyle\lim_{x\to 0} \frac{2x - \arctan 2x}{x^3}$

Solution:

(a) When $x \to 0$, the numerator $\to 0$, but the denominator $\to 1 \neq 0$. L'Hôpital's Rule is not applicable, and

$$\lim_{x\to 0} \frac{\sin x}{x + \cos x} = 0.$$

(b) When $x \to 0$, numerator and denominator both $\to 0$, so we may try l'Hôpital's Rule:

$$\lim_{x\to 0} \frac{2x - \arctan 2x}{x^3} = \lim_{x\to 0} \frac{2 - 2/(1+4x^2)}{3x^2},$$

provided this latter limit exists. But

$$\lim_{x\to 0} \frac{2 - 2/(1+4x^2)}{3x^2} = \lim_{x\to 0} \frac{2(1+4x^2) - 2}{3x^2(1+4x^2)}) = \lim_{x\to 0} \frac{8\cancel{x^2}}{3\cancel{x^2}(1+4x^2)} = \frac{8}{3}.$$

Hence, $\displaystyle\lim_{x\to 0} \frac{2x - \arctan 2x}{x^3} = \frac{8}{3}$.

5. Find the absolute maximum and the absolute minimum of the function $f(x) = 3x^4 - 4x^3$ when $-1 \leq x \leq 2$. Where do these extremes occur?

Solution: We have $f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$, so the critical numbers for f are $x = 0$ and $x = 1$, both of which lie in the interval of interest. We therefore compute:

$$\begin{aligned} f(-1) &= 7; \\ f(0) &= 0; \\ f(1) &= -1; \\ f(2) &= 48 - 32 = 16. \end{aligned}$$

We conclude that the absolute maximum we seek is 16, when $x = 2$, and the absolute minimum is -1, when $x = 1$.

6. A culture of bacteria starts at time $t = 0$ with 760 bacteria and grows at a rate that is proportional to its size. After 5 hours (that is, when $t = 5$) there are 3800 bacteria in the culture.

   (a) Express the population as a function of t.

   (b) What will the population be when $t = 6$?

   (c) How long will it take for the population to reach 2170?

   Solution:

   (a) We know that, for such growth, population, $P(t)$ is given by an equation of the form $P(t) = P_0 e^{kt}$, where P_0 is the initial population—which is 760 in this instance. Thus, $P(t) = 760e^{kt}$, and it remains for us to find k. But when $t = 5$ we have $3800 = P(5) = 760e^{5k}$, whence we conclude that $e^{5k} = 3800/760 = 5$, so that $5k = \ln 5$ and $k = (\ln 5)/5$. Thus,

   $$P(t) = 760 \exp\left[\frac{t \ln 5}{5}\right] = 760 \cdot 5^{t/5}.$$

   (b) When $t = 6$, population will be

   $$P(6)7 = \exp\left[\frac{6 \ln 5}{5}\right] = 760 \cdot 5^{6/5} = 3800 \cdot 5^{1/5}.$$

   (c) To learn when $P(t) = 2170$, we must solve the equation $760 \cdot 5^{t/5} = 2170$ for t:

   $$\begin{aligned} 760 \cdot 5^{t/5} &= 2170; \\ 5^{t/5} &= \frac{2170}{760} = \frac{217}{76}; \\ \frac{t}{5} &= \log_5(217/76) = \frac{\ln(217/76)}{\ln 5}; \\ t &= \frac{5 \ln(217/76)}{\ln 5}. \end{aligned}$$

7. (a) Use the definition of the hyperbolic sine function to show that

   $$\sinh^{-1} x = \ln\left(x + \sqrt{1 + x^2}\right).$$

   (b) Show how to use the formula from part (a) of this problem to derive the fact that

   $$\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{1 + x^2}}.$$

   Solution:

(a) If $y = \sinh^{-1} x$, then

$$x = \sinh y = \frac{e^y - e^{-y}}{2}. \tag{1}$$

Thus,

$$2xe^y = (e^y)^2 - 1, \tag{2}$$

or

$$(e^y)^2 - 2xe^y - 1 = 0. \tag{3}$$

The Quadratic Formula now assures us that

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x + \sqrt{1 + x^2}. \tag{4}$$

But $e^y > 0$ and $\sqrt{1 + x^2} > x$, so we must choose the "+" sign, meaning that

$$e^y = x + \sqrt{1 + x^2}. \tag{5}$$

whence it follows that

$$\sinh^{-1} x = y = \ln\left(x + \sqrt{1 + x^2}\right). \tag{6}$$

(b)

$$\begin{aligned}
\frac{d}{dx}\left(\sinh^{-1} x\right) &= \frac{d}{dx} \ln\left(x + \sqrt{1 + x^2}\right) \\
&= \frac{1}{x + \sqrt{1 + x^2}} \cdot \frac{d}{dx}\left(x + \sqrt{1 + x^2}\right) \\
&= \frac{1}{x + \sqrt{1 + x^2}} \cdot \left[1 + \frac{1}{2} \cdot (1 + x^2)^{-1/2} \cdot \frac{d}{dx}(1 + x^2)\right] \\
&= \frac{1}{x + \sqrt{1 + x^2}} \cdot \left[1 + \frac{1}{\cancel{2}} \cdot (1 + x^2)^{-1/2} \cdot (\cancel{2}x)\right] \\
&= \frac{1}{x + \sqrt{1 + x^2}} \cdot \left[1 + \frac{x}{\sqrt{1 + x^2}}\right] \\
&= \frac{1}{\cancel{x + \sqrt{1 + x^2}}} \cdot \frac{\cancel{\sqrt{1 + x^2} + x}}{\sqrt{1 + x^2}} = \frac{1}{\sqrt{1 + x^2}}.
\end{aligned}$$

MTH 1410, Calculus I—Exam III **April 22, 2010**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 8:50 pm. Complete solutions to the exam problems will be available from the course web-site later this evening.

1. Show how to use the Limit Laws to evaluate the following limits. You may use l'Hôpital's Rule when it is applicable.

   (a) $\lim_{t\to 0}\left[\frac{1}{t}-\frac{1}{t^2+t}\right]$

   (b) $\lim_{x\to 0}\frac{\tan x - x}{x^3}$

   (c) $\lim_{x\to\infty}\left[\sqrt{9x^2+2x+8}-3x\right]$

2. Give the midpoint Riemann sum approximation, with $n = 7$, for $\int_{-3/2}^{11/2} x^3\,dx$. Show your calculations.

3. Let g be the function given by $g(x) = \frac{x}{6x+8}$. Give the domain of g. Then find g^{-1} and give its domain.

4. Let $f(x) = \frac{x+1}{x-2}$. Show how to use the definition of derivative to find $f'(3)$.

5. The equation $2x = 3\sin x$ has a positive root. Show how to use Newton's Method to give an approximation, correct to at least six digits to the right of the decimal, for that root.

6. Let f be the function given by

   $$f(x) = \frac{2x}{(x^2+9)^2}.$$

   Then

   $$f'(x) = \frac{6(\sqrt{3}-x)(\sqrt{3}+x)}{(x^2+9)^3},$$

   and

   $$f''(x) = \frac{24x(x-3)(x+3)}{(x^2+9)^4}.$$

   Give (along with supporting reasoning)

   (a) the intervals where f is increasing and those where it is decreasing.

   (b) the intervals where f is concave upward and those where it is concave downward.

   (c) the critical numbers of f and determine whether each yields a local maximum, a local minimum, or neither.

   (d) the inflection points of f.

7. A rectangle is inscribed with its base on the x-axis and its upper corners on the parabola $y = a - x^2$ (where a is a positive constant). Find the dimensions of the rectangle if its area is to be as large as possible.

MTH 1410, Calculus I—Exam III **April 22, 2010**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 8:50 pm. Complete solutions to the exam problems will be available from the course web-site later this evening.

1. Show how to use the Limit Laws to evaluate the following limits. You may use l'Hôpital's Rule when it is applicable.

   (a) $\lim_{t\to 0}\left[\frac{1}{t}-\frac{1}{t^2+t}\right]$

   (b) $\lim_{x\to 0}\frac{\tan x - x}{x^3}$

   (c) $\lim_{x\to\infty}\left[\sqrt{9x^2+2x+8}-3x\right]$

   Solution:

   (a)

$$\lim_{t\to 0}\left[\frac{1}{t}-\frac{1}{t^2+t}\right]=\lim_{t\to 0}\frac{(t+1)-1}{t(t+1)}=\lim_{t\to 0}\frac{\not{t}}{\not{t}(t+1)}=1.$$

   (b) Numerator and denominator both $\to 0$ as $x\to 0$, so we may attempt to use l'Hôpital's Rule:

$$\lim_{x\to 0}\frac{\tan x - x}{x^3}=\lim_{x\to 0}\frac{\sec^2 x-1}{3x^2}.$$

   We may attempt l'Hôpital's Rule again:

$$\lim_{x\to 0}\frac{\sec^2 x-1}{3x^2}=\lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}.$$

   And still a third time:

$$\lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}=\lim_{x\to 0}\frac{4\sec^2 x\tan^2 x+2\sec^4 x}{6}=\frac{1}{3}.$$

   Thus, $\lim_{x\to 0}\frac{\tan x - x}{x^3}=\frac{1}{3}$.

   (c)

$$\begin{aligned}\lim_{x\to\infty}\left[\sqrt{9x^2+2x+8}-3x\right] &= \lim_{x\to\infty}\left[\left(\left[\sqrt{9x^2+2x+8}-3x\right)\cdot\frac{\left[\sqrt{9x^2+2x+8}+3x\right]}{\left[\sqrt{9x^2+2x+8}+3x\right]}\right]\\ &= \lim_{x\to\infty}\frac{(9x^2+2x+8)-9x^2}{\sqrt{9x^2+2x+8}+3x}=\lim_{x\to\infty}\frac{2x+8}{\sqrt{9x^2+2x+8}+3x}\\ &= \lim_{x\to\infty}\frac{2+(8/x)}{\sqrt{9+(2/x)+(8/x^2)}+3}=\frac{1}{3}.\end{aligned}$$

2. Give the midpoint Riemann sum approximation, with $n=7$, for $\int_{-3/2}^{11/2} x^3\,dx$. Show your calculations.

   Solution: Dividing $[-3/2, 11/2]$, which has length 7, into $n=7$ equal subdivisions, we find that we must take $x_0=-3/2$, $x_1=-1/2$, $x_2=1/2$, $x_3=3/2$, $x_4=5/2$, $x_5=7/2$, $x_6=9/2$, and $x_7=11/2$, so that $x_k-x_{k-1}=1$ for each $k=1,2,\ldots,7$. The mid-points x_k^* of these intervals are $x_1^*=-1$, $x_2^*=0$, $x_3^*=1$,

$x_4^* = 2$, $x_5^* = 3$, $x_6^* = 4$, and $x_7^* = 5$. The desired midpoint Riemann sum approximation for $\int_{-3/2}^{11/2} x^3\,dx$ is therefore

$$\sum_{k=1}^{7} f(x_k^*)(x_k - x_{k-1}) = \sum_{k=1}^{7}[(x_k^*)^3 \cdot 1] = (-1)^3 + 0^3 + 1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 224.$$

3. Let g be the function given by $g(x) = \dfrac{x}{6x+8}$. Give the domain of g. Then find g^{-1} and give its domain.

   Solution: $g(x)$ is meaningful for all x for which the denominator $6x+8$ is non-zero. Hence the domain of g is $(-\infty, -4/3) \cup (-4/3, \infty)$. To find g^{-1}, we solve $y = g(x)$ for x in terms of y:

$$\begin{aligned} y &= \frac{x}{6x+8}; \\ 6xy + 8y &= x; \\ 6xy - x &= -8y; \\ x(6y-1) &= -8y; \\ x &= \frac{8y}{1-6y}. \end{aligned}$$

   Hence, g^{-1} is given by

$$g^{-1}(y) = \frac{8y}{1-6y}.$$

   $g^{-1}(y)$ is meaningful for all values of y for which $1-6y \neq 0$, so the domain of g^{-1} is $(-\infty, 1/6) \cup (1/6, \infty)$.

4. Let $f(x) = \dfrac{x+1}{x-2}$. Show how to use the definition of derivative to find $f'(3)$.

   Solution:

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{(3+h)+1}{(3+h)-2)} - \frac{3+1}{3-2}\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{4+h}{1+h} - \frac{4}{1}\right) = \lim_{h\to 0} \frac{(4+h) - 4(1+h)}{h(1+h)} \\ &= \lim_{h\to 0} \frac{-3\cancel{h}}{\cancel{h}(1+h)} = -3. \end{aligned}$$

5. The equation $2x = 3\sin x$ has a positive root. Show how to use Newton's Method to give an approximation, correct to at least six digits to the right of the decimal, for that root.

   Solution: A quick sketch (see Figure 1) indicates that the desired root is close to $x = 1.5$, so let us take $x_0 = 1.5$. The equation $2x = 3\sin x$ is equivalent to the equation $f(x) = 0$ with $f(x) = 2x - 3\sin x$, so the Newton's Method iteration scheme

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

   becomes

$$x_{k+1} = x_k - \frac{2x_k - 3\sin x_k}{2 - 3\cos x_k}.$$

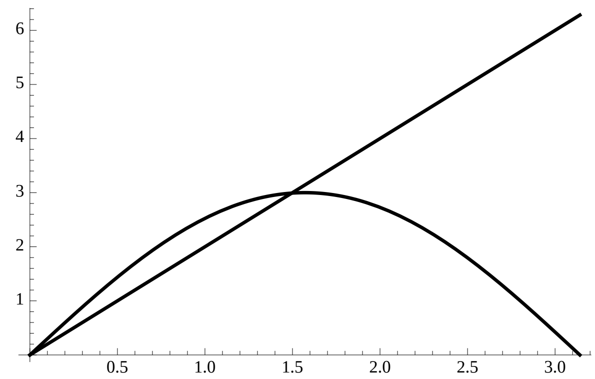


Figure 1: Plot for Problem 5

Thus,

$$\begin{aligned} x_1 &= 1.5 - \frac{3.0 - 3\sin(1.5)}{2 - 3\cos(1.5)} = 1.49579646; \\ x_2 &= x_1 - \frac{2x_1 - 3\sin x_1}{2 - 3\cos x_1} = 1.49578157; \\ x_3 &= x_2 - \frac{2x_2 - 3\sin x_2}{2 - 3\cos x_2} = 1.49578157; \end{aligned}$$

Our approximate solution, correct to 8 digits to the right of the decimal, is 1.49578157.

6. Let f be the function given by

$$f(x) = \frac{2x}{(x^2+9)^2}.$$

Then

$$f'(x) = \frac{6(\sqrt{3}-x)(\sqrt{3}+x)}{(x^2+9)^3},$$

and

$$f''(x) = \frac{24x(x-3)(x+3)}{(x^2+9)^4}.$$

Give (along with supporting reasoning)

(a) the intervals where f is increasing and those where it is decreasing.

(b) the intervals where f is concave upward and those where it is concave downward.

(c) the critical numbers of f and determine whether each yields a local maximum, a local minimum, or neither.

(d) the inflection points of f.

Solution:

(a) The sign of $f'(x)$ is determined by the sign of the product $(\sqrt{3}-x)(\sqrt{3}+x)$ in its numerator. This sign is given by the sign chart of Figure 2. From the chart, we see that f is increasing on $\left[\sqrt{3}, \sqrt{3}\right]$ and decreasing on each of the intervals $\left(-\infty, -\sqrt{3}\right]$ and $\left[\sqrt{3}, \infty\right)$.

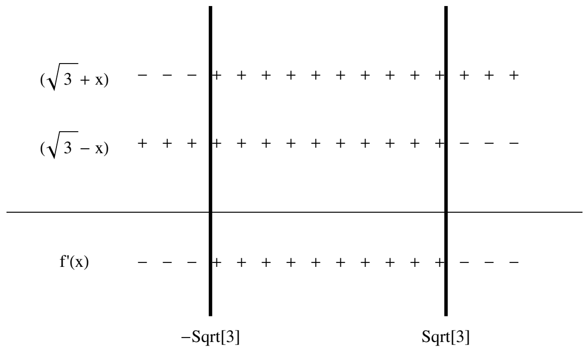


Figure 2: Problem 6, the sign of $f'(x)$

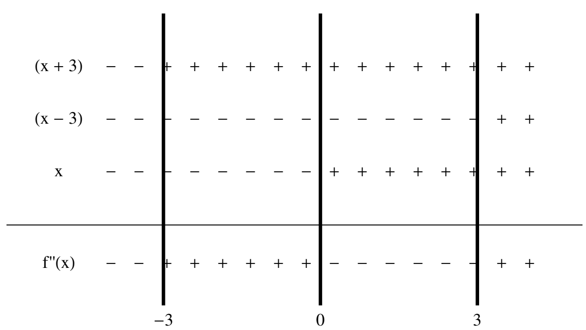


Figure 3: Problem 6, the sign of $f''(x)$

(b) The sign of $f''(x)$ is determined by the sign of the product $x(x-3)(x+3)$ in its numerator. This sign is given by the sign chart of Figure 3. From the chart, we see that f is concave upward on each of the intervals $(-3,0)$ and $(3,\infty)$. f is concave downward on each of the intervals $(-\infty,-3)$, and $(0,-3)$.

(c) The critical numbers of f are the points where $f'(x)=0$, or the points $x=-\sqrt{3}$ and $x=\sqrt{3}$. Because $f'(x)$ changes sign from negative to positive at $x=-\sqrt{3}$, this critical number gives a local minimum for f. On the other hand, $f'(x)$ changes sign from positive to negative at $x=\sqrt{3}$, so this critical number gives a local maximum for f.

(d) The inflection points are the points where the nature of concavity changes. These are the points where $f''(x)$ changes sign, or $x=-3$, $x=0$, and $x=3$.

7. A rectangle is inscribed with its base on the x-axis and its upper corners on the parabola $y=a-x^2$ (where a is a positive constant). Find the dimensions of the rectangle if its area is to be as large as possible.

Solution: Consider Figure 4 in which (x,y) denotes the point where the upper right-hand corner of the inscribed rectangle lies on the curve $y=a-x^2$. The area of the rectangle is $A=2xy$, where $y=a-x^2$ and $0\le x\le\sqrt{a}$. Thus,

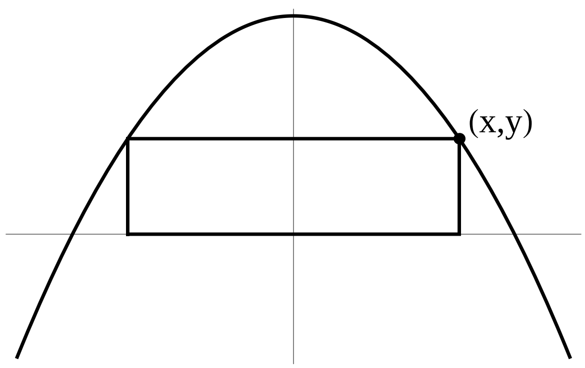


Figure 4: Problem 7

$$A=2xy=2x(a-x^2)=2ax-2x^3,$$

so that

$$\frac{dA}{dx}=2a-6x^2=0$$

when

$$x=\pm\sqrt{\frac{a}{3}}.$$

We need $x\ge 0$, so we take the plus sign. It is clear from the nature of the problem that the critical point gives the maximum, so the width of the rectangle of maximal area is $2x=2\sqrt{a/3}$, while the height is $a-x^2=a-a/3=2a/3$.

MTH 1410, Calculus I—Final Exam **May 11, 2010**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 8:50 pm. Complete solutions to the exam problems will be available from the course web-site later this evening.

1. Show how to use the Limit Laws to evaluate the following limits. You may use l'Hôpital's Rule when it is applicable.

   (a) $\lim_{t\to 0}\left[\frac{1}{t}-\frac{1}{t^2+t}\right]$

   (b) $\lim_{x\to 0}\frac{\tan x - x}{x^3}$

   (c) $\lim_{x\to\infty}\left[\sqrt{9x^2+2x+8}-3x\right]$

2. Let f be the function give by the equation $f(x)=\frac{1}{\sqrt{x+1}}$.

   (a) Show how to use the definition of the derivative to find $f'(a)$, where it is given only that a is some real number greater than -1.

   (b) Use the value you have just found for $f'(a)$ to write an equation for the line tangent to the curve $y=\frac{1}{\sqrt{x+1}}$ at the point where $x=3$.

3. Find all local maxima and local minima for the function f given by $f(x)=x^3-3x$ on the interval $[-2,3]$. Then find the absolute maximum and the absolute minimum taken on by $f(x)$ on this interval. Show all of your supporting reasoning.

4. Evaluate the definite integrals:

   (a) $\int_{\pi/2}^{2\pi}\cos x\,dx$

   (b) $\int_{e}^{e^4}\frac{dx}{x\sqrt{\ln x}}$

5. The equation $x^3+2x^2+3x+1=0$ has exactly one root. Show how to use Newton's Method to give an approximation, correct to at least six digits to the right of the decimal, for that root.

6. Consider the curve given by the equation $2x^2y-y^2=8x-2xy^2$.

   (a) Verify that the point $(-1,2)$ lies on this curve.

   (b) Write an equation for the line tangent to the curve at $(-1,2)$.

   (c) Show how to use the equation you have written for the tangent line to estimate the value y must have if the point $(-1.05,y)$ is to lie on the curve near the point $(-1,2)$.

7. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?

8. A rancher wants to erect a fence around a rectangular area comprising 1,500,000 square feet and then divide it in half with a fence down the middle parallel to one of the sides. What is the shortest length of fence she can use?

MTH 1410, Calculus I—Final Exam **May 11, 2010**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 8:50 pm. Complete solutions to the exam problems will be available from the course web-site later this evening.

1. Show how to use the Limit Laws to evaluate the following limits. You may use l'Hôpital's Rule when it is applicable.

   (a) $\displaystyle\lim_{t\to 0}\left[\frac{1}{t}-\frac{1}{t^2+t}\right]$

   (b) $\displaystyle\lim_{x\to 0}\frac{\tan x - x}{x^3}$

   (c) $\displaystyle\lim_{x\to\infty}\left[\sqrt{9x^2+2x+8}-3x\right]$

   Solution:

   (a)

$$\lim_{t\to 0}\left[\frac{1}{t}-\frac{1}{t^2+t}\right]=\lim_{t\to 0}\frac{(t+1)-1}{t(t+1)}=\lim_{t\to 0}\frac{\not{t}}{\not{t}(t+1)}=1.$$

   (b) Numerator and denominator both $\to 0$ as $x\to 0$, so we may attempt to use l'Hôpital's Rule:

$$\lim_{x\to 0}\frac{\tan x - x}{x^3}=\lim_{x\to 0}\frac{\sec^2 x-1}{3x^2}.$$

   We may attempt l'Hôpital's Rule again:

$$\lim_{x\to 0}\frac{\sec^2 x-1}{3x^2}=\lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}.$$

   And still a third time:

$$\lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}=\lim_{x\to 0}\frac{4\sec^2 x\tan^2 x+2\sec^4 x}{6}=\frac{1}{3}.$$

   Thus, $\displaystyle\lim_{x\to 0}\frac{\tan x - x}{x^3}=\frac{1}{3}.$

   (c)

$$\begin{aligned}\lim_{x\to\infty}\left[\sqrt{9x^2+2x+8}-3x\right]&=\lim_{x\to\infty}\left[\left([\sqrt{9x^2+2x+8}-3x\right)\cdot\frac{[\sqrt{9x^2+2x+8}+3x]}{[\sqrt{9x^2+2x+8}+3x]}\right]\\&=\lim_{x\to\infty}\frac{(9x^2+2x+8)-9x^2}{\sqrt{9x^2+2x+8}+3x}=\lim_{x\to\infty}\frac{2x+8}{\sqrt{9x^2+2x+8}+3x}\\&=\lim_{x\to\infty}\frac{2+(8/x)}{\sqrt{9+(2/x)+(8/x^2)}+3}=\frac{1}{3}.\end{aligned}$$

2. Let f be the function give by the equation $f(x)=\dfrac{1}{\sqrt{x+1}}$.

   (a) Show how to use the definition of the derivative to find $f'(a)$, where it is given only that a is some real number greater than -1.

   (b) Use the value you have just found for $f'(a)$ to write an equation for the line tangent to the curve $y=\dfrac{1}{\sqrt{x+1}}$ at the point where $x=3$.

Solution:

(a)

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{\sqrt{a+h+1}} - \frac{1}{\sqrt{a+1}}\right) \\
&= \lim_{h\to 0} \frac{\sqrt{a+1}-\sqrt{a+h+1}}{h\sqrt{a+1}\sqrt{a+h+1}} = \lim_{h\to 0} \frac{(a+1)-(a+h+1)}{h\sqrt{a+1}\sqrt{a+h+1}(\sqrt{a+1}+\sqrt{a+h+1})} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{a+1}\sqrt{a+h+1}(\sqrt{a+1}+\sqrt{a+h+1})} = -\frac{1}{2(a+1)^{3/2}}.
\end{aligned}$$

(b) From the calculation above, we have $f'(3) = -1/[2(3+1)^{3/2}] = -1/16$. But $f(3) = 1/2$, so the desired equation is therefore

$$y = \frac{1}{2} - \frac{1}{16}(x-3).$$

3. Find all local maxima and local minima for the function f given by $f(x) = x^3 - 3x$ on the interval $[-2, 3]$. Then find the absolute maximum and the absolute minimum taken on by $f(x)$ on this interval. Show all of your supporting reasoning.

Solution: If $f(x) = x^3 - 3x$, then $f'(x) = 3x^2 - 3$, and this is zero when $x = \pm 1$. We have $f''(x) = 6x$, so $f''(-1) = -6 < 0$, while $f''(1) = 6 > 0$. By the second derivative test, f has a local maximum, which is $f(-1) = 2$, at $x = -1$ and a local minimum, which is $f(1) = -2$, at $x = 1$. The absolute extrema must lie either at one of these critical points or at an endpoint. But $f(-2) = -2$, while $f(3) = 18$. Therefore f takes on its absolute minimum value at both of the points $x = -2$ and $x = 1$, and its absolute maximum at the point $x = 3$.

4. Evaluate the definite integrals:

(a) $\displaystyle\int_{\pi/2}^{2\pi} \cos x\, dx$

(b) $\displaystyle\int_{e}^{e^4} \frac{dx}{x\sqrt{\ln x}}$

Solution:

(a)

$$\int_{\pi/2}^{2\pi} \cos x\, dx = \sin x \Big|_{\pi/2}^{2\pi} = \sin 2\pi - \sin \pi/2 = -1.$$

(b) Let $u = \ln x$. Then $du = dx/x$. Also, $x = e \Rightarrow u = 1$, while $x = e^4 \Rightarrow u = 4$. Thus

$$\int_{e}^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_{1}^{4} u^{-1/2}\, du = 2u^{1/2}\Big|_{1}^{4} = 2.$$

5. The equation $x^3 + 2x^2 + 3x + 1 = 0$ has exactly one root. Show how to use Newton's Method to give an approximation, correct to at least six digits to the right of the decimal, for that root.

Solution: Let $f(x) = x^3 + 2x^2 + 3x + 1$. We seek the only root for the equation $f(x) = 0$. Because $f(-1) = -1$ while $f(0) = 1$, the root must lie between $x = -1$ and $x = 0$. We therefore put $x_0 = -0.5$ and iterate the relation

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^3 + 2x_k^2 + 3x_k + 1}{3x_k^2 + 4x_k + 3}.$$

Thus,

$$\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = -0.4285714286 \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = -0.4301587302 \\
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = -0.4301597090 \\
x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} = -0.4301597090.
\end{aligned}$$

We conclude that, to about 10 places accuracy, the desired root is -0.4301597090.

6. Consider the curve given by the equation $2x^2y - y^2 = 8x - 2xy^2$.
   (a) Verify that the point $(-1, 2)$ lies on this curve.
   (b) Write an equation for the line tangent to the curve at $(-1, 2)$.
   (c) Show how to use the equation you have written for the tangent line to estimate the value y must have if the point $(-1.05, y)$ is to lie on the curve near the point $(-1, 2)$.

   Solution:

   (a) Setting $x = -1$, $y = 2$ in the equation for the curve gives

   $$\begin{aligned}
   2 \cdot (-1)^2 \cdot 2 - (2)^2 &= 8 \cdot (-1) - 2 \cdot (-1) \cdot (2)^2; \\
   4 - 4 &= -8 + 8 \\
   0 &= 0,
   \end{aligned}$$

   and we conclude that the point $(-1, 2)$ lies on the curve whose equation we were given.

   (b) Differentiating the equation of the curve implicitly while treating y as a function of x gives

   $$\begin{aligned}
   4x + 2x^2y' - 2yy' &= 8 - 2y^2 - 4xyy'; \\
   (4xy + 2x^2 - 2y)y' &= 8 - 2y^2 - 4x; \\
   y' &= \frac{8 - 2y^2 - 4xy}{4xy + 2x^2 - 2y}.
   \end{aligned}$$

   Therefore,

   $$y'\big|_{(-1,2)} = \frac{8 - 2 \cdot (2)^2 - 4 \cdot (-1) \cdot (2)}{4 \cdot (-1) \cdot (2) + 2 \cdot (-1)^2 - 2 \cdot (2)} = -\frac{4}{5},$$

   and it follows that an equation for the line tangent to the curve at the point $(-1, 2)$ is

   $$y = 2 - \frac{4}{5}(x + 1).$$

   (c) For values of x near -1, the y-coordinate for a corresponding point on the tangent line is a good approximation to the y-coordinate of the corresponding point on the curve. Thus, when $x = -1.05$ the y value of the corresponding point on the curve near $(-1, 2)$ is $y = 2 - 0.8(-1.05 + 1) = 2.04$. (Note: To 10 decimals, the correct value for y is 2.0397356874.)

7. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?

Solution: The area A of a triangle whose base is b and whose altitude is h is given by $A = bh/2$. Treating all three variables as functions of time and differentiating this equation implicitly leads to the following derived equation:

$$\frac{dA}{dt} = \frac{1}{2}\frac{d}{dt}(bh) = \frac{1}{2}\left(h\frac{db}{dt} + b\frac{dh}{dt}\right)$$

At the critical instant, we know that $\frac{dh}{dt} = 1$, $\frac{dA}{dt} = 2$, $h = 10$, and $A = 100$. But $A = bh/2$, so $100 = 10b/2 = 5b$ at the critical instant, and we find that $b = 20$ at that instant. Substituting this information into the derived equation above, we learn that

$$2 = \frac{1}{2}\left(10\frac{db}{dt} + 20 \cdot 1\right) = 5\frac{db}{dt} + 10,$$

whence

$$\frac{db}{dt} = -\frac{8}{5}.$$

We conclude that the base of the triangle is decreasing at the rate of $8/5$ cm/min at the given instant.

8. A rancher wants to erect a fence around a rectangular area comprising 1,500,000 square feet and then divide it in half with a fence down the middle parallel to one of the sides. What is the shortest length of fence she can use?

Solution: Let x half the length of the rectangle and y be the width, so that her fencing arrangement appears as in Figure 1. The total area, which is to be $1,500,000$ ft^2, is then $2xy$, while T, the total amount of fencing she needs, will be $T = 4x + 3y$. Thus, we are to minimize $T = 4x + 3y$ given the constraint $1500000 = 2xy$. We treat y and T as functions of x and we differentiate both equations implicitly with

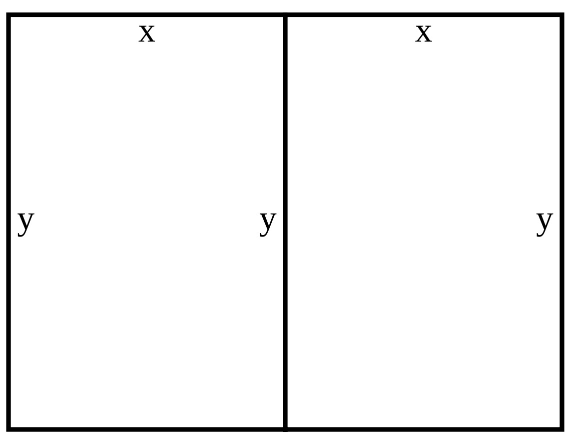


Figure 1: The Rancher's Fence

respect to x, setting $dT/dx = 0$ as we do so in order to locate the critical numbers for T. We find that we must solve the system of equations

$$1500000 = 2xy;$$
$$\frac{dT}{dx} = 4 + 3\frac{dy}{dx} = 0;$$
$$2y + 2x\frac{dy}{dx} = 0.$$

From the second line above, we see that $dy/dx = -4/3$. Substituting this value into the equation that appears on the third line and multiplying the result through by 3 yields $6y - 8x = 0$, from which we conclude that $y = 4x/3$. Substituting this expression for y into the constraint (which is the first equation of the system), we find that

$$\begin{aligned} 1500000 = 2xy = 2x\left(\frac{4}{3}x\right) &= \frac{8}{3}x^2; \\ x^2 &= 562500; \\ x &= 750, \end{aligned}$$

where we must take the positive square root because we know that $x > 0$. But then $y = 4x/3 = 1000$, and $T = 4x + 3y = 4 \cdot 750 + 3 \cdot 1000 = 6000$ ft is the shortest length of fence that will meet her requirements.

MTH 1410, Calculus I—Exam I **Feb. 16, 2011**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Your paper is due at 2:50 pm.

1. Evaluate the following limits. Use the Limit Laws. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $$\lim_{x \to 4} \frac{x^2 - 2x - 8}{3x^2 - 11x - 4}$$

   (b) $$\lim_{x \to \infty} \frac{(4 - x)(7 + 11x)}{(3 - 3x)(3 + 10x)}$$

2. A rock is thrown off of a 100-meter cliff with an upward velocity of 50 m/s. As a result, its height (above the ground at the base of the cliff) after t seconds is given by the formula $h(t) = 100 + 50t - 5t^2$.

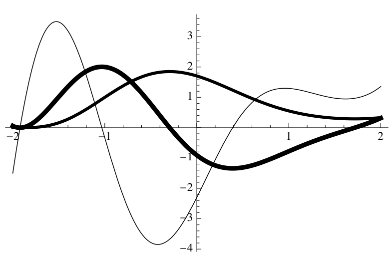
   (a) What is its height after 7 seconds?

   (b) What is its average velocity over the first seven seconds?

   (c) What is its velocity after 7 seconds?

   (d) What is its velocity when it hits the ground at the base of the cliff?

3. Here is a graph showing three functions—a skinny one, a middle-weight one, and a fat one—on the same pair of axes:



   One of the curves is f, one is f', and one is f''. Explain which is which and how you know.

4. Find $f'(x)$ if

   (a) $f(x) = 5x^2 - 3x + 2$

   (b) $f(x) = \dfrac{x^2 + x - 4\sqrt{x}}{x^{2/3}}$

5. Find $f'(x)$ if

   (a) $f(x) = (4x^2 - 2x + 1)^3(4x + 5)^9$

   (b) $f(x) = \dfrac{a}{x^8} + \sin^5 bx$, where a and b are fixed but unspecified constants.

6. A function f is given by

$$f(x) = \begin{cases} 3x + 2c & ; \quad x \leq 8 \\ \\ 12 - cx & ; \quad 8 < x. \end{cases}$$

   For what values of the constant c is f continuous on $(-\infty, \infty)$? Be sure to give your reasoning.

7. (a) Use the definition of the derivative to find $f'(x)$ if $f(x) = \sqrt{3x}$.

   (b) Use the derivative you calculated in part (a) of this problem to write an equation for the line tangent to the curve $y = \sqrt{3x}$ at $x = 3$.

8. Let F be the function given by

$$F(x) = f(x)g(x),$$

   where f and g are functions for which $f'(x)$ and $g'(x)$ are both defined for all real values of x.

   (a) What is $F'(x)$?

   (b) Show how to use the definition of the derivative to obtain the formula you have given in part 8a.

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 1410, Calculus I—Exam I Solutions **Feb. 16, 2011**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Your paper is due at 2:50 pm.

1. Evaluate the following limits. Use the Limit Laws. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a)
   $$\lim_{x\to 4} \frac{x^2-2x-8}{3x^2-11x-4}$$

   (b)
   $$\lim_{x\to\infty} \frac{(4-x)(7+11x)}{(3-3x)(3+10x)}$$

   Solution:

   (a)
   $$\lim_{x\to 4} \frac{x^2-2x-8}{3x^2-11x-4} = \lim_{x\to 4} \frac{\cancel{(x-4)}(x+2)}{\cancel{(x-4)}(3x+1)} = \frac{6}{13}.$$

   (b)
   $$\begin{aligned}\lim_{x\to\infty} \frac{(4-x)(7+11x)}{(3-3x)(3+10x)} &= \lim_{x\to\infty} \frac{[(4-x)(7+11x)]/x^2}{[(3-3x)(3+10x)]/x^2} \\ &= \lim_{x\to\infty} \frac{(4/x-1)(7/x+11)}{(3/x-3)(3/x+10)} = \frac{(-1)\cdot(11)}{(-3)\cdot(10)} = \frac{11}{30}.\end{aligned}$$

2. A rock is thrown off of a 100-meter cliff with an upward velocity of 50 m/s. As a result, its height (above the ground at the base of the cliff) after t seconds is given by the formula $h(t) = 100 + 50t - 5t^2$.

   (a) What is its height after 7 seconds?
   (b) What is its average velocity over the first seven seconds?
   (c) What is its velocity after 7 seconds?
   (d) What is its velocity when it hits the ground at the base of the cliff?

   Solution:

   (a) $h(7) = 100 + 50\cdot 7 - 5\cdot 7^2 = 205$ meters.

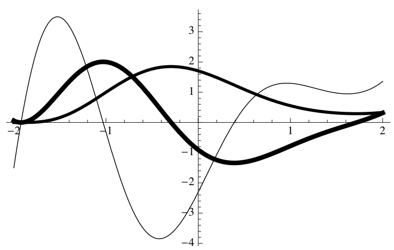
   (b) Average velocity over the first seven seconds is
   $$\frac{h(7)-h(0)}{7-0} = \frac{205-100}{7} = 15 \text{ meters per second.}$$

(c) Velocity is the derivative $h'(t) = 50-10t$, so when $t = 7$, velocity is $h'(7) = 50-10\cdot 7 = -20$ meters per second.

(d) The rock hits the ground when $t > 0$ satisfies $h(t) = 0$. But the only positive solution of the equation $100 + 50t - 5t^2 = 0$ is $t = 5 + 3\sqrt{5}$. So the rock hits the ground at the base of the cliff with velocity

$$h'(5 + 3\sqrt{5}) = 50 - 10(5 + 3\sqrt{5}) = -30\sqrt{5} \text{ meters per second.}$$

3. Here is a graph showing three functions—a skinny one, a middle-weight one, and a fat one—on the same pair of axes:



One of the curves is f, one is f', and one is f''. Explain which is which and how you know.

Solution: The fat curve lies above the x-axis exactly where the middle-weight curve is increasing, touches the x-axis exactly where the middle-weight curve has a horizontal tangent, and lies below the x-axis exactly where the fat curve is decreasing. The skinny curve lies above the x-axis exactly where the fat one is increasing, touches the x-axis exactly where the fat one has a horizontal tangent, and lies below the x-axis exactly where the fat one is decreasing. Thus, the middle-weight curve is f, the fat curve is f', and the skinny curve is f''.

4. Find $f'(x)$ if

(a) $f(x) = 5x^2 - 3x + 2$

(b) $f(x) = \dfrac{x^2 + x - 4\sqrt{x}}{x^{2/3}}$

Solution:

(a) $f'(x) = 10x - 3$.

(b) Carrying out the division, we find that $f(x) = x^{2-2/3} + 2x^{1-2/3} - 4x^{1/2-2/3} = x^{4/3} + x^{1/3} - 4x^{-1/6}$. Consequently, $f'(x) = \frac{4}{3}x^{1/3} + \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-7/6}$.

5. (a) $f(x) = (4x^2 - 2x + 1)^3(4x + 5)^9$
   (b) $f(x) = \dfrac{a}{x^8} + \sin^5 bx$, where a and b are fixed but unspecified constants.

   Solution:

   (a) $f'(x) = 3(4x^2 - 2x + 1)^2(8x - 2)(4x + 5)^9 + 36(4x^2 - 2x + 1)^3(4x + 5)^8$
   (b) $f(x) = ax^{-8} + [\sin(bx)]^5$, so $f'(x) = -8ax^{-9} + 5b\sin^4(bx)\cos(bx)$.

6. A function f is given by

$$f(x) = \begin{cases} 3x + 2c & ; \quad x \leq 8 \\ 12 - cx & ; \quad 8 < x. \end{cases}$$

   For what values of the constant c is f continuous on $(-\infty, \infty)$? Be sure to give your reasoning.

   Solution: If $a \neq 8$ the values of the function $f(x)$ are given by a polynomial function in some open interval centered at a, so f is a continuous function everywhere except possibly at $x = 8$. In order for f to be continuous at $x = 8$, we need to have $\lim_{x\to 8^-} f(x) = \lim_{x\to 8^+} f(x) = f(8) = 24 + 2c$. Now $\lim_{x\to 8^-} f(x) = 24 + 2c$, but $\lim_{x\to 8^+} = 12 - 8c$. Consequently, f is continuous at $x = 8$ precisely when c is chosen so that $24+2c = 12-8c$, or when $c = -6/5$.

7. (a) Use the definition of the derivative to find $f'(x)$ if $f(x) = \sqrt{3x}$.
   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = \sqrt{3x}$ at $x = 3$, at $x = 12$, and at $x = 27$.

   Solution:

   (a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{3(x+h)} - \sqrt{3x}}{h} \\ &= \lim_{h\to 0} \frac{\left[\sqrt{3(x+h)} - \sqrt{3x}\right]\left[\sqrt{3(x+h)} + \sqrt{3x}\right]}{h\left[\sqrt{3(x+h)} + \sqrt{3x}\right]} \\ &= \lim_{h\to 0} \frac{3(x+h) - 3x}{h\left[\sqrt{3(x+h)} + \sqrt{3x}\right]} \\ &= \lim_{h\to 0} \frac{\cancel{3x} + 3\cancel{h} - \cancel{3x}}{\cancel{h}\left[\sqrt{3(x+h)} + \sqrt{3x}\right]} \\ &= \frac{3}{\sqrt{3x} + \sqrt{3x}} = \frac{3}{2\sqrt{3x}}. \end{aligned}$$

   (b) Tangent line at $x = 1$:

$$y = f(3) + f'(3)(x - 3),$$

which is

$$y = 3 + \frac{3}{2\sqrt{9}}(x-3) = 3 + \frac{1}{2}(x-3).$$

8. Let F be the function given by

$$F(x) = f(x)g(x),$$

where f and g are functions for which $f'(x)$ and $g'(x)$ are both defined for all real values of x.

(a) What is $F'(x)$?

(b) Show how to use the definition of the derivative to obtain the formula you have given in part 8a.

Solution:

(a) $F'(x) = f'(x)g(x) + f(x)g'(x)$.

(b)

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} g(x+h)\right] + \lim_{h\to 0} \left[f(x)\frac{g(x+h) - g(x)}{h}\right] \\
&= \left[\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right] \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \left[\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right],
\end{aligned}$$

provided all of the limits in the latter expression exist. But

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= f'(x), \\
\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} &= g'(x), \text{ and} \\
\lim_{h\to 0} f(x) &= f(x).
\end{aligned}$$

We are given that $g'(x)$ exists for all real x, and this means that the function g is continuous everywhere. Consequently,

$$\lim_{h\to 0} g(x+h) = g(x).$$

It now follows that

$$\begin{aligned}
F'(x) &= \left[\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right] \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x) \left[\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right] \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

MTH 1410, Calculus I—Exam II **March 16, 2011**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 1:50 pm.

1. Find $f'(x)$ if
   (a) $f(x) = 3x^6 - 14x^5 + 12x^3 + 7x^2 - 8x$.
   (b) $f(x) = \dfrac{3x^2 - 5x}{x^2 + x + 1}$.
2. Find $f'(x)$ if
   (a) $f(x) = \cosh^3 x \sin 2x$.
   (b) $f(x) = \ln\left[e^x \sin^3 x\right]$.
3. (a) *Use the definition of the derivative* to find $f'(x)$ if $f(x) = 1/\sqrt{x}$.
   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = 1/\sqrt{x}$ at $x = 1$, at $x = 4$, and at $x = 9$.
4. Let f be the function given by

$$f(x) = \begin{cases} x^2 + 2x, & x \le 2 \\ ax^2 + b, & x > 2. \end{cases}$$

   (a) What condition must the constants a and b satisfy if f is to be a continuous function on $(-\infty, \infty)$?
   (b) Find all pairs of values for a and b which make the function f a differentiable function on $-\infty, \infty)$.
5. Let g be the function given by $g(x) = x^3 + x$. Then g is a one-to-one function from $(-\infty, \infty)$ onto $(-\infty, \infty)$, and so g is invertible. Let $G = g^{-1}$. Find an equation for the line tangent to the curve $y = G(x)$ at the point on the curve where $x = 10$.
6. A man started walking north at 4 feet per second from a point P. Five minutes later, a woman started walking south at 5 feet per second from a point 500 feet due east of P. At what rate is the distance between the two changing 15 minutes after the woman started walking?
7. (a) Find an equation for the line tangent to the curve $x^3 - 4x^2y + 2xy^3 + 4 = 0$ at the point whose coordinates are $(2, 1)$.
   (b) Use the result of part (a) of this problem to find an approximate value for the y-coordinate of the point $(2.04, y)$ that lies on the curve $x^3 - 4x^2y + 2xy^3 + 4 = 0$ near the point $(2, 1)$.

MTH 1410, Calculus I—Exam II **March 15, 2011**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 4:50 pm.

1. Find $f'(x)$ if

   (a) $f(x) = 3x^6 - 14x^5 + 12x^3 + 7x^2 - 8x$.

   (b) $f(x) = \dfrac{3x^2 - 5x}{x^2 + x + 1}$.

   Solution:

   (a) $f'(x) = 18x^5 - 70x^4 + 36x^2 + 14x - 8$.

   (b) $f'(x) = \dfrac{(6x-5)(x^2+x+1) - (3x^2-5x)(2x+1)}{(x^2+x+1)^2} = \dfrac{8x^2+6x-5}{(x^2+x+1)^2}$.

2. Find $f'(x)$ if

   (a) $f(x) = \cosh^3 x \sin 2x$.

   (b) $f(x) = \ln\left[e^x \sin^3 x\right]$.

   Solution:

   (a) $f'(x) = 3\cosh^2 x \sinh x \sin 2x + 2\cosh^3 x \cos 2x$.

   (b) $f'(x) = D_x[\ln(e^x) + \ln(\sin^3 x)] = D_x(x + 3\ln\sin x) = 1 + 3\dfrac{\cos x}{\sin x} = 1 + 3\cot x$.

3. (a) *Use the definition of the derivative* to find $f'(x)$ if $f(x) = 1/\sqrt{x}$.

   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = 1/\sqrt{x}$ at $x = 1$, at $x = 4$, and at $x = 9$.

   Solution:

   (a)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right] \\
&= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \\
&= \lim_{h\to 0} \frac{x - (x+h)}{h\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} \\
&= -\lim_{h\to 0} \frac{1}{\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} = -\frac{1}{2x\sqrt{x}}.
\end{aligned}$$

(b) $f'(1) = -1/2$, so the equation of the tangent line at $x = 1$ is $y = 1-(x-1)/2$. $f'(4) = -1/16$, so the equation of the tangent line at $x = 4$ is $y = (1/2) - (1/16)(x-4)$. $f'(9) = -1/54$; the equation of the tangent line at $x = 9$ is $y = (1/3) - (1/54)(x-9)$.

4. Let f be the function given by

$$f(x) = \begin{cases} x^2 + 2x, & x \leq 2 \\ ax^2 + b, & x > 2. \end{cases}$$

(a) What condition must the constants a and b satisfy if f is to be a continuous function on $(-\infty, \infty)$?

(b) Find all pairs of values for a and b which make the function f a differentiable function on $-\infty, \infty)$.

Solution:

(a) If f is to be continuous, we must have $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$. But

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} ax^2 + b = 4a + b,$$

while

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 + 2x = 8.$$

If f is to be continuous, the constants a and b must therefore satisfy the equation $4a + b = 8$.

(b) If f is to be differentiable, f must be continuous at $x = 2$ and the derivatives from the left and from the right must match at $x = 2$. Thus, we must have

$$\begin{aligned} f'(2) &= \lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0^+} \frac{(2+h)^2 + 2(2+h) - 8}{h} \\ &= \lim_{h \to 0^+} \frac{4 + 4h + h^2 + 4 + 2h - 8}{h} \\ &= \lim_{h \to 0^+} (6 + h) = 6, \end{aligned}$$

and

$$\begin{aligned} f'(2) &= \lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0^-} \frac{[a(2+h)^2 + b] - (4a + b)}{h} \\ &= \lim_{h \to 0^-} (4a + ah) = 4a \end{aligned}$$

together with the equation we derived in part (a): $4a + b = 8$. Thus, $4a = 6$ and $4a + b = 8$, so that $a = 3/2$ and $b = 2$.

5. Let g be the function given by $g(x) = x^3 + x$. Then g is a one-to-one function from $(-\infty, \infty)$ onto $(-\infty, \infty)$, and so g is invertible. Let $G = g^{-1}$. Find an equation for the line tangent to the curve $y = G(x)$ at the point on the curve where $x = 10$.

Solution: We know that $y = G(x)$ iff $x = g(y) = y^3 + y$. Differentiating this latter equation implicitly with respect to x gives $1 = 3y^2y' + y'$, whence $y' = 1/(3y^2 + 1)$. But $G(10) = y$, where $10 = y^3 + y$, so that $y = 2$. Thus $y'(10) = 1/(3 \cdot 2^2 + 1) = 1/13$. It now follows that the equation of the line tangent to the curve $y = G(x)$ at the point $(10, 2)$ is $y = 2 + (x - 10)/13$.

6. A man started walking north at 4 feet per second from a point P. Five minutes later, a woman started walking south at 5 feet per second from a point 500 feet due east of P. At what rate is the distance between the two changing 15 minutes after the woman started walking?

Solution: Let x denote the distance the man has walked north of the point P, and let y denote the distance the woman has walked south of her starting point. Let D be the distance between the two. By the Pythagorean Theorem, $D^2 = (x+y)^2 + 500^2 = (x+y)^2 + 250000$. We differentiate this latter equation implicitly with respect to time to obtain:

$$2D\frac{dD}{dt} = 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right), \text{ or}$$

$$\frac{dD}{dt} = \frac{1}{D}(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right).$$

We are given that $dx/dt = 4$ and $dy/dt = 5$, and this means that

$$\frac{dD}{dt} = \frac{9(x+y)}{D}.$$

At the critical instant, the woman has been walking for 15 minutes, or 900 seconds, so $y = 4500$. At that instant, the man has been walking for 20 minutes, so $x = 4800$. Thus, $D = \sqrt{(4800 + 4500)^2 + 250000} = 100\sqrt{8674}$, and this means that

$$\frac{dD}{dt} = \frac{9(4800 + 4500)}{100\sqrt{8674}} = \frac{837}{\sqrt{8674}} \text{ ft/sec.}$$

7. (a) Find an equation for the line tangent to the curve $x^3 - 4x^2y + 2xy^3 + 4 = 0$ at the point whose coordinates are $(2, 1)$.

   (b) Use the result of part (a) of this problem to find an approximate value for the y-coordinate of the point $(2.04, y)$ that lies on the curve $x^3 - 4x^2y + 2xy^3 + 4 = 0$ near the point $(2, 1)$.

Solution:

(a) Differentiating the equation $x^3 - 4x^2y + 2xy^3 + 4 = 0$ implicitly with respect to x, we find that

$$\begin{aligned} 3x^2 - 8xy - 4x^2y' + 2y^3 + 6xy^2y' &= 0; \\ y' &= \frac{8xy - 3x^2 - 2y^3}{6xy^2 - 4x^2}; \\ y'\Big|_{(2,1)} &= -\frac{1}{2}, \end{aligned}$$

and the equation of the required tangent line is $y = 1 - \dfrac{1}{2}(x-2)$.

(b) The equation $x^3 - 4x^2y + 2xy^3 + 4 = 0$ defines y implicitly as a function φ of x near the point $(2, 1)$; we have just computed the linearization of φ at $x = 2$. When x is near 2, we may estimate $\varphi(x)$ by using the linearization in its place. Consequently,

$$\varphi(2.04) \sim 1 - \frac{1}{2}(2.04 - 2) = 1 - 0.02 = 0.98.$$

MTH 1410, Calculus I—Exam III **April 20, 2011**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 1:50 pm.

1. Find the limits:

   (a) $\lim_{x \to -1} \dfrac{x + x^2}{\ln(2 + x)}$

   (b) $\lim_{x \to 0} \dfrac{xe^{-2x}}{\pi e^{2x} - \pi}$

2. Use the definition of derivative to find $f'(x)$ when $f(x) = \dfrac{x}{x+1}$.

3. Find the absolute maximum and the absolute minimum for the function

$$f(x) = 2x^3 - 3x^2 - 12x + 20$$

   on the interval $[-3, 3]$.

4. The equation $2x = 3\sin x$ has a *positive* root. Show how to use Newton's method to give an approximation for that root which is correct to at least six digits to the right of the decimal. [Warning: Do not forget that we always use *radians* in calculus.]

5. Let F be the function given by

$$F(x) = (x - 1)^2(x + 1)^3.$$

   Then, in fully factored form,

$$F'(x) = (x - 1)(x + 1)^2(5x - 1)$$

   and, also in fully factored form,

$$F''(x) = 20(x + 1)\left[x - \frac{1}{5}\left(1 - \sqrt{6}\right)\right]\left[x - \frac{1}{5}\left(1 + \sqrt{6}\right)\right].$$

   Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

6. Find the points on the ellipse $x^2 + 4y^2 = 4$ whose distance from the point $(1, 0)$ is minimal.

7. Murgatroyd was driving his car toward an intersection at 60 miles per hour. A police cruiser was approaching the same intersection but on the cross-street (which is at right angles to the road that Muratroyd is on), at 50 miles per hour. When both cars were a quarter of a mile from the intersection, a police officer in the cruiser pointed a radar gun at Murgatroyd and measured the speed at which the two cars were approaching each other. What did she get?

MTH 1410, Calculus I—Exam III **April 20, 2011**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 1:50 pm.

1. Find the limits:

   (a) $\displaystyle\lim_{x\to -1}\frac{x+x^2}{\ln(2+x)}$

   (b) $\displaystyle\lim_{x\to 0}\frac{xe^{-2x}}{\pi e^{2x}-\pi}$

   Solution:

   (a) $\lim_{x\to -1}(x+x^2) = 0 = \lim_{x\to -1}\ln(2+x)$, so we may attempt l'Hôpital's rule:

$$\begin{aligned}\lim_{x\to -1}\frac{x+x^2}{\ln(2+x)} &= \lim_{x\to -1}\frac{1+2x}{[1/(2+x)]}\\ &= \lim_{x\to -1}(1+2x)(2+x)\\ &= -1.\end{aligned}$$

   (b) $\lim_{x\to 0} xe^{-2x} = 0 = \lim_{x\to 0}(\pi e^{2x}-\pi)$, so we can attempt l'Hôpital's rule again:

$$\begin{aligned}\lim_{x\to 0}\frac{xe^{-2x}}{\pi e^{2x}-\pi} &= \lim_{x\to 0}\frac{e^{-2x}-2xe^{-2x}}{2\pi e^{2x}}\\ &= \frac{1}{2\pi}.\end{aligned}$$

2. Use the definition of derivative to find $f'(x)$ when $f(x) = \dfrac{x}{x+1}$.

   Solution:

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{h}\left[\frac{x+h}{(x+h)+1}-\frac{x}{x+1}\right]\\ &= \lim_{h\to 0}\frac{(x+h)(x+1)-x(x+h+1)}{h(x+1)(x+h+1)} = \lim_{h\to 0}\frac{\cancel{x^2}+\cancel{hx}+\cancel{x}+h-\cancel{x^2}-\cancel{xh}-\cancel{x}}{h(x+1)(x+h+1)}\\ &= \lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(x+1)(x+h+1)} = \lim_{h\to 0}\frac{1}{(x+1)(x+h+1)} = \frac{1}{(x+1)^2}.\end{aligned}$$

3. Find the absolute maximum and the absolute minimum for the function

$$f(x) = 2x^3 - 3x^2 - 12x + 20$$

   on the interval $[-3, 3]$.

   Solution: Absolute extrema are to be found only at endpoints and critical numbers. We have $f'(x) = 6x^2 - 6x - 12 = 6(x+1)(x-2)$, which is defined everywhere and is zero only when $x = -1$ or $x = 2$. Thus, the extrema are among the numbers $f(-3)$, $f(-1)$, $f(2)$, and $f(3)$. We find that $f(-3) = -25$, $f(-1) = 27$, $f(2) = 0$, and $f(3) = 11$. The absolute minimum is $f(-3) = -25$, and the absolute maximum is $f(-1) = 27$.

4. The equation $2x = 3\sin x$ has a *positive* root. Show how to use Newton's method to give an approximation for that root which is correct to at least six digits to the right of the decimal. [Warning: Do not forget that we always use *radians* in calculus.]

Solution: A quick sketch (see Figure 1) indicates that the desired root is close to $x = 1.5$, so

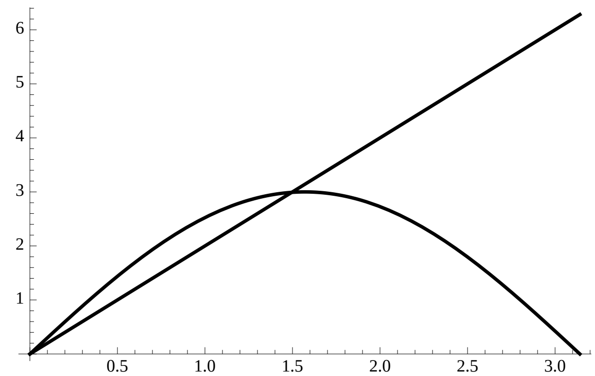


Figure 1: Plot for Problem 4

let us take $x_0 = 1.5$. The equation $2x = 3\sin x$ is equivalent to the equation $f(x) = 0$ with $f(x) = 2x - 3\sin x$, so the Newton's Method iteration scheme

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

becomes

$$x_{k+1} = x_k - \frac{2x_k - 3\sin x_k}{2 - 3\cos x_k}.$$

Thus,

$$\begin{aligned} x_1 &= 1.5 - \frac{3.0 - 3\sin(1.5)}{2 - 3\cos(1.5)} = 1.49579646; \\ x_2 &= x_1 - \frac{2x_1 - 3\sin x_1}{2 - 3\cos x_1} = 1.49578157; \\ x_3 &= x_2 - \frac{2x_2 - 3\sin x_2}{2 - 3\cos x_2} = 1.49578157; \end{aligned}$$

Our approximate solution, correct to 8 digits to the right of the decimal, is 1.49578157.

5. Let F be the function given by

$$F(x) = (x-1)^2(x+1)^3.$$

Then, in fully factored form,

$$F'(x) = (x-1)(x+1)^2(5x-1)$$

and, also in fully factored form,

$$F''(x) = 20(x+1)\left[x - \frac{1}{5}\left(1 - \sqrt{6}\right)\right]\left[x - \frac{1}{5}\left(1 + \sqrt{6}\right)\right].$$

Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

Solution: The quantity $(x-1)$ is positive when $x > 1$ and negative when $x < 1$; $(x+1)^2$ is positive unless $x = -1$; and $(5x-1)$ is positive when $x > 1/5$, negative when $x < 1/5$. Thus, $F'(x) > 0$ when $-\infty < x < -1$, when $-1 < x < 1/5$ and when $1 < x < \infty$. It follows that F is increasing on the intervals $(-\infty, 1/5]$ and $[1, \infty)$, but decreasing on $[1/5, 1]$.

The quantity $(x+1)$ is negative when $x < -1$ and positive when $x > -1$; $[x-(1-\sqrt{6})/5]$ is negative when $x < (1-\sqrt{6})/5$ and positive when $x > (1-\sqrt{6})/5$; $[x-(1+\sqrt{6})/5]$ is negative when $x < (1+\sqrt{6})/5$ and positive when $x > (1+\sqrt{6})/5$. Consequently $F''(x) < 0$ when $-\infty < x < -1$ and when $(1-\sqrt{6})/5 < x < (1+\sqrt{6})/5$, but $F''(x) > 0$ when $-1 < x < (1-\sqrt{6})/5$ and when $(1+\sqrt{6})/5 < x < \infty$. So F is concave upward on $[-1, (1-\sqrt{6})/5]$ and on $[(1+\sqrt{6})/5, \infty)$, but concave downward on $(-\infty, -1]$ and on $[(1-\sqrt{6})/5, (1+\sqrt{6})/5]$.

From the expression for $F'(x)$, we see that the critical numbers for F are $x = 1$, $x = -1$, and $x = 1/5$. Our analysis of the increasing/decreasing behavior of F above, shows that $x = 1$ gives a local minimum for F because F is decreasing just to the left of $x = 1$ but increasing just to the right of $x = 1$. Similarly, $x = 1/5$ gives a local maximum for F, and $x = -1$ gives neither a local maximum nor a local minimum.

6. Find the points on the ellipse $x^2 + 4y^2 = 4$ whose distance from the point $(1, 0)$ is minimal.

Solution: Let (x, y) be any point on the curve. Then $x^2 + 4y^2 = 4$ and the square S of the distance from (x, y) to $(1, 0)$ is $S = (x-1)^2 + y^2$, where $|x| \leq 2$. We can minimize distance by minimizing $S = (x-1)^2 + y^2$ subject to the constraint $x^2 + 4y^2 = 1$. Thus we want to find the critical points of S. Treating y as a function of x and differentiating, we find that $dS/dx = 2(x-1) + 2yy'$, so want to learn where $2(x-1) + 2yy' = 0$. From $x^2 + 4y^2 = 4$, we see that $2x + 8yy' = 0$, or $y' = -x/(4y)$. Thus, we want

$$\begin{aligned} 0 &= 2(x-1) + 2yy' \\ &= x - 1 + y\left(-\frac{x}{4y}\right) \\ &= x - 1 - \frac{1}{4}x \\ &= \frac{3}{4}x - 1, \end{aligned}$$

so that

$$x = \frac{4}{3}.$$

The only critical number for $S(x)$ is thus at $x = 4/3$. The minimum for $S(x)$ must occur either at $x = 4/3$ or at an endpoint $x = \pm 2$. We note that from $x^2 + 4y^2 = 1$ it follows that $y = \pm\sqrt{5}/3$ when $x = 4/3$ and that $y = 0$ when $x = \pm 2$. We therefore have $S(-2) = 9$, $S(4/3) = 2/3$, and $S(2) = 1$. The minimal distance therefore occurs when $x = 4/3$ and $y = \pm\sqrt{5}/3$—that is, at the points $(4/3, \sqrt{5}/3)$ and $(4/3, -\sqrt{5}/3)$.

7. Murgatroyd was driving his car toward an intersection at 60 miles per hour. A police cruiser was approaching the same intersection but on the cross-street (which is at right angles to the

road that Muratroyd is on), at 50 miles per hour. When both cars were a quarter of a mile from the intersection, a police officer in the cruiser pointed a radar gun at Murgatroyd and measured the speed at which the two cars were approaching each other. What did she get?

Solution: Let x denote the distance from Murgatroyd to the intersection, and let y denote the distance from the police car to the interesection. The distance D between the two cars satisfies

$$D^2 = x^2 + y^2,$$

so

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt},$$

or

$$\frac{dD}{dt} = \frac{1}{D}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

At the critical instant, we have $x = y = 1/4$, $D = 1/(2\sqrt{2})$, $dx/dt = -60$, and $dy/dt = -50$. Thus, rate of change of the distance between the two cars was

$$\begin{aligned}\frac{dD}{dt} &= 2\sqrt{2}\left[\frac{1}{4}(-60) + \frac{1}{4}(-50)\right] \\ &= -\frac{110}{\sqrt{2}} \text{ mph.}\end{aligned}$$

The reading on the radar gun was $110/\sqrt{2}$ mph, or about 78 mph.

MTH 1410, Calculus I—Final Exam **May 9, 2011**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 12:50 pm.

1. (a) *Use the definition of the derivative* to find $f'(x)$ if $f(x) = 1/\sqrt{x}$.

   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = 1/\sqrt{x}$ at $x = 1$, at $x = 4$, and at $x = 9$.

2. Evaluate the following definite integrals. Give all of your reasoning.

   (a) $\displaystyle\int_3^5 (3x^2 - 24x + 54)\, dx$

   (b) $\displaystyle\int_0^3 3t\sqrt{9 - t^2}\, dt$

3. Let F be the function given by

$$F(x) = (x-1)(x+1)(x-3)^3.$$

   Then, in fully factored form,

$$F'(x) = 5(x-3)^2 \left[x - \frac{1}{5}(3 - 2\sqrt{6})\right]\left[x - \frac{1}{5}(3 + 2\sqrt{6})\right]$$

   and, also in fully factored form,

$$F''(x) = 20(x-3)\left[x - \frac{1}{5}\left(6 - \sqrt{21}\right)\right]\left[x - \frac{1}{5}\left(6 + \sqrt{21}\right)\right].$$

   Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

4. Suppose that $f(2) = 2$, $f(4) = 4$, $f'(2) = 4$, $f'(4) = -2$, $g(2) = 4$, $g(4) = 2$, $g'(2) = -6$, and $g'(4) = -8$.

   (a) Find $F(4)$ and $F'(4)$, where $F(x) = \dfrac{f(x)}{g(x)}$.

   (b) Find $G(2)$ and $G'(2)$, where $G(x) = g[2f(x)]$.

   (c) Find $H(2)$ and $H'(2)$, where $H(x) = g[f(x^2)]$.

5. (a) Show that the point $(3, 2)$ lies on the curve given by the equation

$$x^3 - 5x^2y^3 + 8y^4 + 205 = 0.$$

   (b) If x and y are related by the equation, $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$, find the value of y' at $(3, 2)$.

   (c) Show how to use the results of parts (a) and (b) of this problem to find an approximate value for y near 2 when $x = 74/25$.

6. Find the points on the hyperbola $4y^2 - x^2 = 1$ whose distance from the point $(5, 0)$ is minimal.

MTH 1410, Calculus I—Final Exam **May 9, 2011**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 1:50 pm.

1. (a) *Use the definition of the derivative* to find $f'(x)$ if $f(x) = 1/\sqrt{x}$.
   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = 1/\sqrt{x}$ at $x = 1$, at $x = 4$, and at $x = 9$.

   Solution:

   (a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right] \\ &= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \\ &= \lim_{h\to 0} \frac{x - (x+h)}{h\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} \\ &= -\lim_{h\to 0} \frac{1}{\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} = -\frac{1}{2x\sqrt{x}}. \end{aligned}$$

   (b) From the immediately preceding equation, $f'(1) = -1/2$, so the equation of the tangent line at $x = 1$ is $y = 1 - (x-1)/2$, or $x + 2y = 3$. $f'(4) = -1/16$, so the equation of the tangent line at $x = 4$ is $y = (1/2) - (1/16)(x-4)$, or $x + 16y = 12$. $f'(9) = -1/54$; the equation of the tangent line at $x = 9$ is $y = (1/3) - (1/54)(x-9)$, or $x + 54y = 27$.

2. Evaluate the following definite integrals. Give all of your reasoning.

   (a) $\displaystyle\int_3^5 (3x^2 - 24x + 54)\,dx$

   (b) $\displaystyle\int_0^3 3t\sqrt{9 - t^2}\,dt$

   Solution:

   (a)

$$\begin{aligned} \int_3^5 (3x^2 - 24x + 54)\,dx &= (x^3 - 12x^2 + 54x)\Big|_3^5 \\ &= (125 - 300 + 270) - (27 - 108 + 162) = 14. \end{aligned}$$

   (b) Let $u = 9 - t^2$. Then $du = -2t\,dt$, or $t\,dt = -(1/2)\,du$. Moreover, $t = 0 \Rightarrow u = 9$ and $t = 3 \Rightarrow u = 0$. Thus

$$\begin{aligned} \int_0^3 3t\sqrt{9 - t^2}\,dt &= -\frac{3}{2}\int_9^0 u^{1/2}\,du = \frac{3}{2}\int_0^9 u^{1/2}\,du \\ &= u^{3/2}\Big|_0^9 = 9^{3/2} - 0^{3/2} = 27. \end{aligned}$$

3. Let F be the function given by

$$F(x) = (x-1)(x+1)(x-3)^3.$$

Then, in fully factored form,

$$F'(x) = 5(x-3)^2\left[x - \frac{1}{5}(3-2\sqrt{6})\right]\left[x - \frac{1}{5}(3+2\sqrt{6})\right]$$

and, also in fully factored form,

$$F''(x) = 20(x-3)\left[x - \frac{1}{5}\left(6-\sqrt{21}\right)\right]\left[x - \frac{1}{5}\left(6+\sqrt{21}\right)\right].$$

Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

Solution: For brevity, let $r_1 = (3-2\sqrt{6})/5$, $r_2 = (3+2\sqrt{6})/5$, and note that $r_1 < r_2 < 3$. The quantity $(x - r_1)$ is positive when $x > r_1$, zero when $x = r_1$ and negative when $x < r_1$. The quantity $(x - r_2)$ is positive when $x > r_2$, zero when $x = r_2$ and negative when $x < r_2$. The quantity $(x-3)^2$ is positive for all $x \neq 3$ and is zero when $x = 3$. Consequently, $F'(x) > 0$ on $(-\infty, r_1)$, on $(r_2, 3)$ and on $(3, \infty)$; $F'(x) < 0$ on (r_1, r_2). Therefore, F is increasing on $(-\infty, r_1)$ and on (r_2, ∞), and F is decreasing on (r_1, r_2).

Again for brevity, we let $s_1 = (6-\sqrt{21})/5$, $s_2 = (6+\sqrt{21})/5$, and we note that $s_1 < s_2 < 3$. Now we have $(x - s_1) > 0$ when $x < s_1$, zero when $x = s_1$, and $x = s_2$. The quantity $(x - s_2)$ is negative when $x < s_2$, zero when $x = s_2$, and positive when $x > s_2$. The quantity $(x-3)$ is negative when $x < 3$, zero when $x = 3$, and positive when $x > 3$. So $F''(x) < 0$ in (∞, s_1), $F''(x) > 0$ in (s_1, s_2), $F''(x) < 0$ in $(s_2, 3)$, and $F''(x) > 0$ in $(3, \infty)$. We conclude that F is concave downward on $(-\infty, s_1)$ and on $(s_2, 3)$, but concave upward on (s_1, s_2) and on $(3, \infty)$.

From the expression for $F'(x)$, we see that the critical numbers for F are $x = r_1$, $x = r_2$, and $x = 3$. Our analysis of the increasing/decreasing behavior of F above, shows that $x = r_1$ gives a local maximum for F because F is increasing just to the left of $x = r_1$ but decreasing just to the right of $x = r_1$. Similarly, $x = r_2$ gives a local minimum for F, and $x = 3$ gives neither a local maximum nor a local minimum.

4. Suppose that $f(2) = 2$, $f(4) = 4$, $f'(2) = 4$, $f'(4) = -2$, $g(2) = 4$, $g(4) = 2$, $g'(2) = -6$, and $g'(4) = -8$.

   (a) Find $F(4)$ and $F'(4)$, where $F(x) = \dfrac{f(x)}{g(x)}$.

   (b) Find $G(2)$ and $G'(2)$, where $G(x) = g[2f(x)]$.

   (c) Find $H(2)$ and $H'(2)$, where $H(x) = g[f(x^2)]$.

Solution:

(a)

$$F(4) = \frac{f(4)}{g(4)} = 2;$$
$$F'(4) = \frac{f'(4)g(4) - f(4)g'(4)}{[g(4)]^2} = 7.$$

(b)

$$G(2) = g[2f(2)] = 2;$$
$$G'(x) = g'[2f(x)]D_x[2f(x)] = 2g'[2f(x)]f'(x),$$

so

$$G'(2) = 2g'[2f(2)]f'(2) = -64.$$

(c)

$$\begin{aligned} H(2) &= g[f(2)] = 2; \\ H'(x) &= g'[f(x^2)] \cdot D_x f(x^2) = g'[f(x^2)]f'(x^2)D_x x^2 = 2xg'[f(x^2)]f'(x^2), \end{aligned}$$

so

$$H'(2) = 4g'[f(4)]f'(4) = 64.$$

5. (a) Show that the point $(3,2)$ lies on the curve given by the equation

$$x^3 - 5x^2y^3 + 8y^4 + 205 = 0.$$

(b) If x and y are related by the equation, $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$, find the value of y' at $(3,2)$.

(c) Show how to use the results of parts (a) and (b) of this problem to find an approximate value for y near 2 when $x = 74/25$.

Solution:

(a) When $x = 3$ and $y = 2$, we have

$$3^3 - 5 \cdot 3^2 \cdot 2^3 + 8 \cdot 2^4 + 205 = 27 - 360 + 128 + 205 = 0,$$

so the point with coordinates $(3,2)$ lies on the curve whose equation is $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$.

(b) Treating y as a function of x and differentiating implicitly gives

$$\begin{aligned} 3x^2 - 10xy^3 - 15x^2y^2y' + 32y^3y' &= 0; \\ (32y^3 - 15x^2y^2)y' &= 10xy^3 - 3x^2; \\ y' &= \frac{10xy^3 - 3x^2}{32y^3 - 15x^2y^2}. \end{aligned}$$

Thus,

$$y'\bigg|_{(3,2)} = \frac{10 \cdot 3 \cdot 2^3 - 3 \cdot 3^2}{32 \cdot 2^3 - 15 \cdot 3^2 \cdot 2^2} = \frac{240 - 27}{256 - 540} = \frac{213}{-284} = -\frac{3}{4}.$$

(c) From the previous part of this problem, we know that the equation of the line tangent to the curve at $(3,2)$ is

$$y = 2 - \frac{3}{4}(x - 3).$$

When a point (x_0, y_0) lies near $(3,2)$ on the curve $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$, it lies near the line tangent to the curve at $(3,2)$. Thus, we can approximate the value of y near 2 that satisfies the equation

$$\left(\frac{74}{25}\right)^3 - 5\left(\frac{74}{25}\right)^2 y^3 + 8y^4 + 205 = 0$$

as

$$y \sim 2 - \frac{3}{4}\left(\frac{74}{25} - 3\right) = 2 + \frac{3}{4} \cdot \frac{1}{25} = \frac{203}{100}.$$

6. Find the points on the hyperbola $4y^2 - x^2 = 1$ whose distance from the point $(5, 0)$ is minimal.

Solution: The distance D from a point (x, y) to the point $(5, 0)$ is given by

$$D^2 = (x - 5)^2 + y^2.$$

If the point (x, y) lies on the given hyperbola, then $4y^2 - x^2 = 1$, so that $y^2 = (1 + x^2)/4$. Substituting this latter equation into the relation for D, we find that

$$D^2 = (x - 5)^2 + \frac{1}{4}(1 + x^2) = \frac{5x^2 - 40x + 101}{4}.$$

We have

$$2D\frac{dD}{dx} = \frac{10x - 40}{4},$$

so $\dfrac{dD}{dx} = 0$ when $x = 4$. This is the only critical point, and we know from the geometry of the situation that there must be a minimum. Consequently, it must lie at a point on the curve where $x = 4$. This requires that $4y^2 - 16 = 1$, or that $y = \pm\sqrt{17}/2$. There are thus two points on the hyperbola whose distance from $(5, 0)$ is minimal: $(4, \pm\sqrt{17}/2)$.

MTH 1410, Calculus I—Exam II **March 14, 2012**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 3:50 pm. Complete solutions to the exam problems will be available from the course web-site later this evening.

1. Show how to use the Limit Laws to evaluate the following limits. *Do not use l'Hôpital's Rule.*

   (a) $\displaystyle\lim_{x\to 3} \frac{x^2-4x+3}{x^3-6x^2+12x-9}$

   (b) $\displaystyle\lim_{x\to\infty} \left[\sqrt{x^2-4x+4}-x\right]$

2. Let f be the function given by the equation $f(x)=\dfrac{1}{x+2}$.

   (a) Show how to use the definition of the derivative to find $f'(a)$, where it is given only that a is some real number other than -2.

   (b) Use the value you have just found for $f'(a)$ of this problem to write an equation for the line tangent to the curve $y=\dfrac{1}{x+2}$ at the point where $x=-1$.

3. Let f and g be differentiable functions for which $f(3)=3$, $g(3)=5$, $f'(3)=4$, and $g'(3)=-2$. Find the following values, being sure to show your reasoning.

   (a) $S'(3)$, where $S(x)=f(x)+g(x)$.

   (b) $P'(3)$, where $P(x)=f(x)g(x)$.

   (c) $Q'(3)$, where $Q(x)=\dfrac{g(x)}{f(x)}$.

   (d) $F'(3)$, where $F(x)=f(x)e^{g(x)}$.

   (e) $G'(3)$, where $G(x)=g[f(x)]$.

4. Evaluate the following limits. You may use l'Hôpital's Rule if it is applicable. Be sure to give your reasoning.

   (a) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x+\cos x}$

   (b) $\displaystyle\lim_{x\to 0} \frac{2\ln(1+x)-2x+x^2}{x^3}$

5. A culture of bacteria starts at time $t=0$ with 760 bacteria and grows at a rate that is proportional to its size. After 5 hours (that is, when $t=5$) there are 3800 bacteria in the culture.

   (a) Express the population as a function of t.

   (b) What will the population be when $t=6$?

   (c) How long will it take for the population to reach 2170?

6. A boat is pulled into a dock by a rope which is attached to its bow and which passes through a pulley on the dock that is one meter higher than the bow of the boat. If the rope is pulled in at a rate of half a meter per second, how fast is the boat approaching the dock when it is nine meters away from the dock?

7. (a) Find an equation for the line tangent to the curve $x^3-4x^2y+2xy^3+4=0$ at the point whose coordinates are $(2,1)$.

   (b) Use the result of part (a) of this problem to find an approximate value for the y-coordinate of the point $(2.04, y)$ that lies on the curve $x^3-4x^2y+2xy^3+4=0$ near the point $(2,1)$.

MTH 1410, Calculus I—Exam II **March 14, 2012**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 3:50 pm. Complete solutions to the exam problems will be available from the course web-site later this evening.

1. Show how to use the Limit Laws to evaluate the following limits. *Do not use l'Hôpital's Rule.*

   (a) $\displaystyle\lim_{x\to 3} \frac{x^2-4x+3}{x^3-6x^2+12x-9}$

   (b) $\displaystyle\lim_{x\to\infty} \left[\sqrt{x^2-4x+4}-x\right]$

   Solution:

   (a)

$$\lim_{x\to 3} \frac{x^2-4x+3}{x^3-6x^2+12x-9} = \lim_{x\to 3} \frac{\cancel{(x-3)}(x-1)}{\cancel{(x-3)}(x^2-3x+3)} = \frac{\lim_{x\to 3}(x-1)}{\lim_{x\to 3}(x^2-3x+3)} = \frac{2}{3}.$$

   (b)

$$\begin{aligned}\lim_{x\to\infty}\left[\sqrt{x^2-4x+4}-x\right] &= \lim_{x\to\infty}\left[\frac{(x^2-4x+4)-x^2}{\sqrt{x^2-4x+4}+x}\right] \\ &= \lim_{x\to\infty}\left[\frac{-4x+4}{\sqrt{x^2-4x+4}+x}\right] \\ &= \lim_{x\to\infty}\left[\frac{-4+(4/x)}{\sqrt{1-(4/x)+(4/x^2)}+1}\right] \\ &= \frac{\lim_{x\to\infty}\left[-4+(4/x)\right]}{\lim_{x\to\infty}\left[\sqrt{1-(4/x)+(4/x^2)}+1\right]} \\ &= \frac{-4}{1+1} = -2.\end{aligned}$$

2. Let f and g be differentiable functions for which $f(3)=3$, $g(3)=5$, $f'(3)=4$, and $g'(3)=-2$. Find the following values, being sure to show your reasoning.

   (a) $S'(3)$, where $S(x)=f(x)+g(x)$.

   (b) $P'(3)$, where $P(x)=f(x)g(x)$.

   (c) $Q'(3)$, where $Q(x)=\dfrac{g(x)}{f(x)}$.

   (d) $F'(3)$, where $F(x)=f(x)e^{g(x)}$.

   (e) $G'(3)$, where $G(x)=g[f(x)]$.

   Solution:

   (a) $S'(3)=f'(3)+g'(3)=4+(-2)=2$.

   (b) $P'(3)=f'(3)g(3)+f(3)g'(3)=4\cdot 5+3\cdot(-2)=14$.

   (c) $Q'(3)=\dfrac{g'(3)f(3)-g(3)f'(3)}{[f(3)]^2}=\dfrac{(-2)\cdot 3-5\cdot 4}{(3)^2}=-\dfrac{26}{9}$.

(d) $F'(3) = f'(3)e^{g(3)} + f(3)g'(3)e^{g(3)} = [f'(3) + f(3)g'(3)]e^{g(3)} = [4 + 3 \cdot (-2)]e^5 = -2e^5$.

(e) $G'(3) = g'[f(3)]f'(3) = g'(3)f'(3) = (-2) \cdot (4) = -8$.

3. Let f be the function given by the equation $f(x) = \dfrac{1}{x+2}$.

(a) Show how to use the definition of the derivative to find $f'(a)$, where it is given only that a is some real number other than -2.

(b) Use the value you have just found for $f'(a)$ of this problem to write an equation for the line tangent to the curve $y = \dfrac{1}{x+2}$ at the point where $x = -1$.

Solution:

(a)

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(a+h)+2} - \frac{1}{a+2}\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{(a+2) - (a+h+2)}{(a+h+2)(a+2)}\right] = -\lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(a+h+2)(a+2)} \\ &= -\frac{1}{(a+2)^2}. \end{aligned}$$

(b) From the previous part of this problem, $f'(-1) = -1$. Consequently, an equation for the line tangent to the curve at the point $(-1, 1)$ where $x = -1$ is $y = 1 - (x+1)$, or just $y = -x$.

4. Evaluate the following limits. You may use l'Hôpital's Rule if it is applicable. Be sure to give your reasoning.

(a) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x + \cos x}$

(b) $\displaystyle\lim_{x\to 0} \frac{2\ln(1+x) - 2x + x^2}{x^3}$

Solution:

(a) When $x \to 0$, the numerator $\to 0$, but the denominator $\to 1 \neq 0$. L'Hôpital's Rule is not applicable, and

$$\lim_{x\to 0} \frac{\sin x}{x + \cos x} = 0.$$

(b) When $x \to 0$, numerator and denominator both $\to 0$, so we may try l'Hôpital's Rule:

$$\lim_{x\to 0} \frac{2\ln(1+x) - 2x + x^2}{x^3} = \lim_{x\to 0} \frac{2/(1+x) - 2 + 2x}{3x^2},$$

provided this latter limit exists. But

$$\lim_{x\to 0} \frac{2/(1+x) - 2 + 2x}{3x^2} = \lim_{x\to 0} \frac{\cancel{2} - \cancel{2} + 2x^2}{3x^2(1+x)}) = \lim_{x\to 0} \frac{2\cancel{x^2}}{3\cancel{x^2}(1+x)} = \frac{2}{3}.$$

Hence, $\displaystyle\lim_{x\to 0} \frac{2\ln(1+x) - 2x + x^2}{x^3} = \frac{2}{3}$.

5. A culture of bacteria starts at time $t = 0$ with 760 bacteria and grows at a rate that is proportional to its size. After 5 hours (that is, when $t = 5$) there are 3800 bacteria in the culture.

(a) Express the population as a function of t.

(b) What will the population be when $t = 6$?

(c) How long will it take for the population to reach 2170?

Solution:

(a) We know that, for such growth, population, $P(t)$ is given by an equation of the form $P(t) = P_0 e^{kt}$, where P_0 is the initial population—which is 760 in this instance. Thus, $P(t) = 760e^{kt}$, and it remains for us to find k. But when $t = 5$ we have $3800 = P(5) = 760e^{5k}$, whence we conclude that $e^{5k} = 3800/760 = 5$, so that $5k = \ln 5$ and $k = (\ln 5)/5$. Thus,

$$P(t) = 760 \exp\left[\frac{t \ln 5}{5}\right] = 760 \cdot 5^{t/5}.$$

(b) When $t = 6$, population will be

$$P(6) = 760 \exp\left[\frac{6 \ln 5}{5}\right] = 760 \cdot 5^{6/5} = 3800 \cdot 5^{1/5}.$$

(c) To learn when $P(t) = 2170$, we must solve the equation $760 \cdot 5^{t/5} = 2170$ for t:

$$\begin{aligned} 760 \cdot 5^{t/5} &= 2170; \\ 5^{t/5} &= \frac{2170}{760} = \frac{217}{76}; \\ \frac{t}{5} &= \log_5(217/76) = \frac{\ln(217/76)}{\ln 5}; \\ t &= \frac{5\ln(217/76)}{\ln 5}. \end{aligned}$$

6. A boat is pulled into a dock by a rope which is attached to its bow and which passes through a pulley on the dock that is one meter higher than the bow of the boat. If the rope is pulled in at a rate of half a meter per second, how fast is the boat approaching the dock when it is nine meters away from the dock?

Solution: Let x denote the distance between the boat and the dock; let h denote the length of rope between the pulley and the boat. By the Pythagorean Theorem, $h^2 = 1^2 + x^2$. We differentiate this latter equation implicitly with respect to time to obtain:

$$\begin{aligned} 2h\frac{dh}{dt} &= 2x\frac{dx}{dt}, \text{ or} \\ \frac{dx}{dt} &= \frac{h}{x}\frac{dh}{dt}. \end{aligned}$$

We are given that $dh/dt = -1/2$, and, at the critical instant, $x = 9$. Because $h^2 = 1 + x^2$, we must have $h = \sqrt{82}$ at that instant. Hence

$$\frac{dx}{dt} = -\frac{h}{x}\frac{dh}{dt} = -\frac{1}{2} \cdot \frac{\sqrt{82}}{9} = -\frac{\sqrt{82}}{18} \text{ m/sec.}$$

The boat is therefore approaching the dock at $\sqrt{82}/18$ m/sec at the critical instant.

7. (a) Find an equation for the line tangent to the curve $x^3 - 4x^2y + 2xy^3 + 4 = 0$ at the point whose coordinates are $(2, 1)$.

(b) Use the result of part (a) of this problem to find an approximate value for the y-coordinate of the point $(2.04, y)$ that lies on the curve $x^3 - 4x^2y + 2xy^3 + 4 = 0$ near the point $(2, 1)$.

Solution:

(a) Differentiating the equation $x^3 - 4x^2y + 2xy^3 + 4 = 0$ implicitly with respect to x, we find that

$$3x^2 - 8xy - 4x^2y' + 2y^3 + 6xy^2y' = 0;$$

$$y' = \frac{8xy - 3x^2 - 2y^3}{6xy^2 - 4x^2};$$

$$y'\Big|_{(2,1)} = -\frac{1}{2},$$

and the equation of the required tangent line is $y = 1 - \frac{1}{2}(x - 2)$.

(b) The equation $x^3 - 4x^2y + 2xy^3 + 4 = 0$ defines y implicitly as a function φ of x near the point $(2, 1)$; we have just computed the linearization of φ at $x = 2$. When x is near 2, we may estimate $\varphi(x)$ by using the linearization in its place. Consequently,

$$\varphi(2.04) \sim 1 - \frac{1}{2}(2.04 - 2) = 1 - 0.02 = 0.98.$$

MTH 1410, Calculus I—Exam III **April 18, 2012**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Give your answers in exact form unless the nature of a problem requires you to do otherwise. Your exam is due at 3:50 pm.

1. Find the limits:

   (a) $\displaystyle\lim_{x\to -1} \frac{x+x^2}{\ln(2+x)}$

   (b) $\displaystyle\lim_{x\to 0} \frac{xe^{-2x}}{\pi e^{2x}-\pi}$

2. Show how to use the definition of derivative and the Limit Laws to find $f'(x)$ when $f(x) = \dfrac{x}{x+1}$.

3. Find the absolute maximum and the absolute minimum for the function

$$f(x) = 2x^3 - 3x^2 - 12x + 20$$

   on the interval $[-3, 3]$.

4. The equation $x^3 + 1 = x$ has a root in the interval $[-2, -1]$. Show how to use Newton's method to give an approximation for that root which is correct to at least six digits to the right of the decimal.

5. Let F be the function given by

$$F(x) = (x-2)^2(x+2)^3.$$

   Then, in fully factored form,

$$F'(x) = (x-2)(x+2)^2(5x-2)$$

   and, also in fully factored form,

$$F''(x) = 20(x+2)\left[x - \frac{2}{5}\left(1-\sqrt{6}\right)\right]\left[x - \frac{2}{5}\left(1+\sqrt{6}\right)\right].$$

   Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

6. Find the points on the hyperbola $x^2 - 4y^2 = 4$ whose distance from the point $(0, 1)$ is minimal.

7. Murgatroyd was driving his car toward an intersection at 60 miles per hour. A police cruiser was approaching the same intersection but on the cross-street (which is at right angles to the road that Muratroyd is on), at 50 miles per hour. When both cars were a quarter of a mile from the intersection, a police officer in the cruiser pointed a radar gun at Murgatroyd and measured the speed at which the two cars were approaching each other. What did she get?

MTH 1410, Calculus I—Exam III **April 18, 2012**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Give your answers in exact form unless the nature of a problem requires you to do otherwise. Your exam is due at 3:50 pm.

1. Find the limits:

   (a) $\displaystyle\lim_{x\to -1} \frac{x+x^2}{\ln(2+x)}$

   (b) $\displaystyle\lim_{x\to 0} \frac{xe^{-2x}}{\pi e^{2x}-\pi}$

   Solution:

   (a) $\lim_{x\to -1}(x+x^2) = 0 = \lim_{x\to -1}\ln(2+x)$, so we may attempt l'Hôpital's rule:

$$\lim_{x\to -1} \frac{x+x^2}{\ln(2+x)} = \lim_{x\to -1} \frac{1+2x}{[1/(2+x)]} = \lim_{x\to -1}(1+2x)(2+x) = -1.$$

   (b) $\lim_{x\to 0} xe^{-2x} = 0 = \lim_{x\to 0}(\pi e^{2x}-\pi)$, so we can attempt l'Hôpital's rule again:

$$\lim_{x\to 0} \frac{xe^{-2x}}{\pi e^{2x}-\pi} = \lim_{x\to 0} \frac{e^{-2x}-2xe^{-2x}}{2\pi e^{2x}} = \frac{1}{2\pi}.$$

2. Show how to use the definition of derivative and the Limit Laws to find $f'(x)$ when $f(x) = \dfrac{x}{x+1}$.

   Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{x+h}{(x+h)+1} - \frac{x}{x+1}\right] \\ &= \lim_{h\to 0} \frac{(x+h)(x+1)-x(x+h+1)}{h(x+1)(x_h+1)} = \lim_{h\to 0} \frac{\cancel{x^2}+\cancel{hx}+\cancel{x}+h-\cancel{x^2}-\cancel{xh}-\cancel{x}}{h(x+1)(x+h+1)} \\ &= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(x+1)(x+h+1)} = \lim_{h\to 0} \frac{1}{(x+1)(x+h+1)} = \frac{1}{(x+1)^2}. \end{aligned}$$

3. Find the absolute maximum and the absolute minimum for the function

$$f(x) = 2x^3 - 3x^2 - 12x + 20$$

   on the interval $[-3, 3]$.

   Solution: Absolute extrema are to be found only at endpoints and critical numbers. We have $f'(x) = 6x^2 - 6x - 12 = 6(x+1)(x-2)$, which is defined everywhere and is zero only when $x = -1$ or $x = 2$. Thus, the extrema are among the numbers $f(-3)$, $f(-1)$, $f(2)$, and $f(3)$. We find that $f(-3) = -25$, $f(-1) = 27$, $f(2) = 0$, and $f(3) = 11$. The absolute minimum is $f(-3) = -25$, and the absolute maximum is $f(-1) = 27$.

4. The equation $x^3 + 1 = x$ has a root in the interval $[-2, -1]$. Show how to use Newton's method to give an approximation for that root which is correct to at least six digits to the right of the decimal.

Solution: A quick sketch indicates that the desired root is not far from $x = -1.5$, so let us take $x_1 = -1.5$. The equation $x^3+1 = x$ is equivalent to the equation $f(x) = 0$ with $f(x) = x^3-x+1$, so the Newton's Method iteration scheme

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

becomes

$$x_{k+1} = x_k - \frac{x_k^3 - x_k + 1}{3x_k^2 - 1} = \frac{2x_k^3 - 1}{3x_k^2 - 1}.$$

Thus,

$$\begin{aligned}
x_2 &= \frac{2x_1^3 - 1}{3x_1^2 - 1} = -1.347826087; \\
x_3 &= \frac{2x_2^3 - 1}{3x_2^2 - 1} = -1.325200399; \\
x_4 &= \frac{2x_3^3 - 1}{3x_3^2 - 1} = -1.324718174; \\
x_5 &= \frac{2x_4^3 - 1}{3x_4^2 - 1} = -1.324717957; \\
x_6 &= \frac{2x_5^3 - 1}{3x_5^2 - 1} = -1.324717957;
\end{aligned}$$

Our approximate solution, correct to 8 digits to the right of the decimal, is -1.32471796. (Note: A more accurate picture suggests taking $x_1 = -4/3$; doing this shortens the procedure by a couple of steps.)

5. Let F be the function given by

$$F(x) = (x - 2)^2(x + 2)^3.$$

Then, in fully factored form,

$$F'(x) = (x - 2)(x + 2)^2(5x - 2)$$

and, also in fully factored form,

$$F''(x) = 20(x + 2)\left[x - \frac{2}{5}\left(1 - \sqrt{6}\right)\right]\left[x - \frac{2}{5}\left(1 + \sqrt{6}\right)\right].$$

Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

Solution: The quantity $(x-2)$ is positive when $x > 2$ and negative when $x < 2$; $(x+2)^2$ is positive unless $x = -2$; and $(5x-2)$ is positive when $x > 2/5$, negative when $x < 2/5$. Thus, $F'(x) > 0$ when $-\infty < x < -2$, when $-2 < x < 2/5$ and when $2 < x < \infty$. It follows that F is increasing on the intervals $(-\infty, 2/5]$ and $[2, \infty)$, but decreasing on $[2/5, 1]$.

The quantity $(x+2)$ is negative when $x < -2$ and positive when $x > -2$; $[x - 2(1-\sqrt{6})/5]$ is negative when $x < (1-2\sqrt{6})/5$ and positive when $x > (1-2\sqrt{6})/5$; $[x - 2(1+\sqrt{6})/5]$ is negative when $x < 2(1+\sqrt{6})/5$ and positive when $x > 2(1+\sqrt{6})/5$. Consequently $F''(x) < 0$ when $-\infty < x < -2$ and when $2(1-\sqrt{6})/5 < x < 2(1+\sqrt{6})/5$, but $F''(x) > 0$ when $-2 < x < 2(1-\sqrt{6})/5$ and when $2(1+\sqrt{6})/5 < x < \infty$. So F is concave upward on $[-2, 2(1-\sqrt{6})/5]$ and on $[2(1+\sqrt{6})/5, \infty)$, but concave downward on $(-\infty, -2]$ and on $[2(1-\sqrt{6})/5, 2(1+\sqrt{6})/5]$.

From the expression for $F'(x)$, we see that the critical numbers for F are $x = 2$, $x = -2$, and $x = 2/5$. Our analysis of the increasing/decreasing behavior of F above, shows that $x = 2$ gives a local minimum for F because F is decreasing just to the left of $x = 2$ but increasing just to the right of $x = 2$. Similarly, $x = 2/5$ gives a local maximum for F, and $x = -2$ gives neither a local maximum nor a local minimum.

6. Find the points on the hyperbola $x^2 - 4y^2 = 4$ whose distance from the point $(0, 1)$ is minimal.

Solution: Let (x, y) be any point on the curve. Then $x^2 - 4y^2 = 4$ and the square S of the distance from (x, y) to $(1, 0)$ is $S = x^2 + (y-1)^2$. We can minimize distance by minimizing $S = x^2 + (y-1)^2$ subject to the constraint $x^2 - 4y^2 = 1$. Thus we want to find the critical points of S. Treating y as a function of x and differentiating implicitly, we find that $dS/dx = 2x + 2(y-1)y'$, so want to learn where $2x + 2(y-1)y' = 0$. From $x^2 - 4y^2 = 4$, we see that $2x - 8yy' = 0$, or $y' = x/(4y)$. Thus, we want

$$\begin{aligned} 0 &= 2x + 2(y-1)y' \\ &= x + (y-1)\left(\frac{x}{4y}\right). \end{aligned}$$

so that

$$4xy + (y-1)x = 0,$$

or

$$x(5y-1) = 0.$$

But we may not have $x = 0$ in the equation $x^2 - 4y^2 = 4$, so

$$y = \frac{1}{5}.$$

We put $y = 1/5$ in the equation $x^2 - 4y^2 = 4$, and we find that $x = \pm 2\sqrt{26}/5$. The only critical numbers for $S(x)$ are thus at $x = \pm 2\sqrt{26}/5$. The minimum for $S(x)$ must occur at one or the other (or both) of these—and it is clear from the symmetry of the graph that both critical points give identical minima. The minimal distance therefore occurs when $x = \pm 2\sqrt{26}/5$ and $y = 1/5$—that is, at the points $(2\sqrt{26}/5, 1/5)$ and $(-2\sqrt{26}/5, 1/5)$. (Note: This minimal distance is $2\sqrt{6/5}$.)

7. Murgatroyd was driving his car toward an intersection at 60 miles per hour. A police cruiser was approaching the same intersection but on the cross-street (which is at right angles to the road that Muratroyd is on), at 50 miles per hour. When both cars were a quarter of a mile from the intersection, a police officer in the cruiser pointed a radar gun at Murgatroyd and measured the speed at which the two cars were approaching each other. What did she get?

   Solution: Let x denote the distance from Murgatroyd to the intersection, and let y denote the distance from the police car to the interesection. The distance D between the two cars satisfies

$$D^2 = x^2 + y^2,$$

   so

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt},$$

   or

$$\frac{dD}{dt} = \frac{1}{D}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

   At the critical instant, we have $x = y = 1/4$, $D = 1/(2\sqrt{2})$, $dx/dt = -60$, and $dy/dt = -50$. Thus, rate of change of the distance between the two cars was

$$\begin{aligned}\frac{dD}{dt} &= 2\sqrt{2}\left[\frac{1}{4}(-60) + \frac{1}{4}(-50)\right]\\ &= -\frac{110}{\sqrt{2}} \text{ mph.}\end{aligned}$$

   The reading on the radar gun was $110/\sqrt{2}$ mph, or about 78 mph.

MTH 1410, Calculus I—Final Exam **May 7, 2012**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 3:50 pm.

1. (a) *Use the definition of the derivative* to find $f'(x)$ if $f(x) = 1/\sqrt{x}$.

   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = 1/\sqrt{x}$ at the point on the curve where $x = 4$.

2. Evaluate the following definite integrals. Give all of your reasoning.

   (a) $\displaystyle\int_3^5 (3x^2 - 24x + 54)\,dx$

   (b) $\displaystyle\int_0^3 3t\sqrt{9 - t^2}\,dt$

3. Let F be the function given by

$$F(x) = (x-2)^2(x+3)^3.$$

   Then, in fully factored form,

$$F'(x) = 5x(x-2)(x+3)^2$$

   and, also in fully factored form,

$$F''(x) = 20(x+3)\left[x - \sqrt{\frac{3}{2}}\right]\left[x + \sqrt{\frac{3}{2}}\right].$$

   Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

4. Suppose that $f(2) = 2$, $f(4) = 4$, $f'(2) = 4$, $f'(4) = -2$, $g(2) = 4$, $g(4) = 2$, $g'(2) = -6$, and $g'(4) = -8$.

   (a) Find $F(4)$ and $F'(4)$, where $F(x) = \dfrac{f(x)}{g(x)}$.

   (b) Find $G(2)$ and $G'(2)$, where $G(x) = g[2f(x)]$.

   (c) Find $H(2)$ and $H'(2)$, where $H(x) = g[f(x^2)]$.

5. (a) Show that the point $(3, 2)$ lies on the curve given by the equation

$$x^3 - 5x^2y^3 + 8y^4 + 205 = 0.$$

   (b) If x and y are related by the equation, $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$, find the value of y' at $(3, 2)$.

   (c) Show how to use the results of parts (a) and (b) of this problem to find an approximate value for y near 2 when $x = 74/25$.

6. Find the points on the ellipse $x^2 + xy + y^2 = 1$ whose distance from the origin is minimal. Give your reasoning.

MTH 1410, Calculus I—Final Exam **May 7, 2012**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 4:50 pm.

1. (a) *Use the definition of the derivative* to find $f'(x)$ if $f(x) = 1/\sqrt{x}$.

   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = 1/\sqrt{x}$ at $x = 1$, at $x = 4$, and at $x = 9$.

   Solution:

   (a)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right] \\
&= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \\
&= \lim_{h\to 0} \frac{x - (x+h)}{h\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} \\
&= -\lim_{h\to 0} \frac{1}{\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} = -\frac{1}{2x\sqrt{x}}.
\end{aligned}$$

   (b) From the immediately preceding equation, $f'(4) = -1/16$, so the equation of the tangent line at $x = 4$ is $y = \frac{1}{2} - \frac{1}{16}(x-4)$, or $x + 16y = 12$.

2. Evaluate the following definite integrals. Give all of your reasoning.

   (a) $\displaystyle\int_3^5 (3x^2 - 24x + 54)\,dx$

   (b) $\displaystyle\int_0^3 3t\sqrt{9-t^2}\,dt$

   Solution:

   (a)

$$\int_3^5 (3x^2 - 24x + 54)\,dx = (x^3 - 12x^2 + 54x)\Big|_3^5 = 14.$$

   (b) Let $u = 9 - t^2$. Then $du = -2t\,dt$, or $t\,dt = -(1/2)\,du$. Moreover, $t = 0 \Rightarrow u = 9$ and $t = 3 \Rightarrow u = 0$. Thus

$$\begin{aligned}
\int_0^3 3t\sqrt{9-t^2}\,dt &= -\frac{3}{2}\int_9^0 u^{1/2}\,du = \frac{3}{2}\int_0^9 u^{1/2}\,du \\
&= u^{3/2}\Big|_0^9 = 9^{3/2} - 0^{3/2} = 27.
\end{aligned}$$

3. Let F be the function given by

$$F(x) = (x-2)^2(x+3)^3.$$

Then, in fully factored form,

$$F'(x) = 5x(x-2)(x+3)^2$$

and, also in fully factored form,

$$F''(x) = 20(x+3)\left[x - \sqrt{\frac{3}{2}}\right]\left[x + \sqrt{\frac{3}{2}}\right].$$

Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

Solution: The quantity x is positive when $x > 0$ and negative when $x < 0$; $(x-2)$ is positive when $x > 2$ and negative when $x < 2$; $(x+3)^2$ is positive unless $x = -3$. Thus, $F'(x) > 0$ when $-\infty < x < -3$, when $-3 < x < 0$ and when $2 < x < \infty$. It follows that F is increasing on the intervals $(-\infty, 0]$ and $[2, \infty)$, but decreasing on $[0, 2]$.

The quantity $(x+3)$ is negative when $x < -3$ and positive when $x > -3$; $\left[x - \sqrt{3/2}\right]$ is negative when $x < \sqrt{3/2}$ and positive when $x > \sqrt{3/2}$; $\left[x + \sqrt{3/2}\right]$ is negative when $x < -\sqrt{3/2}$ and positive when $x > -\sqrt{3/2}$. Consequently $F''(x) < 0$ when $-\infty < x < -3$ and when $-\sqrt{3/2} < x < \sqrt{3/2}$, but $F''(x) > 0$ when $-3 < x < -\sqrt{3/2}$ and when $\sqrt{3/2} < x < \infty$. So F is concave upward on $\left(-3, -\sqrt{3/2}\right)$ and on $\left(\sqrt{3/2}, \infty\right)$, but concave downward on $(-\infty, -1]$ and on $\left(-\sqrt{3/2}, \sqrt{3/2}\right]$.

From the expression for $F'(x)$, we see that the critical numbers for F are $x = 2$, $x = 0$, and $x = -3$. Our analysis of the increasing/decreasing behavior of F above, shows that $x = 2$ gives a local minimum for F because F is decreasing just to the left of $x = 2$ but increasing just to the right of $x = 2$. Similarly, $x = 0$ gives a local maximum for F, and $x = -3$ gives neither a local maximum nor a local minimum.

4. Suppose that $f(2) = 2$, $f(4) = 4$, $f'(2) = 4$, $f'(4) = -2$, $g(2) = 4$, $g(4) = 2$, $g'(2) = -6$, and $g'(4) = -8$.

   (a) Find $F(4)$ and $F'(4)$, where $F(x) = \dfrac{f(x)}{g(x)}$.

   (b) Find $G(2)$ and $G'(2)$, where $G(x) = g[2f(x)]$.

   (c) Find $H(2)$ and $H'(2)$, where $H(x) = g[f(x^2)]$.

Solutions:

(a)

$$F(4) = \frac{f(4)}{g(4)} = 2;$$
$$F'(4) = \frac{f'(4)g(4) - f(4)g'(4)}{[g(4)]^2} = 7.$$

(b)

$$G(2) = g[2f(2)] = 2;$$
$$G'(x) = g'[2f(x)]D_x[2f(x)] = 2g'[2f(x)]f'(x),$$

so

$$G'(2) = 2g'[2f(2)]f'(2) = -64.$$

(c)

$$H(2) = g[f(2^2)] = 2;$$
$$H'(x) = g'[f(x^2)] \cdot D_x f(x^2) = g'[f(x^2)]f'(x^2)D_x x^2 = 2xg'[f(x^2)]f'(x^2),$$

so

$$H'(2) = 4g'[f(4)]f'(4) = 64.$$

5. (a) Show that the point $(3, 2)$ lies on the curve given by the equation

$$x^3 - 5x^2y^3 + 8y^4 + 205 = 0.$$

(b) If x and y are related by the equation, $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$, find the value of y' at $(3, 2)$.

(c) Show how to use the results of parts (a) and (b) of this problem to find an approximate value for y near 2 when $x = 74/25$.

Solution:

(a) When $x = 3$ and $y = 2$, we have

$$3^3 - 5 \cdot 3^2 \cdot 2^3 + 8 \cdot 2^4 + 205 = 27 - 360 + 128 + 205 = 0,$$

so the point with coordinates $(3, 2)$ lies on the curve whose equation is $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$.

(b) Treating y as a function of x and differentiating implicitly gives

$$3x^2 - 10xy^3 - 15x^2y^2y' + 32y^3y' = 0;$$
$$(32y^3 - 15x^2y^2)y' = 10xy^3 - 3x^2;$$
$$y' = \frac{10xy^3 - 3x^2}{32y^3 - 15x^2y^2}.$$

Thus,

$$y'\Big|_{(3,2)} = \frac{10 \cdot 3 \cdot 2^3 - 3 \cdot 3^2}{32 \cdot 2^3 - 15 \cdot 3^2 \cdot 2^2} = \frac{240 - 27}{256 - 540} = \frac{213}{-284} = -\frac{3}{4}.$$

(c) From the previous part of this problem, we know that the equation of the line tangent to the curve at $(3, 2)$ is

$$y = 2 - \frac{3}{4}(x - 3).$$

When a point (x_0, y_0) lies near $(3, 2)$ on the curve $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$, it lies near the line tangent to the curve at $(3, 2)$. Thus, we can approximate the value of y near 2 that satisfies the equation

$$\left(\frac{74}{25}\right)^3 - 5\left(\frac{74}{25}\right)^2 y^3 + 8y^4 + 205 = 0$$

as

$$y \sim 2 - \frac{3}{4}\left(\frac{74}{25} - 3\right) = 2 + \frac{3}{4} \cdot \frac{1}{25} = \frac{203}{100}.$$

6. Find the points on the ellipse $x^2 + xy + y^2 = 1$ whose distance from the origin is minimal. Give your reasoning.

Solution: The squared distance D from a point (x, y) to the origin is given by

$$D = x^2 + y^2,$$

and minimizing the squared distance will also minimize the distance itself. The critical numbers for D are the solutions of $D' = 0$, or, by implicit differentiation, of

$$2x + 2yy' = 0. \tag{1}$$

But $x^2 + xy + y^2 = 1$, so by implicit differentiation again,

$$y' = -\frac{2x + y}{x + 2y}. \tag{2}$$

Combining (2) with (1), we find that we must solve

$$x + y\left(-\frac{2x + y}{x + 2y}\right) = 0, \tag{3}$$

or

$$x^2 - y^2 = 0 \tag{4}$$

simultaneously with the original constraint

$$x^2 + xy + y^2 = 1. \tag{5}$$

From (4), we find that $y = \pm x$. Substituting this latter into (5), we obtain

$$x^2 \pm x^2 + x^2 = 1,$$

so that either

$$y = x \text{ and } 3x^2 = 1$$

or

$$y = -x \text{ and } x^2 = 1.$$

The critical numbers for D are therefore $x = \pm 1/\sqrt{3}$ and $x = \pm 1$. The corresponding points on the curve are the points $(1/\sqrt{3}, 1/\sqrt{3})$, $(-1/\sqrt{3}, -1/\sqrt{3})$, $(1, -1)$, and $(-1, 1)$. It is clear that the minima occur at the first two of these.

MTH 1410, Calculus I—Exam I **Sept. 12, 2013**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Your paper is due at 3:50 pm.

1. Show how to evaluate the following limits.

   (a)
   $$\lim_{x\to -3} \frac{x^2+x-6}{x^2+4x+3}$$

   (b)
   $$\lim_{x\to\infty} \frac{(5-x)(10+8x)}{(3-3x)(3+10x)}$$

2. Show how to evaluate the following limits.

   (a)
   $$\lim_{x\to\infty} \left[\sqrt{x^2+5x+3}-x\right]$$

   (b)
   $$\lim_{x\to 3} \left[\frac{1}{x-3}\left(\frac{1}{x^2}-\frac{1}{9}\right)\right]$$

3. Let g be the function given by $g(t) = 100 + 50t - 5t^2$ for all real numbers t.

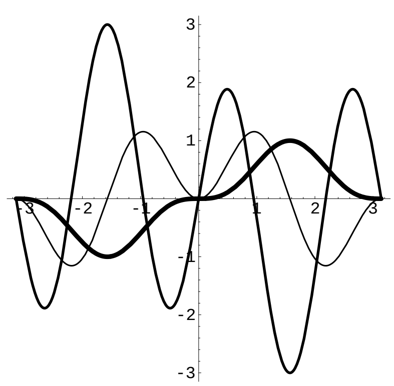
   (a) What are $g(0)$ and $g(7)$?

   (b) What is the slope of the secant line to the curve $s = g(t)$ through the points $\big(0, g(0)\big)$ and $\big(7, g(7)\big)$?

   (c) What is the slope of the line tangent to the curve $s = g(t)$ at the point $\big(7, g(7)\big)$?

   (d) Let t_0 be the *positive* number for which $g(t_0) = 0$. What is the slope of the line tangent to the curve at the point $\big(t_0, g(t_0)\big)$?

4. Here is a graph showing three functions—a skinny one, a middle-weight one, and a fat one—on the same pair of axes:



One of the curves is f, one is f', and one is f''. Explain which is which and how you know.

5. Find $f'(x)$ if

   (a) $f(x) = -2x^2 + 8x + 5$

   (b) $f(x) = \dfrac{x^2 - 2\sqrt{x}}{x^{2/3}}$

6. Find $f'(x)$ if

   (a) $f(x) = (2x^2 - 2x + 1)^2(3x + 4)^{12}$

   (b) $f(x) = \dfrac{a}{x^{12}} + \sin^4 bx$, where a and b are fixed but unspecified constants.

7. A function f is given by

$$f(x) = \begin{cases} -2cx + 2 & ; \quad x \leq 3 \\ 3 - cx & ; \quad 3 < x. \end{cases}$$

   For what values of the constant c is f continuous on $(-\infty, \infty)$? Be sure to give your reasoning.

8. (a) Show how to use the definition of the derivative to find $f'(x)$ if $f(x) = \dfrac{9 - 7x}{3 + 3x}$.

   (b) Use the derivative you calculated in part (a) of this problem to write an equation for the line tangent to the curve $y = f(x)$ at $x = 3$.

9. It is given that $f(3) = -1$, $f'(3) = -1/3$, $f'(11) = -1/27$, $g'(11) = 60$, $g(3) = 11$, and $g'(3) = 12$. Let $F(x) = f(x) + g(x)$, $G(x) = f(x)g(x)$, $H(x) = f(x)/g(x)$, and $K(x) = f[g(x)]$. Find

   (a) $F'(3)$

   (b) $G'(3)$

   (c) $H'(3)$

   (d) $K'(3)$

   Be sure to give your reasoning.

Complete solutions to the exam problems will eventually be available from the course web-site.

MTH 1410, Calculus I—Exam I **Sept. 12, 2013**

Instructions: Work the following problems; give your reasoning and show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Your paper is due at 3:50 pm.

1. Show how to evaluate the following limits.

   (a)
   $$\lim_{x\to -3} \frac{x^2+x-6}{x^2+4x+3}$$

   (b)
   $$\lim_{x\to\infty} \frac{(5-x)(10+8x)}{(3-3x)(3+10x)}$$

   Solution:

   (a)
   $$\lim_{x\to -3} \frac{x^2+x-6}{x^2+4x+3} = \lim_{x\to -3} \frac{\cancel{(x+3)}(x-2)}{(x+1)\cancel{(x+3)}} = \lim_{x\to -3} \frac{x-2}{x+1} = \frac{-5}{-2} = \frac{5}{2}. \tag{1}$$

   (b)
   $$\lim_{x\to\infty} \frac{(5-x)(10+8x)}{(3-3x)(3+10x)} = \lim_{x\to\infty} \frac{\left(\frac{5}{x}-1\right)\left(\frac{10}{x}+8\right)}{\left(\frac{3}{x}-3\right)\left(\frac{3}{x}+10\right)} = \frac{-8}{-30} = \frac{4}{15} \tag{2}$$

2. Show how to evaluate the following limits.

   (a)
   $$\lim_{x\to\infty} \left[\sqrt{x^2+5x+3}-x\right]$$

   (b)
   $$\lim_{x\to 3} \left[\frac{1}{x-3}\left(\frac{1}{x^2}-\frac{1}{9}\right)\right]$$

   Solution:

   (a)
   $$\lim_{x\to\infty} \left[\sqrt{x^2+5x+3}-x\right] = \lim_{x\to\infty} \frac{(x^2+5x+3)-x^2}{\sqrt{x^2+5x+3}+x} \tag{3}$$
   $$= \lim_{x\to\infty} \frac{5x+3}{\sqrt{x^2+5x+3}+x} \tag{4}$$
   $$= \lim_{x\to\infty} \frac{5+\frac{3}{x}}{\sqrt{1+\frac{5}{x}+\frac{3}{x^2}}+1} = \frac{5}{\sqrt{1}+1} = \frac{5}{2}. \tag{5}$$

   (b)
   $$\lim_{x\to 3} \left[\frac{1}{x-3}\left(\frac{1}{x^2}-\frac{1}{9}\right)\right] = \lim_{x\to 3} \frac{1}{x-3}\,\frac{9-x^2}{9x^2} = -\lim_{x\to 3} \frac{(x+3)\cancel{(x-3)}}{9x^2\cancel{(x-3)}} = -\frac{6}{81} = -\frac{2}{27}. \tag{6}$$

3. Let g be the function given by $g(t) = 100 + 50t - 5t^2$ for all real numbers t.

   (a) What are $g(0)$ and $g(7)$?

   (b) What is the slope of the secant line to the curve $s = g(t)$ through the points $\bigl(0, g(0)\bigr)$ and $\bigl(7, g(7)\bigr)$?

   (c) What is the slope of the line tangent to the curve $s = g(t)$ at the point $\bigl(7, g(7)\bigr)$?

   (d) Let t_0 be the *positive* number for which $g(t_0) = 0$. What is the slope of the line tangent to the curve at the point $\bigl(t_0, g(t_0)\bigr)$?

   Solution:

   (a) $g(0) = 100$ and $g(7) = 100 + 50 \cdot 7 - 5 \cdot 49 = 205$.

   (b) The required slope is

$$\frac{g(7) - g(0)}{7 - 0} = \frac{205}{7}. \tag{7}$$

   (c) The required slope is

$$\lim_{h\to 0} \frac{g(7+h) - g(7)}{h} = \lim_{h\to 0} \frac{\cancel{205} - 20h - 5h^2 - \cancel{205}}{h} = \lim_{h\to 0} (-20 - 5h) = -20. \tag{8}$$

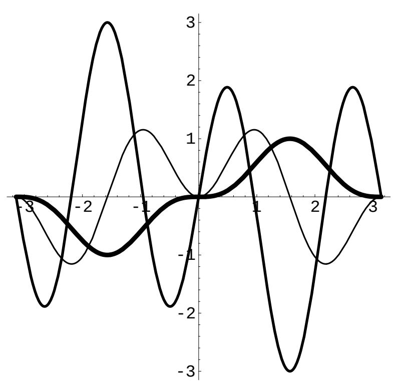
   (d) For the required slope, we have

$$\lim_{h\to 0} \frac{g(t_0 + h) - g(t_0)}{h} = \lim_{h\to 0} \frac{(\cancel{100} + 50h - 5h^2 + \cancel{50t_0} - 10ht_0 - \cancel{5t_0^2}) - (\cancel{100} + \cancel{50t_0} - \cancel{5t_0^2})}{h} \tag{9}$$

$$= \lim_{h\to 0} \frac{(50 - 5h - 10t_0)\cancel{h}}{\cancel{h}} = 50 - 10t_0. \tag{10}$$

   But $g(t_0) = 0$ iff $t_0 = 5 \pm 3\sqrt{5}$, and we must take the plus sign to have $t_0 > 0$. Hence the required slope is $50 - 10(5 + 3\sqrt{5}) = -30\sqrt{5}$.

4. Here is a graph showing three functions—a skinny one, a middle-weight one, and a fat one—on the same pair of axes:



   One of the curves is f, one is f', and one is f''. Explain which is which and how you know.

   Solution: The fat curve slopes upward (has a horizontal tangent) exactly where the thin one is positive (hits the x-axis, and the thin one behaves in the same way with respect to the middle-weight one. So the thin one is f, the fat one is f' and the middle-weight curve is f''.

5. Find $f'(x)$ if

(a) $f(x) = -2x^2 + 8x + 5$

(b) $f(x) = \dfrac{x^2 - 2\sqrt{x}}{x^{2/3}}$

Solution:

(a) $f'(x) = -4x + 8$.

(b) $f(x) = x^{4/3} - 2x^{-1/6}$, so $f'(x) = \dfrac{4}{3}x^{1/3} + \dfrac{1}{3}x^{-7/6}$.

6. Find $f'(x)$ if

(a) $f(x) = (2x^2 - 2x + 1)^2(3x + 4)^{12}$

(b) $f(x) = \dfrac{a}{x^{12}} + \sin^4 bx$, where a and b are fixed but unspecified constants.

Solution:

(a)

$$f'(x) = 2(2x^2 - 2x + 1)(4x - 2)(2x + 4)^{12} + 36(2x^2 - 2x + 1)^2(3x + 4)^{11}. \qquad (11)$$

7. A function f is given by

$$f(x) = \begin{cases} -2cx + 2 & ; \quad x \leq 3 \\ 3 - cx & ; \quad 3 < x. \end{cases}$$

For what values of the constant c is f continuous on $(-\infty, \infty)$? Be sure to give your reasoning.

Solution: If f is a piecewise polynomial function, so inside each of the intervals where f is given by a single polynomial, f is continuous, because polynomials are continuous functions. Thus, f is continuous on $(-\infty, 3) \cup (3, \infty)$ regardless of the value of c, and the only point where continuity depends on c is at $x = 3$, where the definition of continuity requires that f satisfy the condition $\lim_{x\to 3} f(x) = f(3) = -6c+2$. (The latter equality is given.) This means that we must have $-6c+2 = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 3-3c$. But if $-6c+2 = 3-3c$, then $c = -\dfrac{1}{3}$. We conclude that f is continuous on the whole real line precisely when $c = -\dfrac{1}{3}$.

8. (a) Show how to use the definition of the derivative to find $f'(x)$ if $f(x) = \dfrac{9 - 7x}{3 + 3x}$.

(b) Use the derivative you calculated in part (a) of this problem to write an equation for the line tangent to the curve $y = f(x)$ at $x = 3$.

9. It is given that $f(3) = -1$, $f'(3) = -1/3$, $f'(11) = -1/27$, $g'(11) = 60$, $g(3) = 11$, and $g'(3) = 12$. Let $F(x) = f(x) + g(x)$, $G(x) = f(x)g(x)$, $H(x) = f(x)/g(x)$, and $K(x) = f[g(x)]$. Find

(a) $F'(3)$

(b) $G'(3)$

(c) $H'(3)$

(d) $K'(3)$

Be sure to give your reasoning.

Complete solutions to the exam problems will eventually be available from the course web-site.

MTH 1410, Calculus I—Exam II **October 10, 2013**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 3:50 pm.

1. Find $f'(x)$ if
   (a) $f(x) = 3x^6 - 14x^5 + 12x^3 + 7x^2 - 8x$.
   (b) $f(x) = \sqrt{\dfrac{3x^2 - 5x}{x^2 + x + 1}}$.
2. Find $f'(x)$ if
   (a) $f(x) = \cos^3 x \sin 2x$.
   (b) $f(x) = \ln\left[\cos^2 2x \sin^4 x\right]$.
3. (a) *Use the definition of the derivative* to find $f'(x)$ if $f(x) = 1/\sqrt{x}$.
   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = 1/\sqrt{x}$ at $x = 1$, at $x = 4$, and at $x = 9$.
4. Let f be the function given by

$$f(x) = \begin{cases} x^2 + 2x, & x \le 2 \\ ax^2 + b, & x > 2. \end{cases}$$

   (a) What condition must the constants a and b satisfy if f is to be a continuous function?
   (b) Find all pairs of values for a and b which make the function f a differentiable function.
5. A man started walking north at 4 feet per second from a point P. Five minutes later, a woman started walking south at 5 feet per second from a point 500 feet due east of P. At what rate were the people moving apart 15 minutes after the woman started walking.
6. (a) Find an equation for the line tangent to the curve $x^3 - 4x^2y + 2xy^3 + 4 = 0$ at the point whose coordinates are $(2, 1)$.
   (b) Use the result of part (a) of this problem to find an approximate value for the y-coordinate of the point $(2.04, y)$ that lies on the curve $x^3 - 4x^2y + 2xy^3 + 4 = 0$ near the point $(2, 1)$.

MTH 1410, Calculus I—Exam II **October 10, 2013**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 3:50 pm.

1. Find $f'(x)$ if

   (a) $f(x) = 3x^6 - 14x^5 + 12x^3 + 7x^2 - 8x$.

   (b) $f(x) = \sqrt{\dfrac{3x^2 - 5x}{x^2 + x + 1}}$.

   Solution:

   (a) $f'(x) = 18x^5 - 70x^4 + 36x^2 + 14x - 8$.

   (b) We have

   $$\ln f(x) = \ln \sqrt{\frac{3x^2 - 5x}{x^2 + x + 1}}$$

   so that

   $$\ln f(x) = \frac{1}{2}\left[\ln x + \ln(3x - 5) - \ln\left(x^2 + x + 1\right)\right]$$

   Hence,

   $$\frac{f'(x)}{f(x)} = \frac{1}{2}\left[\frac{1}{x} + \frac{3}{3x - 5} - \frac{2x + 1}{x^2 + x + 1}\right],$$

   and

   $$f'(x) = \frac{1}{2}\sqrt{\frac{3x^2 - 5x}{x^2 + x + 1}}\left[\frac{1}{x} + \frac{3}{3x - 5} - \frac{2x + 1}{x^2 + x + 1}\right].$$

2. Find $f'(x)$ if

   (a) $f(x) = \cos^3 x \sin 2x$.

   (b) $f(x) = \ln\left[\cos^2 2x \sin^4 x\right]$.

   Solution:

   (a) $f'(x) = -3\cos^2 x \sin x \sin 2x + 2\cos^3 x \cos 2x$.

   (b) $f'(x) = D_x(2\ln\cos 2x + 4\ln\sin x) = -\dfrac{4\sin 2x}{\cos 2x} + \dfrac{4\cos x}{\sin x} = -4\tan 2x + 4\cot x$.

3. (a) *Use the definition of the derivative* to find $f'(x)$ if $f(x) = 1/\sqrt{x}$.

(b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = 1/\sqrt{x}$ at $x = 1$, at $x = 4$, and at $x = 9$.

Solution:

(a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right] \\ &= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \\ &= \lim_{h\to 0} \frac{x - (x+h)}{h\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} \\ &= -\lim_{h\to 0} \frac{1}{\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} = -\frac{1}{2x\sqrt{x}}. \end{aligned}$$

$f(1) = 1$ and $f'(1) = -1/2$, so the equation of the tangent line at $x = 1$ is

$$y = 1 - \frac{1}{2}(x-1).$$

$f(4) = 1/2$ and $f'(4) = -1/16$, so the equation of the tangent line at $x = 4$ is

$$y = \frac{1}{2} - \frac{1}{16}(x-4).$$

$f(9) = 1/3$ and $f'(9) = -1/54$, so the equation of the tangent line at $x = 9$ is

$$y = \frac{1}{3} - \frac{1}{54}(x-9).$$

4. Let f be the function given by

$$f(x) = \begin{cases} x^2 + 2x, & x \le 2 \\ ax^2 + b, & x > 2. \end{cases}$$

(a) What condition must the constants a and b satisfy if f is to be a continuous function?

(b) Find all pairs of values for a and b which make the function f a differentiable function.

Solution:

(a) If f is to be continuous, we must have $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$. But

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} ax^2 + b = 4a + b,$$

while

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 + 2x = 8.$$

If f is to be continuous, the constants a and b must therefore satisfy the equation $4a + b = 8$.

(b) If f is to be differentiable, f must be continuous at $x = 2$ and the derivatives from the left and from the right must match at $x = 2$. Thus, we must have

$$\begin{aligned} f'(2) &= \lim_{h\to 0^+} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0^+} \frac{(2+h)^2 + 2(2+h) - 8}{h} \\ &= \lim_{h\to 0^+} \frac{4 + 4h + h^2 + 4 + 2h - 8}{h} \\ &= \lim_{h\to 0^+} (6 + h) = 6, \end{aligned}$$

and

$$\begin{aligned} f'(2) &= \lim_{h\to 0^-} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0^-} \frac{[a(2+h)^2 + b] - (4a + b)}{h} \\ &= \lim_{h\to 0^-} (4a + ah) = 4a \end{aligned}$$

together with the equation we derived in part (a): $4a + b = 8$. Thus, $4a = 6$ and $4a + b = 8$, so that $a = 3/2$ and $b = 2$.

5. A man started walking north at 4 feet per second from a point P. Five minutes later, a woman started walking south at 5 feet per second from a point 500 feet due east of P. At what rate were the people moving apart 15 minutes after the woman started walking.

Solution: Let x denote the distance the man has walked north of the point P, and let y denote the distance the woman has walked south of her starting point. Let D be the distance between the two. By the Pythagorean Theorem, $D^2 = (x+y)^2 + 500^2 = (x+y)^2 + 250000$. We differentiate this latter equation implicitly with respect to time to obtain:

$$\begin{aligned} 2D\frac{dD}{dt} &= 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right), \text{ or} \\ \frac{dD}{dt} &= \frac{1}{D}(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right). \end{aligned}$$

We are given that $dx/dt = 4$ and $dy/dt = 5$, and this means that

$$\frac{dD}{dt} = \frac{9(x+y)}{D}.$$

At the critical instant, the woman has been walking for 15 minutes, or 900 seconds, so $y = 4500$. At that instant, the man has been walking for 20 minutes, so $x = 4800$. Thus, $D = \sqrt{(4800+4500)^2 + 250000} = 100\sqrt{8674}$, and this means that

$$\frac{dD}{dt} = \frac{9(4800+4500)}{100\sqrt{8674}} = \frac{837}{\sqrt{8674}} \text{ ft/sec.}$$

6. (a) Find an equation for the line tangent to the curve $x^3 - 4x^2y + 2xy^3 + 4 = 0$ at the point whose coordinates are $(2, 1)$.
   (b) Use the result of part (a) of this problem to find an approximate value for the y-coordinate of the point $(2.04, y)$ that lies on the curve $x^3 - 4x^2y + 2xy^3 + 4 = 0$ near the point $(2, 1)$.

Solution:

(a) Differentiating the equation $x^3 - 4x^2y + 2xy^3 + 4 = 0$ implicitly with respect to x, we find that

$$\begin{aligned} 3x^2 - 8xy - 4x^2y' + 2y^3 + 6xy^2y' &= 0; \\ y' &= \frac{8xy - 3x^2 - 2y^3}{6xy^2 - 4x^2}; \\ y'\Big|_{(2,1)} &= -\frac{1}{2}, \end{aligned}$$

and the equation of the required tangent line is $y = 1 - \frac{1}{2}(x - 2)$.

(b) The equation $x^3 - 4x^2y + 2xy^3 + 4 = 0$ defines y implicitly as a function φ of x near the point $(2, 1)$; we have just computed the linearization of φ at $x = 2$. When x is near 2, we may estimate $\varphi(x)$ by using the linearization in its place. Consequently,

$$\varphi(2.04) \sim 1 - \frac{1}{2}(2.04 - 2) = 1 - 0.02 = 0.98.$$

MTH 1410, Calculus I—Exam III **November 14, 2013**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 3:50 pm.

1. Find the limits:

   (a) $\lim_{x \to -1} \frac{x + x^2}{\ln(2 + x)}$

   (b) $\lim_{x \to 0} \frac{xe^{-2x}}{\pi e^{2x} - \pi}$

2. Find the absolute maximum and the absolute minimum for the function

$$f(x) = 2x^3 - 3x^2 - 12x + 20$$

   on the interval $[-3, 3]$.

3. Take $x_1 = 2$ as your initial guess in the Newton's method approximation of a root of the equation $x^3 - 11 = 0$ and find x_2 and x_3. Give your answers as fractions of integers.

4. Evaluate:

   (a) $\int_{-\pi/4}^{\pi/4} \sec^2 x \, dx$

   (b) $\frac{d}{dx} \left[\int_0^x \frac{1}{\sqrt{1 - t^3}} \, dt \right]$

5. Let F be the function given by

$$F(x) = (x - 1)^2 (x + 1)^3.$$

   Then, in fully factored form,

$$F'(x) = (x - 1)(x + 1)^2 (5x - 1)$$

   and, also in fully factored form,

$$F''(x) = 20(x + 1) \left[x - \frac{1 - \sqrt{6}}{5} \right] \left[x - \frac{1 + \sqrt{6}}{5} \right].$$

   Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward.

6. Find the points on the ellipse $4x^2 + y^2 = 4$ whose distance from the point $(1, 0)$ is maximal.

7. Murgatroyd was driving his car toward an intersection at 60 miles per hour. A police cruiser was approaching the same intersection but on the cross-street (which is at right angles to the road that Muratroyd is on), at 50 miles per hour. When both cars were a quarter of a mile from the intersection, a police officer in the cruiser pointed a radar gun at Murgatroyd and measured the speed at which the two cars were approaching each other. What did she get?

MTH 1410, Calculus I—Exam III, Solutions **November 14, 2013**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 3:50 pm.

1. Find the limits:

   (a) $\lim_{x\to -1} \dfrac{x+x^2}{\ln(2+x)}$

   (b) $\lim_{x\to 0} \dfrac{xe^{-2x}}{\pi e^{2x}-\pi}$

   Solution:

   (a) $\lim_{x\to -1}(x+x^2) = 0 = \lim_{x\to -1}\ln(2+x)$, so we may attempt l'Hôpital's rule:

$$\begin{aligned}\lim_{x\to -1} \frac{x+x^2}{\ln(2+x)} &= \lim_{x\to -1} \frac{1+2x}{[1/(2+x)]}\\ &= \lim_{x\to -1}(1+2x)(2+x)\\ &= -1.\end{aligned}$$

   (b) $\lim_{x\to 0} xe^{-2x} = 0 = \lim_{x\to 0}(\pi e^{2x}-\pi)$, so we can attempt l'Hôpital's rule again:

$$\begin{aligned}\lim_{x\to 0} \frac{xe^{-2x}}{\pi e^{2x}-\pi} &= \lim_{x\to 0} \frac{e^{-2x}-2xe^{-2x}}{2\pi e^{2x}}\\ &= \frac{1}{2\pi}.\end{aligned}$$

2. Find the absolute maximum and the absolute minimum for the function

$$f(x) = 2x^3 - 3x^2 - 12x + 20$$

   on the interval $[-3, 3]$.

   Solution: Absolute extrema are to be found only at endpoints and critical numbers. We have $f'(x) = 6x^2 - 6x - 12 = 6(x+1)(x-2)$, which is defined everywhere and is zero only when $x = -1$ or $x = 2$. Thus, the extrema are among the numbers $f(-3)$, $f(-1)$, $f(2)$, and $f(3)$. We find that $f(-3) = -25$, $f(-1) = 27$, $f(2) = 0$, and $f(3) = 11$. The absolute minimum is $f(-3) = -25$, and the absolute maximum is $f(-1) = 27$.

3. Take $x_1 = 2$ as your initial guess in the Newton's method approximation of a root of the equation $x^3 - 11 = 0$ and find x_2 and x_3. Give your answers as fractions of integers.

   Solution: The Newton's method relation is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

   We have $f(x) = x^3 - 11$, so $f'(x) = 3x^2$. Thus,

$$x_2 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-3}{12} = \frac{9}{4},$$

and

$$x_3 = \frac{9}{4} - \frac{f(9/4)}{f'(9/4)} = \frac{9}{4} - \frac{25/64}{243/16} = \frac{1081}{486}.$$

4. Evaluate:

   (a) $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^2 x\,dx$

   (b) $\displaystyle\frac{d}{dx}\left[\int_0^x \frac{1}{\sqrt{1-t^3}}\,dt\right]$

   Solution:

   (a)

$$\int_{-\pi/4}^{\pi/4} \sec^2 x\,dx = \tan x\Big|_{-\pi/4}^{\pi/4} = 1-(-1) = 2.$$

   (b) By the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\left[\int_0^x \frac{1}{\sqrt{1-t^3}}\,dt\right] = \frac{1}{\sqrt{1-x^3}}.$$

5. Let F be the function given by

$$F(x) = (x-1)^2(x+1)^3.$$

   Then, in fully factored form,

$$F'(x) = (x-1)(x+1)^2(5x-1)$$

   and, also in fully factored form,

$$F''(x) = 20(x+1)\left[x - \frac{1}{5}\left(1-\sqrt{6}\right)\right]\left[x - \frac{1}{5}\left(1+\sqrt{6}\right)\right].$$

   Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward.

   Solution: The quantity $(x-1)$ is positive when $x > 1$ and negative when $x < 1$; $(x+1)^2$ is positive unless $x = -1$; and $(5x-1)$ is positive when $x > 1/5$, negative when $x < 1/5$. Thus, $F'(x) > 0$ when $-\infty < x < -1$, when $-1 < x < 1/5$ and when $1 < x < \infty$. It follows that F is increasing on the intervals $(-\infty, 1/5]$ and $[1, \infty)$, but decreasing on $[1/5, 1]$.

   The quantity $(x+1)$ is negative when $x < -1$ and positive when $x > -1$; $[x-(1-\sqrt{6})/5]$ is negative when $x < (1-\sqrt{6})/5$ and positive when $x > (1-\sqrt{6})/5$; $[x-(1+\sqrt{6})/5]$ is negative when $x < (1+\sqrt{6})/5$ and positive when $x > (1+\sqrt{6})/5$. Consequently $F''(x) < 0$ when $-\infty < x < -1$ and when $(1-\sqrt{6})/5 < x < (1+\sqrt{6})/5$, but $F''(x) > 0$ when $-1 < x < (1-\sqrt{6})/5$ and when $(1+\sqrt{6})/5 < x < \infty$. So F is concave upward on $[-1, (1-\sqrt{6})/5]$ and on $[(1+\sqrt{6})/5, \infty)$, but concave downward on $(-\infty, -1]$ and on $[(1-\sqrt{6})/5, (1+\sqrt{6})/5]$.

6. Find the points on the ellipse $4x^2 + y^2 = 4$ whose distance from the point $(1, 0)$ is maximal.

Solution: The distance D from a point (x, y) to the point $(1, 0)$ is given by

$$D^2 = (x-1)^2 + y^2.$$

If the point (x, y) lies on the given ellipse, then $4x^2 + y^2 = 4$, so that $y^2 = 4 - 4x^2$. Substituting this latter equation into the relation for D, we find that

$$D^2 = (x-1)^2 + 4 - 4x^2 = 5 - 2x - 3x^2.$$

Because $y^2 = 4 - 4x^2$ must not be negative, we are interested only in those values of x for which $-1 \le x \le 1$. We have

$$2D\frac{dD}{dx} = -2 - 6x.$$

Thus, $dD/dx = 0$ when $x = -1/3$. We must check the value of D when $x = -1$, when $x = -1/3$, and when $x = 1$. The corresponding values of D^2 are 4, $16/3$, and 0. Thus, D takes on the maximal value $4/\sqrt{3}$ when $x = -1/3$. But $4x^2 + y^2 = 4$, so $y = \pm 4\sqrt{2}/3$ when $x = -1/3$. The points on the curve that are farthest from $(1, 0)$ are therefore $(-1/3, 4\sqrt{2}/3)$ and $(-1/3, -4\sqrt{2}/3)$.

7. Murgatroyd was driving his car toward an intersection at 60 miles per hour. A police cruiser was approaching the same intersection but on the cross-street (which is at right angles to the road that Muratroyd is on), at 50 miles per hour. When both cars were a quarter of a mile from the intersection, a police officer in the cruiser pointed a radar gun at Murgatroyd and measured the speed at which the two cars were approaching each other. What did she get?

Solution: Let x denote the distance from Murgatroyd to the intersection, and let y denote the distance from the police car to the intersection. The distance D between the two cars satisfies

$$D^2 = x^2 + y^2,$$

so

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt},$$

or

$$\frac{dD}{dt} = \frac{1}{D}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

At the critical instant, we have $x = y = 1/4$, $D = 1/(2\sqrt{2})$, $dx/dt = -60$, and $dy/dt = -50$. Thus, rate of change of the distance between the two cars was

$$\begin{aligned}\frac{dD}{dt} &= 2\sqrt{2}\left[\frac{1}{4}(-60) + \frac{1}{4}(-50)\right] \\ &= -\frac{110}{\sqrt{2}} \text{ mph.}\end{aligned}$$

The reading on the radar gun was about 78 mph.

MTH 1410, Calculus I—Final Exam **December 10, 2013**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 3:50 pm.

1. (a) *Use the definition of the derivative* to find $f'(x)$ if $f(x) = 1/\sqrt{x}$.

   (b) Use the derivative you calculated in part (a) of this problem to write equations for the line tangent to the curve $y = 1/\sqrt{x}$ at $x = 4$.

2. Evaluate the following definite integrals. Give all of your reasoning.

   (a) $\displaystyle\int_3^5 (3x^2 - 24x + 54)\,dx$

   (b) $\displaystyle\int_0^5 3t\sqrt{25 - t^2}\,dt$

3. Let F be the function given by

$$F(x) = (x-1)(x+1)(x-3)^3.$$

   Then, in fully factored form,

$$F'(x) = 5(x-3)^2\left[x - \frac{1}{5}(3 - 2\sqrt{6})\right]\left[x - \frac{1}{5}(3 + 2\sqrt{6})\right]$$

   and, also in fully factored form,

$$F''(x) = 20(x-3)\left[x - \frac{1}{5}\left(6 - \sqrt{21}\right)\right]\left[x - \frac{1}{5}\left(6 + \sqrt{21}\right)\right].$$

   Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

4. Suppose that $f(2) = 2$, $f(4) = 4$, $f'(2) = 4$, $f'(4) = -2$, $g(2) = 4$, $g(4) = 2$, $g'(2) = -6$, and $g'(4) = -8$.

   (a) Find $F(4)$ and $F'(4)$, where $F(x) = \dfrac{f(x)}{g(x)}$.

   (b) Find $G(2)$ and $G'(2)$, where $G(x) = g[2f(x)]$.

   (c) Find $H(2)$ and $H'(2)$, where $H(x) = g[f(x^2)]$.

5. (a) Show that the point $(3, 2)$ lies on the curve given by the equation

$$x^3 - 5x^2y^3 + 8y^4 + 205 = 0.$$

   (b) If x and y are related by the equation, $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$, find the value of y' at $(3, 2)$.

   (c) Show how to use the results of parts (a) and (b) of this problem to find an approximate value for y near 2 when $x = 74/25$.

6. Find the points on the hyperbola $4y^2 - x^2 = 1$ whose distance from the point $(5, 0)$ is minimal.

MTH 1410, Calculus I—Final Exam **Dec. 10, 2013**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 1:50 pm.

1. (a) *Use the definition of the derivative* to find $f'(x)$ if $f(x) = 1/\sqrt{x}$.

   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = 1/\sqrt{x}$ at $x = 4$.

   Solution:

   (a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right] \\ &= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \\ &= \lim_{h\to 0} \frac{x-(x+h)}{h\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})} \\ &= -\lim_{h\to 0} \frac{1}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})} = -\frac{1}{2x\sqrt{x}}. \end{aligned}$$

   (b) From the immediately preceding equation, $f'(4) = -1/16$, so the equation of the tangent line at $x = 4$ is $y = (1/2) - (1/16)(x-4)$, or $x + 16y = 12$.

2. Evaluate the following definite integrals. Give all of your reasoning.

   (a) $\displaystyle\int_3^5 (3x^2 - 24x + 54)\,dx$

   (b) $\displaystyle\int_0^5 3t\sqrt{25-t^2}\,dt$

   Solution:

   (a)

$$\begin{aligned} \int_3^5 (3x^2 - 24x + 54)\,dx &= (x^3 - 12x^2 + 54x)\Big|_3^5 \\ &= (125 - 300 + 270) - (27 - 108 + 162) = 14. \end{aligned}$$

   (b) Let $u = 25 - t^2$. Then $du = -2t\,dt$, or $t\,dt = -(1/2)\,du$. Moreover, $t = 0 \Rightarrow u = 25$ and $t = 5 \Rightarrow u = 0$. Thus

$$\begin{aligned} \int_0^5 3t\sqrt{25-t^2}\,dt &= -\frac{3}{2}\int_{25}^0 u^{1/2}\,du = \frac{3}{2}\int_0^{25} u^{1/2}\,du \\ &= u^{3/2}\Big|_0^{25} = 25^{3/2} - 0^{3/2} = 125. \end{aligned}$$

3. Let F be the function given by

$$F(x) = (x-1)(x+1)(x-3)^3.$$

Then, in fully factored form,

$$F'(x) = 5(x-3)^2\left[x - \frac{1}{5}(3-2\sqrt{6})\right]\left[x - \frac{1}{5}(3+2\sqrt{6})\right]$$

and, also in fully factored form,

$$F''(x) = 20(x-3)\left[x - \frac{1}{5}\left(6-\sqrt{21}\right)\right]\left[x - \frac{1}{5}\left(6+\sqrt{21}\right)\right].$$

Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

Solution: For brevity, let $r_1 = (3-2\sqrt{6})/5$, $r_2 = (3+2\sqrt{6})/5$, and note that $r_1 < r_2 < 3$. The quantity $(x - r_1)$ is positive when $x > r_1$, zero when $x = r_1$ and negative when $x < r_1$. The quantity $(x - r_2)$ is positive when $x > r_2$, zero when $x = r_2$ and negative when $x < r_2$. The quantity $(x-3)^2$ is positive for all $x \neq 3$ and is zero when $x = 3$. Consequently, $F'(x) > 0$ on $(-\infty, r_1)$, on $(r_2, 3)$ and on $(3, \infty)$; $F'(x) < 0$ on (r_1, r_2). Therefore, F is increasing on $(-\infty, r_1)$ and on (r_2, ∞), and F is decreasing on (r_1, r_2).

Again for brevity, we let $s_1 = (6-\sqrt{21})/5$, $s_2 = (6+\sqrt{21})/5$, and we note that $s_1 < s_2 < 3$. Now we have $(x - s_1) > 0$ when $x < s_1$, zero when $x = s_1$, and $x = s_2$. The quantity $(x - s_2)$ is negative when $x < s_2$, zero when $x = s_2$, and positive when $x > s_2$. The quantity $(x - 3)$ is negative when $x < 3$, zero when $x = 3$, and positive when $x > 3$. So $F''(x) < 0$ in (∞, s_1), $F''(x) > 0$ in (s_1, s_2), $F''(x) < 0$ in $(s_2, 3)$, and $F''(x) > 0$ in $(3, \infty)$. We conclude that F is concave downward on $(-\infty, s_1)$ and on $(s_2, 3)$, but concave upward on (s_1, s_2) and on $(3, \infty)$.

From the expression for $F'(x)$, we see that the critical numbers for F are $x = r_1$, $x = r_2$, and $x = 3$. Our analysis of the increasing/decreasing behavior of F above, shows that $x = r_1$ gives a local maximum for F because F is increasing just to the left of $x = r_1$ but decreasing just to the right of $x = r_1$. Similarly, $x = r_2$ gives a local minimum for F, and $x = 3$ gives neither a local maximum nor a local minimum.

4. Suppose that $f(2) = 2$, $f(4) = 4$, $f'(2) = 4$, $f'(4) = -2$, $g(2) = 4$, $g(4) = 2$, $g'(2) = -6$, and $g'(4) = -8$.

   (a) Find $F(4)$ and $F'(4)$, where $F(x) = \dfrac{f(x)}{g(x)}$.

   (b) Find $G(2)$ and $G'(2)$, where $G(x) = g[2f(x)]$.

   (c) Find $H(2)$ and $H'(2)$, where $H(x) = g[f(x^2)]$.

Solution:

(a)

$$F(4) = \frac{f(4)}{g(4)} = 2;$$
$$F'(4) = \frac{f'(4)g(4) - f(4)g'(4)}{[g(4)]^2} = 7.$$

(b)

$$G(2) = g[2f(2)] = 2;$$
$$G'(x) = g'[2f(x)]D_x[2f(x)] = 2g'[2f(x)]f'(x),$$

so

$$G'(2) = 2g'[2f(2)]f'(2) = -64.$$

(c)

$$H(2) = g[f(2)] = 2;$$
$$H'(x) = g'[f(x^2)] \cdot D_x f(x^2) = g'[f(x^2)]f'(x^2)D_x x^2 = 2xg'[f(x^2)]f'(x^2),$$

so

$$H'(2) = 4g'[f(4)]f'(4) = 64.$$

5. (a) Show that the point $(3,2)$ lies on the curve given by the equation

$$x^3 - 5x^2y^3 + 8y^4 + 205 = 0.$$

(b) If x and y are related by the equation, $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$, find the value of y' at $(3,2)$.

(c) Show how to use the results of parts (a) and (b) of this problem to find an approximate value for y near 2 when $x = 74/25$.

Solution:

(a) When $x = 3$ and $y = 2$, we have

$$3^3 - 5 \cdot 3^2 \cdot 2^3 + 8 \cdot 2^4 + 205 = 27 - 360 + 128 + 205 = 0,$$

so the point with coordinates $(3,2)$ lies on the curve whose equation is $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$.

(b) Treating y as a function of x and differentiating implicitly gives

$$3x^2 - 10xy^3 - 15x^2y^2y' + 32y^3y' = 0;$$
$$(32y^3 - 15x^2y^2)y' = 10xy^3 - 3x^2;$$
$$y' = \frac{10xy^3 - 3x^2}{32y^3 - 15x^2y^2}.$$

Thus,

$$y'\Big|_{(3,2)} = \frac{10 \cdot 3 \cdot 2^3 - 3 \cdot 3^2}{32 \cdot 2^3 - 15 \cdot 3^2 \cdot 2^2} = \frac{240 - 27}{256 - 540} = \frac{213}{-284} = -\frac{3}{4}.$$

(c) From the previous part of this problem, we know that the equation of the line tangent to the curve at $(3,2)$ is

$$y = 2 - \frac{3}{4}(x - 3).$$

When a point (x_0, y_0) lies near $(3,2)$ on the curve $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$, it lies near the line tangent to the curve at $(3,2)$. Thus, we can approximate the value of y near 2 that satisfies the equation

$$\left(\frac{74}{25}\right)^3 - 5\left(\frac{74}{25}\right)^2 y^3 + 8y^4 + 205 = 0$$

as

$$y \sim 2 - \frac{3}{4}\left(\frac{74}{25} - 3\right) = 2 + \frac{3}{4} \cdot \frac{1}{25} = \frac{203}{100}.$$

6. Find the points on the hyperbola $4y^2 - x^2 = 1$ whose distance from the point $(5,0)$ is minimal.

Solution: The distance D from a point (x,y) to the point $(5,0)$ is given by

$$D^2 = (x-5)^2 + y^2.$$

If the point (x, y) lies on the given hyperbola, then $4y^2 - x^2 = 1$, so that $y^2 = (1 + x^2)/4$. Substituting this latter equation into the relation for D, we find that

$$D^2 = (x-5)^2 + \frac{1}{4}(1 + x^2) = \frac{5x^2 - 40x + 101}{4}.$$

We have

$$2D\frac{dD}{dx} = \frac{10x - 40}{4},$$

so $\dfrac{dD}{dx} = 0$ when $x = 4$. This is the only critical point, and we know from the geometry of the situation (The curve is a hyperbola opening vertically and centered on the y-axis.) that there must be a minimum. Consequently, it must lie at a point on the curve where $x = 4$. This requires that $4y^2 - 16 = 1$, or that $y = \pm\sqrt{17}/2$. There are thus two points on the hyperbola whose distance from $(5, 0)$ is minimal: $(4, \pm\sqrt{17}/2)$.

MTH 1410, Calculus I (MW)—Exam I **Sept. 17, 2014**

Instructions: Work the following problems on your own paper; give your reasoning and show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Your paper is due at 5:50 pm.

1. Show how to use the Limit Laws to evaluate the following limits. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $\displaystyle\lim_{x\to 1/2} \frac{2x^3 + 7x^2 - 14x + 5}{2x^3 - 5x^2 - 4x + 3}$

   (b) $\displaystyle\lim_{x\to 3} \frac{\left(\dfrac{1}{x} - \dfrac{1}{3}\right)}{x^2 - 9}$

2. Show how to use the Limit Laws to evaluate the following limits. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $\displaystyle\lim_{x\to\infty} \frac{4x^2 - 3x + 12}{9x^2 + 12x - 15}$

   (b) $\displaystyle\lim_{x\to\infty} \left[\sqrt{x^2 - 3x + 12} - x\right].$

3. Show how to use the definition of the derivative to find $f'(x)$ when $f(x) = \dfrac{1}{1 + 3x}$.

4. Find $f'(x)$ when f is given by

   (a) $f(x) = (5x^2 - 4x)(3x - 2)$.

   (b) $f(x) = \dfrac{x + \tan x}{x^2 - \cos x}$.

5. Show how to find an equation for the line that is tangent to the curve $y = \dfrac{x^2 + 1}{x - 1}$ at the point on the curve where $x = -1$.

6. Let f be the function given by

$$f(x) = \begin{cases} 3x - 7, & \text{when } x \le a, \\ \\ x^2 - 12x + 37, & \text{when } x > a, \end{cases}$$

   where a is a certain constant. Find all values of a for which the function f is continuous everywhere. Be sure to give the reasoning that supports your conclusions.

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 1410, Calculus I (MW)—Exam I **Sept. 17, 2014**

Instructions: Work the following problems on your own paper; give your reasoning and show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Your paper is due at 5:50 pm.

1. Show how to use the Limit Laws to evaluate the following limits. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $\displaystyle\lim_{x\to 1/2} \frac{2x^3+7x^2-14x+5}{2x^3-5x^2-4x+3}$

   (b) $\displaystyle\lim_{x\to 3} \frac{\left(\frac{1}{x}-\frac{1}{3}\right)}{x^2-9}$

   Solution:

   (a)

$$\lim_{x\to 1/2} \frac{2x^3+7x^2-14x+5}{2x^3-5x^2-4x+3} = \lim_{x\to 1/2} \frac{\cancel{(2x-1)}(x^2+4x-5)}{\cancel{(2x-1)}(x^2-2x-3)}$$
$$= \frac{\lim_{x\to 1/2}(x^2+4x-5)}{\lim_{x\to 1/2}(x^2-2x-3)} = \frac{(-11/4)}{(-15/4)} = \frac{11}{15}.$$

   (b)

$$\lim_{x\to 3} \frac{\left(\frac{1}{x}-\frac{1}{3}\right)}{x^2-9} = \lim_{x\to 3} \frac{3-x}{3x(x^2-9)}$$
$$= \lim_{x\to 3} \frac{-\cancel{(x-3)}}{3x\cancel{(x-3)}(x+3)}$$
$$= \frac{-1}{\lim_{x\to 3}[3x(x+3)]} = -\frac{1}{54}.$$

2. Show how to use the Limit Laws to evaluate the following limits. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $\displaystyle\lim_{x\to\infty} \frac{4x^2-3x+12}{9x^2+12x-15}$

   (b) $\displaystyle\lim_{x\to\infty} \left[\sqrt{x^2-3x+12}-x\right]$.

   Solution:

   (a)

$$\lim_{x\to\infty} \frac{4x^2-3x+12}{9x^2+12x-15} = \lim_{x\to\infty} \frac{\left(4-\frac{3}{x}+\frac{12}{x^2}\right)}{\left(9+\frac{12}{x}-\frac{15}{x^2}\right)} = \frac{4}{9}.$$

(b)

$$\lim_{x\to\infty}\left[\sqrt{x^2-3x+12}-x\right] = \lim_{x\to\infty}\frac{(\cancel{x^2}-3x+12)-\cancel{x^2}}{\sqrt{x^2-3x+12}+x}$$
$$= -\lim_{x\to\infty}\frac{3-\left(\dfrac{12}{x}\right)}{\sqrt{1-\dfrac{3}{x}+\dfrac{12}{x^2}}+1}$$
$$= -\frac{\lim\limits_{x\to\infty}\left[3-\left(\dfrac{12}{x}\right)\right]}{\lim\limits_{x\to\infty}\left[\sqrt{1-\dfrac{3}{x}+\dfrac{12}{x^2}}+1\right]} = -\frac{3}{2}$$

3. Show how to use the definition of the derivative to find $f'(x)$ when $f(x) = \dfrac{1}{1+3x}$.

Solution:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{1+3(x+h)}-\frac{1}{1+3x}\right]$$
$$= \lim_{h\to 0}\frac{\cancel{1}+\cancel{3x}-\cancel{1}-\cancel{3x}-3\cancel{h}}{\cancel{h}[1+3(x+h)](1+3x)} = -\frac{3}{(1+3x)^2}.$$

4. Find $f'(x)$ when f is given by

(a) $f(x) = (5x^2-4x)(3x-2)$.

(b) $f(x) = \dfrac{x+\tan x}{x^2-\cos x}$.

Solution

(a)

$$\frac{d}{dx}[(5x^2-4x)(3x-2)] = \left[\frac{d}{dx}(5x^2-4x)\right](3x-2) + (5x^2-4x)\frac{d}{dx}(3x-2)$$
$$= (10x-4)(3x-2) + 3(5x^2-4x).$$

(b)

$$\frac{d}{dx}\left[\frac{x+\tan x}{x^2-\cos x}\right] = \frac{(x+\tan x)'(x^2-\cos x)-(x^2-\cos x)'(x+\tan x)}{(x^2-\cos x)^2}$$
$$= \frac{(1+\sec^2 x)(x^2-\cos x)-(2x+\sin x)(x+\tan x)}{(x^2-\cos x)^2}.$$

5. Show how to find an equation for the line that is tangent to the curve $y = \dfrac{x^2+1}{x-1}$ at the point on the curve where $x=-1$.

Solution: If f is as given, then $f(-1) = -1$, while

$$f'(x) = \frac{2x(x-1)-(x^2+1)\cdot 1}{(x-1)^2}$$

so that

$$f'(-1) = \frac{2(-1)(-2)-[(-1)^2+1]}{(-2)^2} = \frac{1}{2}.$$

Thus, an equation for the tangent line at $x=-1$ is $y = -1+\dfrac{1}{2}(x+1)$, or $y = \dfrac{1}{2}x - \dfrac{1}{2}$.

6. Let f be the function given by

$$f(x) = \begin{cases} 3x - 7, & \text{when } x \leq a, \\ \\ x^2 - 12x + 37, & \text{when } x > a, \end{cases}$$

where a is a certain constant. Find all values of a for which the function f is continuous everywhere. Be sure to give the reasoning that supports your conclusions.

Solution: The function f is given by a polynomial function throughout some open interval that contains each point, with the exception of the point $x = a$, so the latter point is the only one where f may fail to be continuous.

At $x = a$, continuity requires that $\lim_{x\to a^-} f(x) = 3a - 7 = f(a) = \lim_{x\to a^-} f(x) = a^2 - 12a + 37$. Thus, for continuity of f at $x = a$, we must have

$$a^2 - 12a + 37 = 3a - 7$$

or

$$a^2 - 15a + 44 = 0.$$

Thus $(a - 4)(a - 11) = 0$, so that the values of a for which f is continuous at $x = a$ are $a = 4$ and $a = 11$.

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 1410, Calculus I ($T\theta$)—Exam I **Sept.11, 2014**

Instructions: Work the following problems on your own paper; give your reasoning and show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Your paper is due at 1:50 pm.

1. Show how to use the Limit Laws to evaluate the following limits. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $\lim_{x \to 2} \dfrac{x^3 - 4x^2 + x + 6}{x^3 + 2x^2 - 13x + 10}$

   (b) $\lim_{x \to 9} \dfrac{\left(\dfrac{1}{x} - \dfrac{1}{9}\right)}{x - 9}$

2. Show how to use the Limit Laws to evaluate the following limits. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $\lim_{x \to \infty} \dfrac{4x^2 - 3x + 12}{9x^2 + 12x - 15}$

   (b) $\lim_{x \to \infty} \left[\sqrt{x^2 - 5x + 12} - x\right]$.

3. Show how to use the definition of the derivative to find $f'(x)$ when $f(x) = \sqrt{1 + 2x}$.

4. Find $f'(x)$ when f is given by

   (a) $f(x) = (3x^2 - 2x)(5x - 4)$.

   (b) $f(x) = \dfrac{x + \sin x}{x^2 - \sec x}$.

5. Show how to find an equation for the line that is tangent to the curve $y = \dfrac{x - 1}{x^2 + 1}$ at the point on the curve where $x = -1$.

6. Let f be the function given by

$$f(x) = \begin{cases} ax + b, & \text{when } 2 < x < 9, \\ x^2 - 8x + 22, & \text{when } x \le 2 \text{ or } x \ge 9, \end{cases}$$

   where a and b are certain constants. Find all values of a and b for which the function f is continuous everywhere. Be sure to give the reasoning that supports your conclusions.

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 1410, Calculus I ($T\theta$)—Exam I, Solutions **Sept 11, 2014**

Instructions: Work the following problems on your own paper; give your reasoning and show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Your paper is due at 1:50 pm.

1. Show how to use the Limit Laws to evaluate the following limits. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $\displaystyle\lim_{x\to 2}\frac{x^3-4x^2+x+6}{x^3+2x^2-13x+10}$

   (b) $\displaystyle\lim_{x\to 9}\frac{\left(\frac{1}{x}-\frac{1}{9}\right)}{x-9}$

   Solution:

   (a) We note first that $\lim_{x\to 2}(x^3-4x^2+x+6)=0=\lim_{x\to 2}(x^3+2x^2-13x+10)$, so we may not use the division rule for limits. However,

   $$\begin{aligned}\lim_{x\to 2}\frac{x^3-4x^2+x+6}{x^3+2x^2-13x+10} &= \lim_{x\to 2}\frac{\cancel{(x-2)}(x^2-2x-3)}{\cancel{(x-2)}(x^2+4x-5)}\\ &= \lim_{x\to 2}\frac{x^2-2x-3}{x^2+4x-5}=\frac{4-4-3}{4+8-5}=-\frac{3}{7}.\end{aligned}$$

   (b) We have $\lim_{x\to 9}\left[\frac{1}{x}-\frac{1}{9}\right]=0=\lim\limits_{x\to 9}(x-9)$, so we may not use the division rule. However,

   $$\begin{aligned}\lim_{x\to 9}\frac{\left(\frac{1}{x}-\frac{1}{9}\right)}{x-9} &= \lim_{x\to 9}\left(\left[\frac{1}{x-9}\right]\left[\frac{9-x}{9x}\right]\right)\\ &= -\lim_{x\to 9}\left(\left[\frac{1}{\cancel{x-9}}\right]\left[\frac{\cancel{x-9}}{9x}\right]\right)=-\lim_{x\to 9}\frac{1}{9x}=-\frac{1}{81}.\end{aligned}$$

2. Show how to use the Limit Laws to evaluate the following limits. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $\displaystyle\lim_{x\to\infty}\frac{4x^2-3x+12}{9x^2+12x-15}$

   (b) $\displaystyle\lim_{x\to\infty}\left[\sqrt{x^2-5x+12}-x\right]$.

   Solution:

   (a) We may not use the division rule for limits because neither the limit in the numerator nor the limit in the denominator exists. But

   $$\lim_{x\to\infty}\frac{4x^2-3x+12}{9x^2+12x-15}=\lim_{x\to\infty}\frac{\left(4-\frac{3}{x}-\frac{12}{x^2}\right)}{\left(9+\frac{12}{x}-\frac{15}{x^2}\right)}=\frac{4}{9}$$

(b) We may not use the subtraction rule for limits, because neither $\lim_{x\to\infty} \sqrt{x^2 - 5x + 12}$ nor $\lim_{x\to\infty} x$ exists. Nevertheless,

$$\begin{aligned}\lim_{x\to\infty}\left[\sqrt{x^2-5x+12}-x\right] &= \lim_{x\to\infty}\frac{(\cancel{x^2}-5x+12)-\cancel{x^2}}{\sqrt{x^2-5x+12}+x}\\ &= \lim_{x\to\infty}\frac{-5x+12}{\sqrt{x^2-5x+12}+x}\\ &= \lim_{x\to\infty}\frac{\left(-5+\dfrac{12}{x}\right)}{\sqrt{1-\dfrac{5}{x}+\dfrac{12}{x^2}}+1} = -\frac{5}{2}.\end{aligned}$$

3. Show how to use the definition of the derivative to find $f'(x)$ when $f(x) = \sqrt{1+2x}$.

Solution:

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{1+2(x+h)}-\sqrt{1+2x}}{h}\\ &= \lim_{h\to 0}\frac{[1+2(x+h)]-[1+2x]}{h[\sqrt{1+2(x+h)}+\sqrt{1+2x}]}\\ &= \lim_{h\to 0}\frac{\cancel{1}+\cancel{2x}+2h-\cancel{1}-\cancel{2x}}{h[\sqrt{1+2x+2h}+\sqrt{1+2x}]}\\ &= \lim_{h\to 0}\frac{2\cancel{h}}{\cancel{h}[\sqrt{1+2x+2h}+\sqrt{1+2x}]} = \frac{\cancel{2}}{\cancel{2}\sqrt{1+2x}} = \frac{1}{\sqrt{1+2x}}.\end{aligned}$$

4. Find $f'(x)$ when f is given by

(a) $f(x) = (3x^2 - 2x)(5x - 4)$.

(b) $f(x) = \dfrac{x + \sin x}{x^2 - \sec x}$.

Solution:

(a)

$$\begin{aligned}\frac{d}{dx}\left[(3x^2-2x)(5x-4)\right] &= (3x^2-2x)'(5x-4)+(3x^2-2x)(5x-4)'\\ &= (6x-2)(5x-4)+5(3x^2-2x).\end{aligned}$$

(b)

$$\begin{aligned}f'(x) &= \frac{(x+\sin x)'(x^2-\sec x)-(x+\sin x)(x^2-\sec x)'}{(x^2-\sec x)^2}\\ &= \frac{(1+\cos x)(x^2-\sec x)-(x+\sin x)(2x-\sec x\tan x)}{(x^2-\sec x)^2}\end{aligned}$$

5. Show how to find an equation for the line that is tangent to the curve $y = \dfrac{x-1}{x^2+1}$ at the point on the curve where $x = -1$.

Solution: If $f(x) = \dfrac{x-1}{x^2+1}$, then $f(-1) = -1$. Also,

$$f'(x) = \frac{1\cdot(x^2+1)-(x-1)\cdot 2x}{(x^2+1)^2} = \frac{1+2x-x^2}{(1+x^2)^2},$$

so that

$$f'(-1) = -\frac{1}{2}.$$

The tangent line whose equation we seek is therefore the line that passes through the point with coordinates $(-1,-1)$ with slope $-\frac{1}{2}$. One equation for this line is

$$y = -1 - \frac{1}{2}(x+1).$$

6. Let f be the function given by

$$f(x) = \begin{cases} ax+b, & \text{when } 2 < x < 9, \\ \\ x^2 - 8x + 22, & \text{when } x \leq 2 \text{ or } x \geq 9, \end{cases}$$

where a and b are certain constants. Find all values of a and b for which the function f is continuous everywhere. Be sure to give the reasoning that supports your conclusions.

Solution: The only places where there is a question are $x = 2$ and $x = 9$. For continuity at $x = 2$ we must have $\lim_{x\to 2} f(x) = f(2) = -6$. This can be so if and only if

$$\lim_{x\to 2^+} f(x) = 10 = \lim_{x\to 2^-} f(x).$$

But

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 - 8x + 22) = 10,$$

and

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (ax+b) = 2a + b,$$

So the condition

$$2a + b = 10$$

must be satisfied if f is to be continuous at $x = 2$. A similar analysis at $x = 9$ leads to the condition

$$9a + b = 31,$$

which must be satisfied if f is to be continuous at $x = 9$.

If both these conditions are satisfied, then $(9a+b) - (2a+b) = 31 - 10 = 21$, or $7a = 21$. It then follows that $a = 3$ and $b = 4$.

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 1410, Calculus I (MW)—Exam II **Oct. 15, 2014**

Instructions: Work the following problems on your own paper; give your reasoning and show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Your paper is due at1:50 pm.

1. Show how to use the Limit Laws to evaluate the following limits. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $\lim_{x\to 0} \dfrac{\sqrt{x+a}-\sqrt{a}}{x}$.

   (b) $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$

2. Show how to use the definition of the derivative to find $f'(x)$ when f is given by

   $$f(x) = \frac{x}{1+x}.$$

3. Let $G(x) = 3x^5 - 85x^3 + 240x$. Show how to determine the intervals where G is increasing and those where G is decreasing.

4. Show how to find an equation for the line tangent to the curve

   $$3(x^2+y^2)^2 = 100(y^2-x^2)$$

   at the point $(2, -4)$.

5. (a) Suppose that $h(-2) = -15$, $k(-2) = -4$, $h'(-2) = -12$, $k'(-2) = -3$, and $H(x) = \dfrac{h(x)}{k(x)}$. Show how to find $H'(-2)$.

   (b) Let $F(x) = f(f(x))$ and $G(x) = [F(x)]^2$, and suppose that $f(3) = 15$, $f(15) = 255$, $f'(15) = 32$, and $f'(3) = 8$. Show how to find $F'(3)$ and $G'(3)$.

6. A ladder 10 feet long rests against a vertical wall, with its bottom sliding away from the wall at the rate of 2 feet per second.

   (a) Show how to determine the velocity of the top of the ladder at the instant when its distance from the floor is the same as the distance from the wall to the bottom of the ladder.

   (b) Let θ denote the acute angle, from the wall to the ladder at the ladder's top. Show how to determine how fast θ is changing when its value is $\pi/4$ radians.

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 1410, Calculus I (*MW*)—Exam II **Oct. 15, 2014**

Instructions: Work the following problems on your own paper; give your reasoning and show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Your paper is due at1:50 pm.

1. Show how to use the Limit Laws to evaluate the following limits. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $\lim_{x\to 0} \dfrac{\sqrt{x+a}-\sqrt{a}}{x}$.

   (b) $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$

   Solution:

   (a)

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{x+a}-\sqrt{a}}{x} &= \lim_{x\to 0} \frac{(x+\not{a})-\not{a}}{x(\sqrt{x+a}+\sqrt{a})} \\ &= \lim_{x\to 0} \frac{\not{x}}{\not{x}(\sqrt{x+a}+\sqrt{a})} \\ &= \lim_{x\to 0} \frac{1}{\sqrt{x+a}+\sqrt{a}} = \frac{1}{2\sqrt{a}}.\end{aligned}$$

   (b)

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos x}{x^2} &= \lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)} \\ &= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 \cdot \left(\lim_{x\to 0} \frac{1}{1+\cos x}\right) = 1^2 \cdot \frac{1}{2} = \frac{1}{2}.\end{aligned}$$

2. Show how to use the definition of the derivative to find $f'(x)$ when f is given by

$$f(x) = \frac{x}{1+x}.$$

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{x+h}{1+x+h} - \frac{x}{1+x}\right) \\ &= \lim_{h\to 0} \frac{(\cancel{x}+h+\cancel{x^2}+\cancel{hx}) - (\cancel{x}+\cancel{x^2}+\cancel{hx})}{h(1+x+h)(1+x)} \\ &= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(1+x+h)(1+x)} \\ &= \lim_{h\to 0} \frac{1}{(1+x+h)(1+x)} = \frac{1}{(1+x)^2}. \end{aligned}$$

3. Let $G(x) = 3x^5 - 85x^3 + 240x$. Show how to determine the intervals where G is increasing and those where G is decreasing.

   Solution: We have

   $$\begin{aligned} G'(x) &= 15x^4 - 255x^2 + 240 = 15(x^4 - 17x^2 + 16) \\ z &= 15(x^2-1)(x^2-16) = 15(x+4)(x+1)(x-1)(x-4). \end{aligned}$$

   A sign picture now shows that $G'(x) > 0$ on $(-\infty, -4)$, on $(-1, 1)$, and on $(4, \infty)$, while $G'(x) < 0$ on $(-4, -1)$ and on $(1, 4)$. It follows that G is increasing on $(-\infty, -4]$, on $[-1, 1]$, and on $[4, \infty)$, and that G is decreasing on $[-4, -1]$ and on $[1, 4]$.

4. Show how to find an equation for the line tangent to the curve

   $$3(x^2+y^2)^2 = 100(y^2 - x^2)$$

   at the point $(2, -4)$.

   Solution: Treating y as a function of x and differentiating the equation implicitly, we obtain

   $$6(x^2+y^2)(2x+2yy') = 100(2yy' - 2x).$$

   Putting $x = 2$ and $y = -4$ now gives

   $$\begin{aligned} 6\cdot 20\cdot(4-8y') &= 100(-8y'-4); \\ 480 - 960y' &= -800y' - 400; \\ -160y' &= -880; y' = \frac{11}{2}. \end{aligned}$$

   It now follows that the desired equation is $y = -4 + \frac{11}{2}(x-2)$, which can be rewritten as $11x - 2y - 30 = 0$.

5. (a) Suppose that $h(-2) = -15$, $k(-2) = -4$, $h'(-2) = -12$, $k'(-2) = -3$, and $H(x) = \dfrac{h(x)}{k(x)}$. Show how to find $H'(-2)$.

(b) Let $F(x) = f(f(x))$ and $G(x) = [F(x)]^2$, and suppose that $f(3) = 15$, $f(15) = 255$, $f'(15) = 32$, and $f'(3) = 8$. Show how to find $F'(3)$ and $G'(3)$.

Solution:

(a) If $H(x) = \dfrac{h(x)}{k(x)}$, then

$$H'(x) = \frac{h'(x)k(x) - h(x)k'(x)}{[k(x)]^2},$$

so that

$$H'(-2) = \frac{h'(-2)k(-2) - h(-2)k'(-2)}{[k(-2)]^2}$$
$$= \frac{(-12)(-4) - (-15)(-3)}{(-4)^2} = \frac{3}{16}.$$

(b) If $F(x) = f(f(x))$ and $G(x) = [F(x)]^2$, then

$$F'(x) = f'[f(x)] \cdot f'(x),$$

so that

$$F'(3) = f'[f(3)] \cdot f'(3) = f'(15) \cdot 8 = 32 \cdot 8 = 256;$$

and

$$G'(x) = 2F(x)F'(x) = 2f[f(x)] \cdot F'(x),$$

so that

$$G'(3) = 2f[f(3)] \cdot F'(3) = 2 \cdot f[15] \cdot 8160 = 2 \cdot 255 \cdot 256 = 130560.$$

6. A ladder 10 feet long rests against a vertical wall, with its bottom sliding away from the wall at the rate of 2 feet per second.

(a) Show how to determine the velocity of the top of the ladder at the instant when its distance from the floor is the same as the distance from the wall to the bottom of the ladder.

(b) Let θ denote the acute angle, from the wall to the ladder at the ladder's top. Show how to determine how fast θ is changing when its value is $\pi/4$ radians.

Solution:

(a) Let x denote the distance from the wall to the bottom of the ladder, y the distance from the floor to the top of the ladder. By the Pythagorean Theorem, $x^2 + y^2 = 100$, so, treating x and y both as functions of t and differentiating this relationship implicitly, we obtain

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0, \text{ or}$$
$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

But it is given that $\frac{dx}{dt} = 2$, so at the instant when $x = y$, we have $x = y = 5\sqrt{2}$ ft. and $\frac{dy}{dt} = -2$ ft/sec.

(b) With the notation above, we have

$$y \tan\theta = x, \text{ so that}$$
$$\frac{dy}{dt}\tan\theta + y\frac{d\theta}{dt}\sec^2\theta = \frac{dx}{dt}.$$

Taking $\theta = \pi/4$ and using the information from part (a), above, this gives

$$(-2)\cdot 1 + 5\sqrt{2}\cdot 2\frac{d\theta}{dt} = 2, \text{ or}$$
$$\frac{d\theta}{dt} = \frac{\sqrt{2}}{5} \text{ radians/sec.}$$

MTH 1410, Calculus I—Exam III **November 19, 2014**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 5:50 pm.

1. Show how to find the limits:
   (a) $\lim_{x\to 1} \dfrac{\ln(x^2)}{x-1}$
   (b) $\lim_{x\to 0} \dfrac{e^x + e^{-x} - 2}{\cos x - 1}$
2. Show how to use the definition of derivative and the Limit Laws to find $f'(x)$ when $f(x) = \dfrac{1}{\sqrt{x+1}}$.
3. Let f be the function whose second derivative is $f''(x) = \dfrac{3}{x^2} - 8e^x$.
   (a) Show how to find $f'(x)$, given that $f'(1) = -8e$.
   (b) Show how to find $f(x)$, given also that $f(1) = 4 - 8e$.
4. Let F be the function given by $F(x) = \cos^3 x$, and whose domain is the interval $-\dfrac{\pi}{4} < x < \dfrac{9\pi}{4}$. Show how to determine the intervals where F is increasing, and the intervals where F is decreasing. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*
5. Let g be the function given by
   $$g(x) = \frac{4x}{1+x^2}$$
   on the interval $[-4, 0]$. Show how to find the absolute maximum and the absolute minimum values of $g(x)$ on this interval. *Give your reasoning.*
6. A dock lies on shore that runs east-west. A lighthouse lies exactly one mile offshore directly north of the dock. The light rotates twice every minute, counterclockwise as seen from above. How fast is the spot of light cast by the lighthouse moving along the beach when it is one mile east of the dock? *Give your reasoning.*

MTH 1410, Calculus I—Exam III **November 19, 2014**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 5:50 pm.

1. Show how to find the limits:

   (a) $\lim_{x\to 1} \dfrac{\ln(x^2)}{x-1}$

   (b) $\lim_{x\to 0} \dfrac{e^x + e^{-x} - 2}{\cos x - 1}$

   Solution:

   (a) We note first that $\lim_{x\to 1} \ln x^2 = \ln 1 = 0$, and that $\lim_{x\to 1}(x-1) = 0$. We may attempt l'Hôpital's rule:

$$\lim_{x\to 1} \frac{\ln(x^2)}{x-1} = \lim_{x\to 1} \frac{2\ln(x)}{x-1} = \lim_{x\to 1} \frac{\left(\frac{2}{x}\right)}{1} = 2.$$

   (b) We have $\lim_{x\to 0}(e^x + e^{-x} - 2) = 1 + 1 - 2 = 0$ and $\lim_{x\to 0}(\cos x - 1) = 1 - 1 = 0$, so we may attempt l'Hôpital's rule:

$$\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{\cos x - 1} = \lim_{x\to 0} \frac{e^x - e^{-x}}{-\sin x},$$

   provided the latter limit exists. Now we note that $\lim_{x\to 0}(e^x - e^{-x}) = 0$ and $\lim_{x\to 0} \sin x = 0$, so that l'Hôpital's rule is applicable again:

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{-\sin x} = \lim_{x\to 0} \frac{e^x + e^{-x}}{-\cos x} = -2,$$

   and we conclude that the desired limit is 2.

2. Show how to use the definition of derivative and the Limit Laws to find $f'(x)$ when $f(x) = \dfrac{1}{\sqrt{x+1}}$.

   Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \left[\frac{1}{h}\left(\frac{1}{\sqrt{x+h+1}} - \frac{1}{\sqrt{x+1}}\right)\right] \\
&= \lim_{h\to 0} \frac{\sqrt{x+1} - \sqrt{x+h+1}}{h\sqrt{x+h+1}\sqrt{x+1}} \\
&= \lim_{h\to 0} \frac{(x+1) - (x+h+1)}{h\sqrt{x+h+1}\sqrt{x+1}\left(\sqrt{x+1} + \sqrt{x+h+1}\right)} \\
&= \lim_{h\to 0} \frac{\cancel{x} + \cancel{1} - \cancel{x} - h - \cancel{1}}{h\sqrt{x+h+1}\sqrt{x+1}\left(\sqrt{x+1} + \sqrt{x+h+1}\right)} \\
&= -\lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\sqrt{x+h+1}\sqrt{x+1}\left(\sqrt{x+1} + \sqrt{x+h+1}\right)} \\
&= -\frac{1}{\sqrt{x+1}\sqrt{x+1}\left[2\sqrt{x+1}\right]} = -\frac{1}{2(x+1)^{3/2}}.
\end{aligned}$$

3. Let f be the function whose second derivative is $f''(x) = \dfrac{3}{x^2} - 8e^x$.

(a) Show how to find $f'(x)$, given that $f'(1) = -8e$.

(b) Show how to find $f(x)$, given also that $f(1) = 4 - 8e$.

Solution:

(a) If $f''(x) = 3x^{-2} - 8e^x$, there is a constant, c_1 such that $f'(x) = -3x^{-1} - 8e^x + c_1$. But then

$$-8e = f'(1) = -3(1)^{-1} - 8e^1 + c_1 = -3 - 8e + c_1,$$

and it follows that $c_1 = 3$. Thus, $f'(x) = 3 - 3x^{-1} - 8e^x = 3 - \dfrac{3}{x} - 8e^x$.

(b) From what we have seen in part (a), above, we now know that there is a constant, c_2, such that $f(x) = 3x - 3\ln x - 8e^x + c_2$. But

$$4 - 8e = f(1) = 3 \cdot 1 - 3\ln 1 - 8e^1 + c_2 = 3 - 8e + c_2,$$

so that $c_2 = 1$. Thus, $f(x) = 3x - 3\ln x - 8e^x + 1$.

4. Let F be the function given by $F(x) = \cos^3 x$, and whose domain is the interval $-\dfrac{\pi}{4} < x < \dfrac{9\pi}{4}$. Show how to determine the intervals where F is increasing, and the intervals where F is decreasing. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

Solution: We have

$$f'(x) = -3\cos^2 x \sin x,$$

which is defined on all of $(-\pi/4, 9\pi/4)$, zero where either $\cos x$ or $\sin x$ is zero, and otherwise positive where $\sin x$ is negative, and negative where $\sin x$ is positive. But $\sin x$ is negative on $(-\pi/4, 0)$ and on $(\pi, 2\pi)$ and positive on $(0, \pi)$ and on $(2\pi, 9\pi/4)$. Thus, f is decreasing on the intervals $[0, \pi]$ and $[2\pi, 9\pi/4)$, and f is increasing on the intervals $(-\pi/4, 0]$ and $[\pi, 2\pi]$.

The zeros of $f'(x)$, as we have seen, are at the zeros of the functions $\sin x$ and $\cos x$. In the interval $(-\pi/4, 9\pi/4)$, these are $x = 0$, $x = \pi/2$, $x = \pi$, $x = 3\pi/2$ and $x = 2\pi$. These are the critical numbers for f. We note that $f'(x)$ changes sign when $\sin x$ changes sign (although the sign of $f(x)$ is always the opposite of the sign of the sine function), but not when $\cos x$ changes sign. At $x = 0$, $f'(x)$ changes sign from positive to negative, so, by the First Derivative Test, f has a local maximum at $x = 0$. The sine function also undergoes a sign change at $x = \pi$, and the sign of $f'(x)$ changes from negative to positive there. Again, by the First Derivative Test, f has a local minimum at $x = \pi$. At $x = 2\pi$, $f'(x)$ undergoes another sigh change from positive to negative, giving another local maximum. At the critical points for f where $\cos x = 0$, the derivative doesn't undergo a sign change, so, by the First Derivative Test, again, the critical points $x = \pi/2$ and $x = 3\pi/2$ give neither a local maximum nor a local minimum.

5. Let g be the function given by

$$g(x) = \frac{4x}{1 + x^2}$$

on the interval $[-4, 0]$. Show how to find the absolute maximum and the absolute minimum values of $g(x)$ on this interval. *Give your reasoning.*

Solution: The absolute extremes of a differentiable function on a closed, bounded interval are to be found either at endpoints of the interval or at critical points. Here, the endpoints are $x = -4$, and $x = 0$. To find the critical points, we must solve $f'(x) = 0$. This leads to

$$0 = f'(x) = \frac{4(1 + x^2) - 8x^2}{(1 + x^2)^2} = \frac{4 - 4x^2}{(1 + x^2)^2}.$$

The only solution to this equation in the interval $[-4, 0]$ is $x = -1$. We have

$$f(-4) = \frac{4 \cdot (-4)}{1 + (-4)^2} = -\frac{16}{17};$$
$$f(-1) = \frac{4 \cdot (-1)}{1 + (-1)^2} = -2;$$
$$f(0) = \frac{4 \cdot 0}{1 + 0^2} = 0.$$

Thus, the absolute minimum value of $f(x)$ in the interval $[-4, 0]$ is $f(-1) = -2$, and the absolute maximum value of $f(x)$ in the interval $[-4, 0]$ is $f(0) = 0$.

6. A dock lies on shore that runs east-west. A lighthouse lies exactly one mile offshore directly north of the dock. The light rotates twice every minute, counterclockwise as seen from above. How fast, in miles per hour, is the spot of light cast by the lighthouse moving along the beach when it is one mile east of the dock? *Give your reasoning.*

   Solution: Let x denote the distance along the shore from the dock, measured in miles to the east, to the spot of light cast by the lighthouse. Let θ be the angle, measured counterclockwise in radians, that the light beam makes with the line that connects the lighthouse to the dock. The light rotates twice every minute, or 120 times every hour. But a full rotation is 2π radians, so $\frac{d\theta}{dt} = 240\pi$ radians per hour. We have

$$\arctan x = \theta, \text{ so that}$$
$$\frac{1}{1+x^2}\frac{dx}{dt} = \frac{d\theta}{dt}.$$

   It follows, now, that at the critical instant we have

$$\frac{1}{2}\frac{dx}{dt} = 240\pi, \text{ or}$$
$$\frac{dx}{dt} = 480\pi.$$

   The spot of light cast by the lighthouse is moving at 480π miles per hour when it is one mile east of the dock.

MTH 1410, Calculus I—Exam III **November 13, 2014**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 1:50 pm.

1. Show how to find the limits:

   (a) $\displaystyle\lim_{x\to -1} \frac{x+x^2}{\ln(2+x)}$

   (b) $\displaystyle\lim_{x\to 0} \frac{xe^{-2x}}{\pi e^{2x}-\pi}$

2. Show how to use the definition of derivative and the Limit Laws to find $f'(x)$ when $f(x) = \dfrac{x}{x+1}$.

3. Let f be the function whose second derivative is $f''(x) = 3x + 6\sin x$.

   (a) Show how to find $f'(x)$, given that $f'(0) = 3$.

   (b) Show how to find $f(x)$, given that $f(0) = 3$ as well.

4. Let F be the function given by

   $$F(x) = (x-1)^2(x+1)^3.$$

   Then, in fully factored form,

   $$F'(x) = (x-1)(x+1)^2(5x-1)$$

   and, also in fully factored form,

   $$F''(x) = 20(x+1)\left[x - \frac{1}{5}\left(1-\sqrt{6}\right)\right]\left[x - \frac{1}{5}\left(1+\sqrt{6}\right)\right].$$

   Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

5. Find the points on the ellipse $x^2 + 4y^2 = 4$ whose distance from the point $(1, 0)$ is minimal.

6. Murgatroyd was driving his car toward an intersection at 60 miles per hour. A police cruiser was approaching the same intersection but on the cross-street (which is at right angles to the road that Muratroyd is on), at 50 miles per hour. When both cars were a quarter of a mile from the intersection, a police officer in the cruiser pointed a radar gun at Murgatroyd and measured the speed at which the two cars were approaching each other. What did she get?

MTH 1410, Calculus I—Exam III **November 13, 2014**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 1:50 pm.

1. Find the limits:

   (a) $\lim\limits_{x\to -1} \dfrac{x+x^2}{\ln(2+x)}$

   (b) $\lim\limits_{x\to 0} \dfrac{xe^{-2x}}{\pi e^{2x}-\pi}$

   Solution:

   (a) $\lim_{x\to -1}(x+x^2) = 0 = \lim_{x\to -1}\ln(2+x)$, so we may attempt l'Hôpital's rule:

$$\begin{aligned}\lim_{x\to -1}\frac{x+x^2}{\ln(2+x)} &= \lim_{x\to -1}\frac{1+2x}{[1/(2+x)]}\\ &= \lim_{x\to -1}(1+2x)(2+x)\\ &= -1.\end{aligned}$$

   (b) $\lim_{x\to 0} xe^{-2x} = 0 = \lim_{x\to 0}(\pi e^{2x}-\pi)$, so we can attempt l'Hôpital's rule again:

$$\begin{aligned}\lim_{x\to 0}\frac{xe^{-2x}}{\pi e^{2x}-\pi} &= \lim_{x\to 0}\frac{e^{-2x}-2xe^{-2x}}{2\pi e^{2x}}\\ &= \frac{1}{2\pi}.\end{aligned}$$

2. Use the definition of derivative to find $f'(x)$ when $f(x) = \dfrac{x}{x+1}$.

   Solution:

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{h}\left[\frac{x+h}{(x+h)+1}-\frac{x}{x+1}\right]\\ &= \lim_{h\to 0}\frac{(x+h)(x+1)-x(x+h+1)}{h(x+1)(x+h+1)} = \lim_{h\to 0}\frac{\cancel{x^2}+\cancel{hx}+\cancel{x}+h-\cancel{x^2}-\cancel{xh}-\cancel{x}}{h(x+1)(x+h+1)}\\ &= \lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(x+1)(x+h+1)} = \lim_{h\to 0}\frac{1}{(x+1)(x+h+1)} = \frac{1}{(x+1)^2}.\end{aligned}$$

3. Let f be the function whose second derivative is $f''(x) = 3x + 6\sin x$.

   (a) Show how to find $f'(x)$, given that $f'(0) = 3$.

   (b) Show how to find $f(x)$, given that $f(0) = 3$ as well.

   Solution: If $f''(x) = 3x + 6\sin x$, then

$$f'(x) = \frac{3}{2}x^2 - 6\cos x + c_1,$$

   where c_1 is an as yet to be determined constant. But

$$3 = f'(0) = \frac{3}{2}\cdot 0^2 - 6\cos 0 + c_1 = -6 + c_1,$$

whence $c_1 = 9$. Thus,

$$f'(x) = \frac{3}{2}x^2 - 6\cos x + 9.$$

Now it follows that there is a constant c_2 such that

$$f(x) = \frac{1}{2}x^3 - 6\sin x + 9x + c_2.$$

From the fact that $f(0) = 3$, we now see that

$$3 = f(0) = \frac{1}{2}\cdot 0^3 - 6\sin 0 + 9\cdot 0 + c_2 = c_2,$$

and it follows that $c_2 = 3$ and

$$f(x) = \frac{1}{2}x^3 + 9x + 3 - 6\sin x.$$

4. Let F be the function given by

$$F(x) = (x-1)^2(x+1)^3.$$

Then, in fully factored form,

$$F'(x) = (x-1)(x+1)^2(5x-1)$$

and, also in fully factored form,

$$F''(x) = 20(x+1)\left[x - \frac{1}{5}\left(1-\sqrt{6}\right)\right]\left[x - \frac{1}{5}\left(1+\sqrt{6}\right)\right].$$

Use this information to determine the intervals where F is increasing, the intervals where F is decreasing, the intervals where F is concave upward, and the intervals where F is concave downward. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

Solution: The quantity $(x-1)$ is positive when $x > 1$ and negative when $x < 1$; $(x+1)^2$ is positive unless $x = -1$; and $(5x-1)$ is positive when $x > 1/5$, negative when $x < 1/5$. Thus, $F'(x) > 0$ when $-\infty < x < -1$, when $-1 < x < 1/5$ and when $1 < x < \infty$. It follows that F is increasing on the intervals $(-\infty, 1/5]$ and $[1, \infty)$, but decreasing on $[1/5, 1]$.

The quantity $(x+1)$ is negative when $x < -1$ and positive when $x > -1$; $[x-(1-\sqrt{6})/5]$ is negative when $x < (1-\sqrt{6})/5$ and positive when $x > (1-\sqrt{6})/5$; $[x-(1+\sqrt{6})/5]$ is negative when $x < (1+\sqrt{6})/5$ and positive when $x > (1+\sqrt{6})/5$. Consequently $F''(x) < 0$ when $-\infty < x < -1$ and when $(1-\sqrt{6})/5 < x < (1+\sqrt{6})/5$, but $F''(x) > 0$ when $-1 < x < (1-\sqrt{6})/5$ and when $(1+\sqrt{6})/5 < x < \infty$. So F is concave upward on $[-1, (1-\sqrt{6})/5]$ and on $[(1+\sqrt{6})/5, \infty)$, but concave downward on $(-\infty, -1]$ and on $[(1-\sqrt{6})/5, (1+\sqrt{6})/5]$.

From the expression for $F'(x)$, we see that the critical numbers for F are $x = 1$, $x = -1$, and $x = 1/5$. Our analysis of the increasing/decreasing behavior of F above, shows that $x = 1$ gives a local minimum for F because F is decreasing just to the left of $x = 1$ but increasing just to the right of $x = 1$. Similarly, $x = 1/5$ gives a local maximum for F, and $x = -1$ gives neither a local maximum nor a local minimum.

5. Find the points on the ellipse $x^2 + 4y^2 = 4$ whose distance from the point $(1, 0)$ is minimal.

Solution: Let (x, y) be any point on the curve. Then $x^2 + 4y^2 = 4$ and the square S of the distance from (x, y) to $(1, 0)$ is $S = (x-1)^2 + y^2$, where $|x| \leq 2$. We can minimize distance by minimizing $S = (x-1)^2 + y^2$ subject to the constraint $x^2 + 4y^2 = 1$. Thus we want to find the critical points of S.

Treating y as a function of x and differentiating, we find that $dS/dx = 2(x-1) + 2yy'$, so want to learn where $2(x-1) + 2yy' = 0$. From $x^2 + 4y^2 = 4$, we see that $2x + 8yy' = 0$, or $y' = -x/(4y)$. Thus, we want

$$\begin{aligned} 0 &= 2(x-1) + 2yy' \\ &= x - 1 + y\left(-\frac{x}{4y}\right) \\ &= x - 1 - \frac{1}{4}x \\ &= \frac{3}{4}x - 1, \end{aligned}$$

so that

$$x = \frac{4}{3}.$$

The only critical number for $S(x)$ is thus at $x = 4/3$. The minimum for $S(x)$ must occur either at $x = 4/3$ or at an endpoint $x = \pm 2$. We note that from $x^2 + 4y^2 = 1$ it follows that $y = \pm\sqrt{5}/3$ when $x = 4/3$ and that $y = 0$ when $x = \pm 2$. We therefore have $S(-2) = 9$, $S(4/3) = 2/3$, and $S(2) = 1$. The minimal distance therefore occurs when $x = 4/3$ and $y = \pm\sqrt{5}/3$—that is, at the points $(4/3, \sqrt{5}/3)$ and $(4/3, -\sqrt{5}/3)$.

6. Murgatroyd was driving his car toward an intersection at 60 miles per hour. A police cruiser was approaching the same intersection but on the cross-street (which is at right angles to the road that Muratroyd is on), at 50 miles per hour. When both cars were a quarter of a mile from the intersection, a police officer in the cruiser pointed a radar gun at Murgatroyd and measured the speed at which the two cars were approaching each other. What did she get?

   Solution: Let x denote the distance from Murgatroyd to the intersection, and let y denote the distance from the police car to the interesection. The distance D between the two cars satisfies

$$D^2 = x^2 + y^2,$$

so

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt},$$

or

$$\frac{dD}{dt} = \frac{1}{D}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

At the critical instant, we have $x = y = 1/4$, $D = 1/(2\sqrt{2})$, $dx/dt = -60$, and $dy/dt = -50$. Thus, rate of change of the distance between the two cars was

$$\begin{aligned} \frac{dD}{dt} &= 2\sqrt{2}\left[\frac{1}{4}(-60) + \frac{1}{4}(-50)\right] \\ &= -\frac{110}{\sqrt{2}} \text{ mph.} \end{aligned}$$

The reading on the radar gun was $110/\sqrt{2}$ mph, or about 78 mph.

MTH 1410, Calculus I (TΘ)—FinalExam **December 10, 2014**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 5:30 pm.

1. Show how to use the Limit Laws to evaluate the following limits. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $\lim_{x\to 9} \dfrac{\left(\dfrac{1}{x} - \dfrac{1}{9}\right)}{x-9}$

   (b) $\lim_{x\to\infty} \left[\sqrt{x^2 - 5x + 12} - x\right]$.

2. Show how to use the definition of derivative to find $f'(x)$ when f is given by

$$f(x) = \frac{3x+2}{2x-3}.$$

3. (a) Show how to find an equation for the line tangent to the curve

$$2(x^2+y^2)^2 = 25(x^2-y^2)$$

   at the point $(-3, 1)$.

   (b) Show how to use the equation from part (a), above, to give an approximation for the value of y on the curve near $y = 1$ that corresponds to $x = -3.013$.

4. (a) Suppose that $h(-2) = -15$, $k(-2) = -4$, $h'(-2) = -12$, $k'(-2) = -3$, and $H(x) = \dfrac{h(x)}{k(x)}$. Show how to find $H'(-2)$.

   (b) Let $F(x) = f(f(x))$ and $G(x) = [F(x)]^2$, and suppose that $f(9) = 14$, $f(14) = 3$, $f'(14) = 8$, and $f'(9) = 3$. Show how to find $F'(9)$ and $G'(9)$.

5. Let f be the function given by

$$f(x) = \begin{cases} ax+b, & \text{when } 2 < x < 9, \\ \\ x^2 - 8x + 22, & \text{when } x \le 2 \text{ or } x \ge 9, \end{cases}$$

   where a and b are certain constants. Find all values of a and b for which the function f is continuous everywhere. Be sure to give the reasoning that supports your conclusions.

6. Show how to evaluate:

   (a) $\int_0^3 (x^2 - 2x + 3)\, dx$

   (b) $\int_0^3 \dfrac{x\, dx}{\sqrt{25 - x^2}}$

MTH 1410, Calculus I (TΘ)—FinalExam **December 10, 2014**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 5:30 pm.

1. Show how to use the Limit Laws to evaluate the following limits. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $\lim_{x\to 9} \dfrac{\left(\dfrac{1}{x} - \dfrac{1}{9}\right)}{x-9}$

   (b) $\lim_{x\to\infty} \left[\sqrt{x^2-5x+12} - x\right]$.

   Solution:

   (a) We have $\lim_{x\to 9} \left[\dfrac{1}{x} - \dfrac{1}{9}\right] = 0 = \lim_{x\to 9}(x-9)$, so we may not use the division rule. However,

$$\lim_{x\to 9} \frac{\left(\dfrac{1}{x} - \dfrac{1}{9}\right)}{x-9} = \lim_{x\to 9}\left(\left[\frac{1}{x-9}\right]\left[\frac{9-x}{9x}\right]\right)$$
$$= -\lim_{x\to 9}\left(\left[\frac{1}{\cancel{x-9}}\right]\left[\frac{\cancel{x-9}}{9x}\right]\right) = -\lim_{x\to 9}\frac{1}{9x} = -\frac{1}{81}.$$

   (b) We may not use the subtraction rule for limits, because neither $\lim_{x\to\infty}\sqrt{x^2-5x+12}$ nor $\lim_{x\to\infty} x$ exists. Nevertheless,

$$\lim_{x\to\infty}\left[\sqrt{x^2-5x+12} - x\right] = \lim_{x\to\infty}\frac{(\cancel{x^2}-5x+12)-\cancel{x^2}}{\sqrt{x^2-5x+12}+x}$$
$$= \lim_{x\to\infty}\frac{-5x+12}{\sqrt{x^2-5x+12}+x}$$
$$= \lim_{x\to\infty}\frac{\left(-5+\dfrac{12}{x}\right)}{\sqrt{1-\dfrac{5}{x}+\dfrac{12}{x^2}}+1} = -\frac{5}{2}.$$

2. Show how to use the definition of derivative to find $f'(x)$ when f is given by

$$f(x) = \frac{3x+2}{2x-3}.$$

   Solution:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{3(x+h)+2}{2(x+h)-3} - \frac{3x+2}{2x-3}\right)$$
$$= \lim_{h\to 0}\frac{(\cancel{6x^2}+\cancel{6hx}-\cancel{5x}-9h-\cancel{6})-(\cancel{6x^2}+\cancel{6hx}-\cancel{5x}+4h-\cancel{6})}{h[2(x+h)-3](2x-3)}$$
$$= \lim_{h\to 0}\frac{-13\cancel{h}}{\cancel{h}[2(x+h)-3](2x-3)} = \frac{-13}{(2x-3)^2}.$$

3. (a) Show how to find an equation for the line tangent to the curve

$$2(x^2 + y^2)^2 = 25(x^2 - y^2)$$

at the point $(-3, 1)$.

(b) Show how to use the equation from part (a), above, to give an approximation for the value of y on the curve near $y = 1$ that corresponds to $x = -3.013$.

Solution:

(a) Treating y as a function of x and differentiating implicitly with respect to x, and, we obtain

$$4(x^2 + y^2)(2x + 2yy') = 25(2x - 2yy'),$$

so that when $x = -3$ and $y = 1$,

$$\begin{aligned} 4(9+1)((-6+2y') &= -25(6+2y'); \\ -240 + 80y' &= -150 - 50y'; \\ y' &= \frac{9}{13}. \end{aligned}$$

Thus, the desired equation is

$$y = 1 + \frac{9}{13}(x + 3),$$

or

$$y = \frac{9}{13}x + \frac{40}{13}.$$

(b) The tangent line to a curve is a good approximation to the curve near the point of tangency, so we can approximate the required value of y at $x = -3.013$ by finding the corresponding y-value on the tangent line when $x = -3.013$. Thus, the require approximation is

$$y = 1 + \frac{9}{13}(-3.013 + 3) = 1 - 0.009 = 0.991.$$

[Note for the curious: The approximation is pretty good; a more exact calculation shows that y is about -0.990883525 when $x = -3.013$.]

4. (a) Suppose that $h(-2) = -15$, $k(-2) = -4$, $h'(-2) = -12$, $k'(-2) = -3$, and $H(x) = \dfrac{h(x)}{k(x)}$. Show how to find $H'(-2)$.

(b) Let $F(x) = f(f(x))$ and $G(x) = [F(x)]^2$, and suppose that $f(9) = 14$, $f(14) = 3$, $f'(14) = 8$, and $f'(9) = 3$. Show how to find $F'(9)$ and $G'(9)$.

Solution:

(a) If $H(x) = \dfrac{h(x)}{k(x)}$, then

$$H'(x) = \frac{h'(x)k(x) - h(x)k'(x)}{[k(x)]^2},$$

so that

$$\begin{aligned} H'(-2) &= \frac{h'(-2)k(-2) - h(-2)k'(-2)}{[k(-2)]^2} \\ &= \frac{(-12)(-4) - (-15)(-3)}{(-4)^2} = \frac{3}{16}. \end{aligned}$$

(b) If $F(x) = f(f(x))$ and $G(x) = [F(x)]^2$, then

$$F'(x) = f'[f(x)] \cdot f'(x),$$

so that

$$F'(9) = f'[f(9)] \cdot f'(9) = f'(14) \cdot 3 = 8 \cdot 3 = 24;$$

and

$$G'(x) = 2F(x)F'(x) = 2f[f(x)] \cdot F'(x),$$

so that

$$G'(9) = 2f[f(9)] \cdot F'(9) = 2 \cdot f[14] \cdot 24 = 48 \cdot 3 = 144.$$

5. Let f be the function given by

$$f(x) = \begin{cases} ax + b, & \text{when } 2 < x < 9, \\ \\ x^2 - 8x + 22, & \text{when } x \leq 2 \text{ or } x \geq 9, \end{cases}$$

where a and b are certain constants. Find all values of a and b for which the function f is continuous everywhere. Be sure to give the reasoning that supports your conclusions.

Solution: The only places where there is a question are $x = 2$ and $x = 9$. For continuity at $x = 2$ we must have $\lim_{x\to 2} f(x) = f(2) = -6$. This can be so if and only if

$$\lim_{x\to 2^+} f(x) = 10 = \lim_{x\to 2^-} f(x).$$

But

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 - 8x + 22) = 10,$$

and

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (ax + b) = 2a + b,$$

So the condition

$$2a + b = 10$$

must be satisfied if f is to be continuous at $x = 2$. A similar analysis at $x = 9$ leads to the condition

$$9a + b = 31,$$

which must be satisfied if f is to be continuous at $x = 9$.

If both these conditions are satisfied, then $7a = (9a + b) - (2a + b) = 31 - 10 = 21$, or $7a = 21$. It then follows that $a = 3$ and $b = 4$.

6. Show how to evaluate:

(a) $\displaystyle\int_0^3 (x^2 - 2x + 3)\, dx$

(b) $\displaystyle\int_0^3 \frac{x\, dx}{\sqrt{25 - x^2}}$

Solution:

(a)

$$\begin{aligned}\int_0^3 (x^2 - 2x + 3)\,dx &= \left(\frac{x^3}{3} - x^2 + 3x\right)\Big|_0^3 \\ &= \left(\frac{27}{3} - 9 + 9\right) - 0 = 9.\end{aligned}$$

(b) We know that if $F(x)$ is an antiderivative for $f(x)$, then $\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a)$. Thus, we must find an antiderivative for $x/\sqrt{25 - x^2}$. To this end we let $u = 25 - x^2$. Then $du = -2x\,dx$. Thus,

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{25 - x^2}} &= -\frac{1}{2}\int u^{-1/2}\,du \\ &= -u^{1/2} = -\sqrt{25 - x^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^3 \frac{x\,dx}{\sqrt{25 - x^2}} &= -\sqrt{25 - x^2}\Big|_0^3 \\ &= -\sqrt{25 - 9} + \sqrt{25 - 0} = 1.\end{aligned}$$

MTH 1410, Calculus I (TΘ)—FinalExam **December 9, 2014**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 1:00 pm.

1. Show how to use the Limit Laws to evaluate the following limits. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.
   (a) $\displaystyle\lim_{x\to 3} \frac{\left(\frac{1}{x}-\frac{1}{3}\right)}{x^2-9}$
   (b) $\displaystyle\lim_{x\to\infty} \left[\sqrt{x^2-3x+12}-x\right]$.
2. Show how to use the definition of derivative to find $f'(x)$ when f is given by
$$f(x)=\frac{x-1}{x+1}.$$
3. (a) Suppose that $h(-2)=-15$, $k(-2)=-4$, $h'(-2)=-12$, $k'(-2)=-3$, and $H(x)=\dfrac{h(x)}{k(x)}$. Show how to find $H'(-2)$.
   (b) Let $F(x)=f(f(x))$ and $G(x)=[F(x)]^2$, and suppose that $f(3)=15$, $f(15)=255$, $f'(15)=32$, and $f'(3)=8$. Show how to find $F'(3)$ and $G'(3)$.
4. (a) Show how to find an equation for the line tangent to the curve
$$3(x^2+y^2)^2=100(y^2-x^2)$$
   at the point $(2,-4)$.
   (b) Show how to use the equation from part (a), above, to give an approximate value for the value of y near $y=-4$ on the curve that corresponds to $x=2.02$.
5. Let F be the function given by $F(x)=\cos^3 x$, and whose domain is the interval $-\dfrac{\pi}{4}<x<\dfrac{9\pi}{4}$. Show how to determine the intervals where F is increasing, and the intervals where F is decreasing. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*
6. Show how to evaluate:
   (a) $\displaystyle\int_2^4 (3x^3+5x)\,dx$
   (b) $\displaystyle\int_0^2 \frac{2x}{1+x^2}\,dx$

MTH 1410, Calculus I (TΘ)—FinalExam **December 9, 2014**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 1:00 pm.

1. Show how to use the Limit Laws to evaluate the following limits. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $\displaystyle\lim_{x\to 3} \frac{\left(\frac{1}{x} - \frac{1}{3}\right)}{x^2 - 9}$

   (b) $\displaystyle\lim_{x\to\infty} \left[\sqrt{x^2 - 3x + 12} - x\right]$.

   Solution:

   (a)

$$\lim_{x\to 3} \frac{\left(\frac{1}{x} - \frac{1}{3}\right)}{x^2 - 9} = \lim_{x\to 3} \frac{3 - x}{3x(x^2 - 9)}$$
$$= \lim_{x\to 3} \frac{-\cancel{(x-3)}}{3x\cancel{(x-3)}(x+3)}$$
$$= \frac{-1}{\lim_{x\to 3}[3x(x+3)]} = -\frac{1}{54}.$$

   (b)

$$\lim_{x\to\infty} \left[\sqrt{x^2 - 3x + 12} - x\right] = \lim_{x\to\infty} \frac{(\cancel{x^2} - 3x + 12) - \cancel{x^2}}{\sqrt{x^2 - 3x + 12} + x}$$
$$= -\lim_{x\to\infty} \frac{3 - \left(\frac{12}{x}\right)}{\sqrt{1 - \frac{3}{x} + \frac{12}{x^2}} + 1}$$
$$= -\frac{\lim_{x\to\infty}\left[3 - \left(\frac{12}{x}\right)\right]}{\lim_{x\to\infty}\left[\sqrt{1 - \frac{3}{x} + \frac{12}{x^2}} + 1\right]} = -\frac{3}{2}$$

2. Show how to use the definition of derivative to find $f'(x)$ when f is given by

$$f(x) = \frac{x-1}{x+1}.$$

Solution:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x+h)-1}{(x+h)+1} - \frac{x-1}{x+1}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x-1)+h}{(x+1)+h} - \frac{x-1}{x+1}\right] \\
&= \lim_{h\to 0} \frac{[(x-1)(x+1)+h(x+1)] - [((x+1)(x-1)+h(x-1)]}{h(x+1)[(x+1)+h]} \\
&= \lim_{h\to 0} \frac{\cancel{x^2} - \cancel{1} + \cancel{hx} + h - \cancel{x^2} + \cancel{1} - \cancel{hx} + h}{h(x+1)[(x+1)+h]} \\
&= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}(x+1)[(x+1)+h]} = \frac{2}{(x+1)^2}.
\end{aligned}
$$

3. (a) Suppose that $h(-2) = -15$, $k(-2) = -4$, $h'(-2) = -12$, $k'(-2) = -3$, and $H(x) = \dfrac{h(x)}{k(x)}$. Show how to find $H'(-2)$.

   (b) Let $F(x) = f(f(x))$ and $G(x) = [F(x)]^2$, and suppose that $f(3) = 15$, $f(15) = 255$, $f'(15) = 32$, and $f'(3) = 8$. Show how to find $F'(3)$ and $G'(3)$.

Solution:

(a) If $H(x) = \dfrac{h(x)}{k(x)}$, then

$$H'(x) = \frac{h'(x)k(x) - h(x)k'(x)}{[k(x)]^2},$$

so that

$$
\begin{aligned}
H'(-2) &= \frac{h'(-2)k(-2) - h(-2)k'(-2)}{[k(-2)]^2} \\
&= \frac{(-12)(-4) - (-15)(-3)}{(-4)^2} = \frac{3}{16}.
\end{aligned}
$$

(b) If $F(x) = f(f(x))$ and $G(x) = [F(x)]^2$, then

$$F'(x) = f'[f(x)] \cdot f'(x),$$

so that

$$F'(3) = f'[f(3)] \cdot f'(3) = f'(15) \cdot 8 = 32 \cdot 8 = 256;$$

and

$$G'(x) = 2F(x)F'(x) = 2f[f(x)] \cdot F'(x),$$

so that

$$G'(3) = 2f[f(3)] \cdot F'(3) = 2 \cdot f[15] \cdot 256 = 2 \cdot 255 \cdot 256 = 130560.$$

4. (a) Show how to find an equation for the line tangent to the curve

$$3(x^2 + y^2)^2 = 100(y^2 - x^2)$$

at the point $(2, -4)$.

(b) Show how to use the equation from part (a), above, to give an approximate value for the value of y near $y = -4$ on the curve that corresponds to $x = 2.02$.

Solution:

(a) Treating y as a function of x and differentiating the equation implicitly, we obtain

$$6(x^2 + y^2)(2x + 2yy') = 100(2yy' - 2x).$$

Putting $x = 2$ and $y = -4$ now gives

$$\begin{aligned} 6 \cdot 20 \cdot (4 - 8y') &= 100(-8y' - 4); \\ 480 - 960y' &= -800y' - 400; \\ -160y' &= -880; \\ y' &= \frac{11}{2}. \end{aligned}$$

It now follows that the desired equation is $y = -4 + \frac{11}{2}(x - 2)$, which can be rewritten as

$$11x - 2y - 30 = 0.$$

(b) The tangent line at the point $(2, -4)$ lies close to the curve in the vicinity of $x = 2$, so an approximation for the value of y near $y = -4$ when $x = 2.02$ is

$$y = -4 + \frac{11}{2}(2.02 - 2) = -4 + 0.11 = -3.89.$$

5. Let F be the function given by $F(x) = \cos^3 x$, and whose domain is the interval $-\frac{\pi}{4} < x < \frac{9\pi}{4}$. Show how to determine the intervals where F is increasing, and the intervals where F is decreasing. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

Solution: We have

$$f'(x) = -3\cos^2 x \sin x,$$

which is defined on all of $(-\pi/4, 9\pi/4)$, zero where either $\cos x$ or $\sin x$ is zero, and otherwise positive where $\sin x$ is negative, and negative where $\sin x$ is positive. But $\sin x$ is negative on $(-\pi/4, 0)$ and on $(\pi, 2\pi)$ and positive on $(0, \pi)$ and on $(2\pi, 9\pi/4)$. Thus, f is decreasing on the intervals $[0, \pi]$ and $[2\pi, 9\pi/4)$, and f is increasing on the intervals $(-\pi/4, 0]$ and $[\pi, 2\pi]$.

The zeros of $f'(x)$, as we have seen, are at the zeros of the functions $\sin x$ and $\cos x$. In the interval $(-\pi/4, 9\pi/4)$, these are $x = 0$, $x = \pi/2$, $x = \pi$, $x = 3\pi/2$ and $x = 2\pi$. These are the critical numbers for f. We note that $f'(x)$ changes sign when $\sin x$ changes sign (although the sign of $f(x)$ is always the opposite of the sign of the sine function), but not when $\cos x$ changes sign. At $x = 0$, $f'(x)$ changes sign from positive to negative, so, by the First Derivative Test, f has a local maximum at $x = 0$. The sine function also undergoes a sign change at $x = \pi$, and the sign of $f'(x)$ changes from negative to positive there. Again, by the First Derivative Test, f has a local minimum at $x = \pi$. At $x = 2\pi$, $f'(x)$ undergoes another sigh change from positive to negative, giving another local maximum. At the critical points for f where $\cos x = 0$, the derivative doesn't undergo a sign change, so, by the First Derivative Test, again, the critical points $x = \pi/2$ and $x = 3\pi/2$ give neither a local maximum nor a local minimum.

6. Show how to evaluate:

(a) $\displaystyle\int_2^4 (3x^3 + 5x)\, dx$

(b) $\displaystyle\int_0^2 \frac{2x}{1+x^2}\, dx$

Solution:

(a)

$$\begin{aligned}
\int_2^4 (3x^3 + 5x)\, dx &= \left(\frac{3}{4}x^4 + \frac{5}{2}x^2\right)\Big|_2^4 \\
&= \left(\frac{3}{4}\cdot 4^4 + \frac{5}{2}\cdot 4^2\right) - \left(\frac{3}{4}\cdot 2^4 + \frac{5}{2}\cdot 2^2\right) \\
&= 210.
\end{aligned}$$

(b) Let $u = 1 + x^2$. Then $du = 2x\, dx$, $u = 1$ when $x = 0$, and $u = 5$ when $x = 2$. Thus,

$$\begin{aligned}
\int_0^2 \frac{2x}{1+x^2}\, dx &= \int_1^5 \frac{du}{u} \\
&= \ln|u|\Big|_1^5 \\
&= \ln 5 - \ln 1 \\
&= \ln 5.
\end{aligned}$$

MTH 1410, Calculus I —Exam I **Feb. 12, 2015**

Instructions: Work the following problems on your own paper; give your reasoning and show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Your paper is due at 1:50 pm.

1. Show how to use the Limit Laws to evaluate the following limits. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $\displaystyle\lim_{x\to 1/2} \frac{2x^3 + 7x^2 - 14x + 5}{2x^3 - 5x^2 - 4x + 3}$

   (b) $\displaystyle\lim_{x\to 3} \frac{\left(\frac{1}{x} - \frac{1}{3}\right)}{x^2 - 9}$

2. Show how to use the Limit Laws to evaluate the following limits. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $\displaystyle\lim_{x\to\infty} \frac{4x^2 - 3x + 12}{9x^2 + 12x - 15}$

   (b) $\displaystyle\lim_{x\to\infty} \left[\sqrt{x^2 - 3x + 12} - x\right]$.

3. Show how to use the definition of the derivative to find $f'(x)$ when $f(x) = \dfrac{1}{1 - 2x}$.

4. Find $f'(x)$ when f is given by

   (a) $f(x) = (5x^2 - 4x)(3x - 2)$.

   (b) $f(x) = xe^{-x}$.

5. Show how to find an equation for the line that is tangent to the curve $y = \dfrac{x^2 + 1}{x - 1}$ at the point on the curve where $x = -1$.

6. Let f be the function given by

$$f(x) = \begin{cases} 3x - 7, & \text{when } x \le a, \\ x^2 - 12x + 37, & \text{when } x > a, \end{cases}$$

   where a is a certain constant. Find all values of a for which the function f is continuous everywhere. Be sure to give the reasoning that supports your conclusions.

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 1410, Calculus I—Exam I , Solutions **Feb. 12, 2015**

Instructions: Work the following problems on your own paper; give your reasoning and show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Your paper is due at 1:50 pm.

1. Show how to use the Limit Laws to evaluate the following limits. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $\lim_{x\to 1/2} \dfrac{2x^3+7x^2-14x+5}{2x^3-5x^2-4x+3}$

   (b) $\lim_{x\to 3} \dfrac{\left(\dfrac{1}{x}-\dfrac{1}{3}\right)}{x^2-9}$

   Solution:

   (a)

$$\lim_{x\to 1/2} \frac{2x^3+7x^2-14x+5}{2x^3-5x^2-4x+3} = \lim_{x\to 1/2} \frac{\cancel{(2x-1)}(x^2+4x-5)}{\cancel{(2x-1)}(x^2-2x-3)}$$
$$= \frac{\lim_{x\to 1/2}(x^2+4x-5)}{\lim_{x\to 1/2}(x^2-2x-3)} = \frac{(-11/4)}{(-15/4)} = \frac{11}{15}.$$

   (b)

$$\lim_{x\to 3} \frac{\left(\dfrac{1}{x}-\dfrac{1}{3}\right)}{x^2-9} = \lim_{x\to 3} \frac{3-x}{3x(x^2-9)}$$
$$= \lim_{x\to 3} \frac{-\cancel{(x-3)}}{3x\cancel{(x-3)}(x+3)}$$
$$= \frac{-1}{\lim_{x\to 3}[3x(x+3)]} = -\frac{1}{54}.$$

2. Show how to use the Limit Laws to evaluate the following limits. You need not mention the Limit Laws explicitly, but you must show the calculations they lead you to.

   (a) $\lim_{x\to\infty} \dfrac{4x^2-3x+12}{9x^2+12x-15}$

   (b) $\lim_{x\to\infty} \left[\sqrt{x^2-3x+12}-x\right]$.

   Solution:

   (a)

$$\lim_{x\to\infty} \frac{4x^2-3x+12}{9x^2+12x-15} = \lim_{x\to\infty} \frac{\left(4-\dfrac{3}{x}+\dfrac{12}{x^2}\right)}{\left(9-\dfrac{12}{x}-\dfrac{15}{x^2}\right)} = \frac{4}{9}.$$

(b)

$$\lim_{x\to\infty}\left[\sqrt{x^2-3x+12}-x\right] = \lim_{x\to\infty}\frac{(\cancel{x^2}-3x+12)-\cancel{x^2}}{\sqrt{x^2-3x+12}+x}$$

$$= -\lim_{x\to\infty}\frac{3-\left(\dfrac{12}{x}\right)}{\sqrt{1-\dfrac{3}{x}+\dfrac{12}{x^2}}+1}$$

$$= -\frac{\lim_{x\to\infty}\left[3-\left(\dfrac{12}{x}\right)\right]}{\lim_{x\to\infty}\left[\sqrt{1-\dfrac{3}{x}+\dfrac{12}{x^2}}+1\right]} = -\frac{3}{2}$$

3. Show how to use the definition of the derivative to find $f'(x)$ when $f(x) = \dfrac{1}{1-2x}$.

Solution:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{1-2(x+h)}-\frac{1}{1-2x}\right]$$

$$= \frac{1}{h}\left[\frac{[1-2x]-[1-2(x+h)]}{[1-2(x+h)][1-2x]}\right]$$

$$= \lim_{h\to 0}\frac{\cancel{1}-\cancel{2x}-\cancel{1}+\cancel{2x}+2h}{h[1-2(x+h)][1-2x]}$$

$$= \lim_{h\to 0}\frac{2\cancel{h}}{\cancel{h}[1-2(x+h)][1-2x]} = \frac{2}{(1-2x)^2}.$$

4. Find $f'(x)$ when f is given by

(a) $f(x) = (5x^2-4x)(3x-2)$.

(b) $f(x) = xe^{-x}$.

Solution

(a)

$$\frac{d}{dx}[(5x^2-4x)(3x-2)] = \left[\frac{d}{dx}(5x^2-4x)\right](3x-2)+(5x^2-4x)\frac{d}{dx}(3x-2)$$

$$= (10x-4)(3x-2)+3(5x^2-4x).$$

(b) $f(x) = xe^{-x} = \dfrac{x}{e^x}$, so that

$$f'(x) = \frac{d}{dx}\left[\frac{x}{e^x}\right] = \frac{\left[\dfrac{d}{dx}(x)\right]e^x - x\left[\dfrac{d}{dx}(e^x)\right]}{(e^x)^2}$$

$$= \frac{e^x-xe^x}{e^{2x}} = \frac{1-x}{e^x} = e^{-x}-xe^{-x}.$$

5. Show how to find an equation for the line that is tangent to the curve $y = \dfrac{x^2+1}{x-1}$ at the point on the curve where $x = -1$.

Solution: If f is as given, then $f(-1) = -1$, while

$$f'(x) = \frac{2x(x-1) - (x^2+1) \cdot 1}{(x-1)^2}$$

so that

$$f'(-1) = \frac{2(-1)(-2) - [(-1)^2+1]}{(-2)^2} = \frac{1}{2}.$$

Thus, an equation for the tangent line at $x = -1$ is $y = -1 + \frac{1}{2}(x+1)$, or $y = \frac{1}{2}x - \frac{1}{2}$.

6. Let f be the function given by

$$f(x) = \begin{cases} 3x - 7, & \text{when } x \leq a, \\ \\ x^2 - 12x + 37, & \text{when } x > a, \end{cases}$$

where a is a certain constant. Find all values of a for which the function f is continuous everywhere. Be sure to give the reasoning that supports your conclusions.

Solution: The function f is given by a polynomial function throughout some open interval that contains each point, with the exception of the point $x = a$, so the latter point is the only one where f may fail to be continuous.

At $x = a$, continuity requires that $\lim_{x \to a^-} f(x) = 3a - 7 = f(a) = \lim_{x \to a^-} f(x) = a^2 - 12a + 37$. Thus, for continuity of f at $x = a$, we must have

$$a^2 - 12a + 37 = 3a - 7$$

or

$$a^2 - 15a + 44 = 0.$$

Thus $(a-4)(a-11) = 0$, so that the values of a for which f is continuous at $x = a$ are $a = 4$ and $a = 11$.

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 1410, Calculus I —Exam I **March 19, 2015**

Instructions: Work the following problems on your own paper; give your reasoning and show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Your paper is due at 1:50 pm.

1. (a) If $F(x) = x^2 \sin e^x$, what is $F'(x)$?

   (b) If $G(x) = \dfrac{x^2 + x}{x^2 + 1}$, what is $G'(x)$?

2. Let $P(x) = 3x^5 - 15x^4 - 25x^3 + 37$. Show all of your reasoning as you answer the following questions.

   (a) Where are the critical points of P?

   (b) On what intervals is P an increasing function? A decreasing function?

   (c) Where are the local maxima of P? The local minima?

3. Suppose that the equation $x^3 + y^3 = 126$ gives y implicitly as a function of x in the vicinity of the point $(5, 1)$. Find y' and y'' at the point $(5, 1)$. Give your reasoning.

4. Show how to use the definition of the derivative to find $f'(x)$ when $f(x) = \dfrac{x + 1}{x - 1}$.

5. Show how to find all values of a and b that make the function f, given below, continuous everywhere. Give the resulting values of a and b, and the reasoning that supports your claims.

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & \text{if } x < 2 \\ ax^2 - bx - 18, & \text{if } 2 \leq x < 3 \\ 10x - a + b, & \text{if } x \geq 3 \end{cases}$$

6. Find the absolute maximum and the absolute minimum values of $f(x) = 12x^5 - 75x^4 + 80x^3 + 7$ in the interval $[-1, 5]$. Give all of your reasoning.

7. A 14-foot ladder is leaning against a vertical wall. The top slips down the wall at a rate of 3 ft/s. How fast is the foot of the ladder moving away from the wall when the top of the ladder is 9 feet above the ground. Give all of the reasoning that supports your answer.

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 1410, Calculus II —Exam I **March 19, 2015**

Instructions: Work the following problems on your own paper; give your reasoning and show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Your paper is due at 1:50 pm.

1. (a) If $F(x) = x^2 \sin e^x$, what is $F'(x)$?

   (b) If $G(x) = \dfrac{x^2+x}{x^2+1}$, what is $G'(x)$?

   Solution:

   (a)

$$F'(x) = \frac{d}{dx}\left(x^2 \sin e^x\right) = \frac{d}{dx}\left(x^2\right)\sin e^x + x^2 \frac{d}{dx}\left(\sin e^x\right)$$
$$= 2x\sin e^x + \left(x^2 \cos e^x\right)\frac{d}{dx}e^x = 2x \sin e^x + x^2 e^x \cos x$$

   (b)

$$\frac{d}{dx}\left(\frac{x^2+x}{x^2+1}\right) = \frac{\frac{d}{dx}\left(x^2+x\right)\left(x^2+1\right) - \left(x^2+x\right)\frac{d}{dx}\left(x^2+1\right)}{\left(x^2+1\right)^2}$$
$$= \frac{(2x+1)(x^2+1)-(x^2+x)(2x)}{(x^2+1)^2} = -\frac{x^2-2x-1}{(x^2+1)^2}.$$

2. Let $P(x) = 3x^5 - 15x^4 - 25x^3 + 37$. Show all of your reasoning as you answer the following questions.

   (a) Where are the critical points of P?

   (b) On what intervals is P an increasing function? A decreasing function?

   (c) Where are the local maxima of P? The local minima?

   Solution:

   (a) P is a polynomial, and thus differentiable everywhere. So its only critical points are those where $P'(x) = 0$. But

$$P'(x) = 15x^4 - 60x^3 - 75x^2 = 15x^2(x^2 - 4x - 5)$$
$$= 15x^2(x+1)(x-5).$$

   This latter is zero when $x = -1$, when $x = 0$, and when $x = 5$, and these values of x are the critical points of P.

   (b) $P'(x)$ is the product of 15, $(x+1)$, x^2, and $(x-5)$. The first of these is always positive, $(x+1)$ is negative when $x < -1$ and positive when $x > -1$. x^2 is positive except when $x = 0$, and $(x-5)$ is negative when $x < 5$, but positive when $x > 5$. Hence $P'(x)$ is positive when $x < -1$, negative when $-1 < x < 0$, negative when $0 < x < 5$, and positive when $x > 5$. It follows that P is decreasing on $(-\infty, -1]$ and on $[5, \infty)$, but increasing on $[-1, 5]$.

   (c) $P'(x)$ changes sign from negative to positive at $x = -1$, so this critical point gives a local minimum. $P'(x)$ doesn't change sign at $x = 0$, meaning that this critical point gives neither a local maximum nor a local minimum. $P'(x)$ changes sign from negative to positive at $x = 5$, and this critical point gives a local minimum.

3. Suppose that the equation $x^3 + y^3 = 126$ gives y implicitly as a function of x in the vicinity of the point $(5, 1)$. Find y' and y'' at the point $(5, 1)$. Give your reasoning.

Solution: Differentiating the equation $x^3 + y^3 = 126$ on both sides while treating the variable y as a function of x gives us

$$3x^2 + 3y^2y' = 0,$$

or

$$x^2 + y^2y' = 0.$$

Substituting 5 for x and 1 for y gives $y' = -25$ at $(1, 5)$. Differentiating the equation $x^2 + y^2y' = 0$ again on both sides while treating both y and y' as functions of x gives

$$2x + (2yy')y' + y^2y'' = 0.$$

Now we put $x = 5$, $y = 1$, and $y' = -25$, to obtain $10 + 1250 + 625y'' = 0$ at $(5, 1)$, whence $y'' = -\dfrac{252}{125}$ at that point.

4. Show how to use the definition of the derivative to find $f'(x)$ when $f(x) = \dfrac{x+1}{x-1}$.

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{(x+h)+1}{(x+h)-1} - \frac{x+1}{x-1}\right] \\
&= \lim_{h\to 0} \frac{[(x+h)+1](x-1) - [(x+h)-1](x+1)}{h(x-1)[(x+h)-1]} \\
&= \lim_{h\to 0} \frac{(x^2 + hx + \cancel{x} - \cancel{x} - h - 1) - (x^2 + hx - \cancel{x} + \cancel{x} + h - 1)}{h(x-1)[(x+h)-1]} \\
&= \lim_{h\to 0} \frac{\cancel{x^2} + \cancel{hx} - h - \cancel{1} - \cancel{x^2} - \cancel{hx} - h + \cancel{1}}{h(x-1)[(x+h)-1]} \\
&= \lim_{h\to 0} \frac{-2\cancel{h}}{\cancel{h}(x-1)[(x+h)-1]} = -\frac{2}{(x-1)^2}.
\end{aligned}$$

5. Show how to find all values of a and b that make the function f, given below, continuous everywhere. Give the resulting values of a and b, and the reasoning that supports your claims.

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & \text{if } x < 2 \\[2ex] ax^2 - bx - 18, & \text{if } 2 \le x < 3 \\[2ex] 10x - a + b, & \text{if } x \ge 3 \end{cases}$$

Solution: The only places where the issue of continuity is in doubt are $x = 2$ and $x = 3$, because f is given by continuous functions at all other points. We have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2-4}{x-2} = \lim_{x\to 2^-} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = 4,$$

while

$$f(2) = (ax^2 - bx - 18) = 4a - 2b - 18 = \lim_{x\to 2^+} f(x).$$

Consequently, in order for $\lim_{x\to 2} f(x)$ to exist and equal $f(2)$, we need to have the numbers a and b satisfy the equation $4a - 2b - 18 = 4$, or, equivalently, $2a - b - 9 = 2$. This gives $b = 2a - 11$.

We must also have $9a - 3b - 18 = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) = 30 - a + b$, or $10a - 4b = 48$, which is equivalent to $5a - 2b = 24$. Replacing b in the latter equation with the $2a - 11$ we found earlier, we learn that

$$\begin{aligned} 5a - 2(2a - 11) &= 24; \\ 5a - 4a + 22 &= 24; \\ a &= 2. \end{aligned}$$

Thus, our solution is $a = 2$ and $b = 2a - 11 = -7$.

6. Find the absolute maximum and the absolute minimum values of $f(x) = 12x^5 - 75x^4 + 80x^3 + 7$ in the interval $[-1, 5]$. Give all of your reasoning.

Solution: If $f(x) = 12x^5 - 75x^4 + 80x^3 + 7$, then

$$f'(x) = 60x^4 - 300x^3 + 240x^2 = 60x^2(x - 1)(x - 4).$$

Thus, the absolute maximum and the absolute minimum values of f on the interval $[-1, 5]$ are among the numbers

$$\begin{aligned} f(-1) &= -160 \\ f(0) &= 7 \\ f(1) &= 24 \\ f(4) &= -1785, \text{ and} \\ f(5) &= 632. \end{aligned}$$

The absolute maximum is $f(5) = 632$, and the absolute minimum is $f(4) = -1782$.

7. A 14-foot ladder is leaning against a vertical wall. The top slips down the wall at a rate of 3 ft/s. How fast is the foot of the ladder moving away from the wall when the top of the ladder is 9 feet above the ground. Give all of the reasoning that supports your answer.

Solution: Let y denote the distance from the floor to the point where the top of the ladder rests against the wall, and let x denote the distance from the wall to the point where the foot of the ladder rests on the floor. The ladder is 14 feet long. So, by the Pythagorean Theorem,

$$x^2 + y^2 = 196.$$

Treating x and y both as functions of time t, and differentiating implicitly with respect to t, we obtain

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

At the critical instant, $y = 9$, and this means that $x = \sqrt{196 - 81} = \sqrt{115}$. It is given that $\dfrac{dy}{dx} = -3$ at the critical instant. Consequently, at the critical instant we have

$$\not{2}\sqrt{115}\frac{dx}{dt} + \not{2} \cdot 9(-3) = 0,$$

so that the speed of the foot of the ladder is

$$\frac{dx}{dt} = \frac{27}{\sqrt{115}} \text{ ft/s}$$

away from the wall.

MTH 1410, Calculus I—Exam III **April 30, 2015**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 1:50 pm.

1. Find the area between bounded by the x-axis and the curves $x = 0$, $x = 4$, and $y = 9 + x^2$.

2. The equation $x^3 - x^2 - 3 = 0$ has a root somewhere in the vicinity of $x = 2$. Use Newton's method, with $x = 2$ as your initial guess, to find an approximation of that root, correct to at least three digits to the right of the decimal. Be sure to show what calculations you are doing.

3. Evaluate the following limits, and justify your answers. You may use l'Hôpital's rule if you justify its use.

   (a) $\displaystyle\lim_{x \to -1} \frac{x + x^2}{\ln(2 + x)}$

   (b) $\displaystyle\lim_{x \to 0} \frac{xe^{-2x}}{\pi e^{2x} - \pi}$

4. Let $f(x) = \dfrac{2x + 1}{x - 2}$. Show how to use the definition of derivative to find $f'(3)$.

5. Let F be the function given by

$$F(x) = (x - 1)^2 (x + 1)^3.$$

   Then, in fully factored form,

$$F'(x) = (x - 1)(x + 1)^2 (5x - 1).$$

   Use this information to determine the intervals where F is increasing and the intervals where F is decreasing. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

6. Find the points on the ellipse $x^2 + 4y^2 = 4$ whose distance from the point $(1, 0)$ is minimal.

MTH 1410, Calculus I—Exam III **April 30, 2015**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 1:50 pm.

1. Find the area between bounded by the x-axis and the curves $x = 0$, $x = 4$, and $y = 9 + x^2$.

   Solution: The desired area is

$$\int_0^4 (9 + x^2)\,dx = \left(9x + \frac{x^3}{3}\right)\Big|_0^4 = \left(36 + \frac{64}{3}\right) - (0 + 0) = \frac{172}{3}.$$

2. The equation $x^3 - x^2 - 3 = 0$ has a root somewhere in the vicinity of $x = 2$. Use Newton's Method, with $x = 2$ as your initial guess, to find an approximation of that root, correct to at least three digits to the right of the decimal. Be sure to show what calculations you are doing.

   Solution: We seek the solution of $x^3 - x^2 - 3 = 0$ near $x = 2$, so we take $f(x) = x^3 - x^2 - 3$. The Newton's method iteration scheme for this problem is given by

$$x_0 = 2;$$
$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} = x_{n-1} - \frac{x_{n-1}^3 - x_{n-1}^2 - 3}{3x_{n-1}^2 - 2x_{n-1}}, \text{ or}$$
$$x_n = \frac{2x_{n-1}^3 - x_{n-1}^2 + 3}{3x_{n-1}^2 - 2x_{n-1}}$$

   Thus,

$$x_1 = \frac{2(2)^3 - 2^2 + 3}{3(2)^2 - 2(2)} = \frac{15}{8} = 1.875;$$
$$x_2 = \frac{2x_1^3 - x_1^2 + 3}{3x_1^2 - 2x_1} \sim 1.8637930;$$
$$x_3 = \frac{2x_2^3 - x_2^2 + 3}{3x_2^2 - 2x_2} \sim 1.8637065.$$

   Thus, to three places to the right of the decimal, our root is 1.864.

3. Evaluate the following limits, and justify your answers. You may use l'Hôpital's rule if you justify its use.

   (a) $\lim_{x \to -1} \frac{x + x^2}{\ln(2 + x)}$

   (b) $\lim_{x \to 0} \frac{xe^{-2x}}{\pi e^{2x} - \pi}$

   Solution:

(a) We note that $\lim_{x\to -1}(x + x^2) = 0 = \lim_{x\to -1} \ln(2 + x)$, so we may attempt a l'Hôpital's rule solution:

$$\lim_{x\to -1} \frac{x^2 + x}{\ln(2 + x)} = \lim_{x\to -1} \frac{2x + 1}{\left(\dfrac{1}{2 + x}\right)} = \lim_{x\to -1} [(2x + 1)(2 + x)] = -1.$$

(b)

$$\lim_{x\to 0} \frac{xe^{-2x}}{\pi e^{2x} - \pi} = \frac{1}{\pi}\left(\lim_{x\to 0} e^{-2x}\right)\left(\lim_{x\to 0} \frac{x}{e^{2x} - 1}\right),$$

and the first of the two limits in the product is one. We may apply l'Hôpital's rule to the second limit in the product, because $\lim_{x\to 0} x = 0$ and $\lim_{x\to 0}(e^{2x} - 1) = 0$. Hence,

$$\lim_{x\to 0} \frac{xe^{-2x}}{\pi e^{2x} - \pi} = \frac{1}{\pi} \lim_{x\to 0} \frac{1}{2e^{2x}} = \frac{1}{2\pi}.$$

4. Let $f(x) = \dfrac{2x + 1}{x - 2}$. Show how to use the definition of derivative to find $f'(3)$.

Solution:

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{2(3 + h) + 1}{(3 + h) - 2} - \frac{2\cdot 3 + 1}{3 - 2}\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{7 + 2h}{1 + h} - \frac{7}{1}\right] = \lim_{h\to 0} \frac{(7 + 2h) - 7(1 + h)}{h(1 + h)} \\ &= \lim_{h\to 0} \frac{\cancel{7} + 2h - \cancel{7} - 7h}{h(1 + h)} = \lim_{h\to 0} \frac{-5\cancel{h}}{\cancel{h}(1 + h)} \\ &= \lim_{h\to 0} \frac{-5}{1 + h} = -5. \end{aligned}$$

5. Let F be the function given by

$$F(x) = (x - 1)^2(x + 1)^3.$$

Then, in fully factored form,

$$F'(x) = (x - 1)(x + 1)^2(5x - 1).$$

Use this information to determine the intervals where F is increasing and the intervals where F is decreasing. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

Solution: It is easy to see from the factored form of $f'(x)$, that

(a) $f'(x) > 0$ when $-\infty < x < -1$;
(b) $f'(x) > 0$ when $-1 < x < 1/5$;
(c) $f'(x) < 0$ when $1/5 < x < 1$;
(d) $f'(x) > 0$ when $1 < x < \infty$.

From these observations, we may conclude that F is increasing on the interval $(-\infty, 1/5]$ and on the interval $[1, \infty)$. Also, F is decreasing on the interval $[1/5, 1]$. The critical numbers of F are the numbers where $F'(x)$ is either zero or nonexistent; these are $x = -1$, $x = 1/5$, and $x = 1$. $F'(x)$ doesn't change sign at $x = -1$, so F has neither a local maximum nor a local minimum at $x = -1$. $F'(x)$ changes sign from positive to negative at $x = 1/5$, so F has a local maximum at $x = 1/5$. Finally, $F'(x)$ changes sign from negative to positive at $x = 1$, so F has a local minimum at $x = 1$.

6. Find the points on the ellipse $x^2 + 4y^2 = 4$ whose distance from the point $(1, 0)$ is minimal.

Solution: If the point (x, y) lies on the curve $x^2 + 4y^2 = 4$, the square of its distance from the point $(1, 0)$ is $D = (x-1)^2 + (y-0)^2 = (x-1)^2 + y^2$. Because the distance itself is positive, we can minimize that distance by minimizing D in its place. Thus, we need to find points where $dD/dx = 0$. We treat y as a non-negative (We can obtain any negative values of y by symmetry.) function of x whose domain is $[-2, 2]$. Then, differentiating implicitly, we have

$$\frac{dD}{dx} = 2(x-1) + 2y\frac{dy}{dx}, \tag{1}$$

so, to find critical points, we must solve

$$(x-1) + y\frac{dy}{dx} = 0. \tag{2}$$

But $x^2 + 4y^2 = 4$, so another implicit differentiation gives

$$2x + 8y\frac{dy}{dx} = 0, \tag{3}$$

or

$$\frac{dy}{dx} = -\frac{x}{4y}. \tag{4}$$

Substituting (4) into (2), we obtain

$$x - 1 + \cancel{y}\left(-\frac{x}{4\cancel{y}}\right) = 0,$$

whence

$$x = \frac{4}{3}.$$

The minimum we seek must occur at an endpoint or at a critical point, so we must evaluate D when $x = -2$, when $x = 4/3$, and when $x = 2$. The corresponding points on the ellipse are $(-2, 0)$, $(4/3, \pm\sqrt{5}/3)$, and $(2, 0)$. The corresponding squared distances are

$$\begin{aligned} D\Big|_{(-2,0)} &= (-2-1)^2 + 0^2 = 9; \\ D\Big|_{(4/3,\pm\sqrt{5}/3)} &= \left(\frac{4}{3} - 1\right)^2 + \left(\frac{\sqrt{5}}{3}\right)^2 = \frac{6}{9} = \frac{2}{3}; \\ D\Big|_{(2,0)} &= (2-1)^2 + 0^2 = 1. \end{aligned}$$

The points where the minimum distance of $\sqrt{\frac{2}{3}}$ occurs are $\left(\frac{4}{3}, \frac{\sqrt{5}}{3}\right)$ and $\left(\frac{4}{3}, -\frac{\sqrt{5}}{3}\right)$.

MTH 1410, Calculus I—Final Exam **May 12, 2015**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 1:00 pm.

1. (a) *Use the definition of the derivative* to find $f'(x)$ if $f(x) = 1/\sqrt{x}$.

   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = 1/\sqrt{x}$ at $x = 1$, at $x = 4$, and at $x = 9$.

2. Evaluate the following definite integrals. Give all of your reasoning.

   (a) $\displaystyle\int_3^5 (3x^2 - 24x + 54)\,dx$

   (b) $\displaystyle\int_0^3 3t\sqrt{9 - t^2}\,dt$

3. Let F be the function given by

$$F(x) = \cos^3 x; \text{ where } \frac{\pi}{4} \le x \le \frac{9\pi}{4}.$$

   Determine the intervals where F is increasing and the intervals where F is decreasing. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

4. Suppose that $f(2) = 2$, $f(4) = 4$, $f'(2) = 4$, $f'(4) = -2$, $g(2) = 4$, $g(4) = 2$, $g'(2) = -6$, and $g'(4) = -8$.

   (a) Find $F(4)$ and $F'(4)$, where $F(x) = \dfrac{f(x)}{g(x)}$.

   (b) Find $G(2)$ and $G'(2)$, where $G(x) = g[2f(x)]$.

   (c) Find $H(2)$ and $H'(2)$, where $H(x) = g[f(x^2)]$.

5. (a) Show that the point $(3, 2)$ lies on the curve given by the equation

$$x^3 - 5x^2y^3 + 8y^4 + 205 = 0.$$

   (b) If x and y are related by the equation, $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$, find the value of y' at $(3, 2)$.

   (c) Show how to use the results of parts (a) and (b) of this problem to find an approximate value for y near 2 when $x = 74/25$.

6. Find the points on the hyperbola $4y^2 - x^2 = 1$ whose distance from the point $(5, 0)$ is minimal.

MTH 1410, Calculus I—Final Exam Solutions **May 12, 2015**

Instructions: Work the following problems *on your own paper*; give your reasoning and show your supporting calculations. Do not give decimal approximations unless a problem requires you to do so. Your exam is due at 1:00 pm.

1. (a) *Use the definition of the derivative* to find $f'(x)$ if $f(x) = 1/\sqrt{x}$.

   (b) Use the derivative you calculated in part (a) of this problem to write equations for the lines tangent to the curve $y = 1/\sqrt{x}$ at $x = 1$, at $x = 4$, and at $x = 9$.

   Solution:

   (a)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right] \\ &= \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} = \lim_{h\to 0} \frac{x - (x+h)}{h\sqrt{x}\sqrt{x+h}\left[\sqrt{x} + \sqrt{x+h}\right]} \\ &= \lim_{h\to 0} \frac{\not{x} - \not{x} - h}{h\sqrt{x}\sqrt{x+h}\left[\sqrt{x} + \sqrt{x+h}\right]} = \lim_{h\to 0} \frac{-\not{h}}{\not{h}\sqrt{x}\sqrt{x+h}\left[\sqrt{x} + \sqrt{x+h}\right]} \\ &= -\frac{1}{2x^{3/2}} \end{aligned}$$

   (b) We have

$$\begin{aligned} f'(1) &= -\frac{1}{2}, \\ f'(4) &= -\frac{1}{2\cdot 8} = -\frac{1}{16}, \text{ and} \\ f'(9) &= -\frac{1}{2\cdot 27} = -\frac{1}{54}. \end{aligned}$$

   The desired tangent lines are thus

$$\begin{aligned} y &= 1 - \frac{1}{2}(x-1), \\ y &= \frac{1}{2} - \frac{1}{16}(x-4), \text{ and} \\ y &= \frac{1}{3} - \frac{1}{54}(x-9). \end{aligned}$$

2. Evaluate the following definite integrals. Give all of your reasoning.

   (a) $\displaystyle\int_3^5 (3x^2 - 24x + 54)\, dx$

   (b) $\displaystyle\int_0^3 3t\sqrt{9 - t^2}\, dt$

   Solution:

   (a)

$$\int_3^5 (3x^2 - 24x + 54)\, dx = (x^3 - 12x^2 + 54x)\Big|_3^5 = 14.$$

(b) Let $u = 9 - t^2$. Then $du = -2t\,dt$, or $t\,dt = -\frac{1}{2}du$. Moreover, $u = 9$ when $t = 0$ and $u = 0$ when $t = 3$. Thus

$$\int_0^3 3t\sqrt{9-t^2}\,dt = -\frac{3}{2}\int_9^0 u^{1/2}\,du = \frac{3}{2}\int_0^9 u^{1/2}\,du = u^{3/2}\Big|_0^9 = 27.$$

3. Let F be the function given by

$$F(x) = \cos^3 x; \text{ where } \frac{\pi}{4} \le x \le \frac{9\pi}{4}.$$

Determine the intervals where F is increasing, and the intervals where F is decreasing. What are the critical numbers of F? What is the nature of each of the critical points (local maximum, local minimum, or neither)? *Give your reasoning.*

Solution: $F'(x) = -3\cos^2 x \sin x$, so $f'(x) < 0$ when $\frac{\pi}{4} \le x < \frac{\pi}{2}$, when $\frac{\pi}{2} < x < \pi$, and when $2\pi < x \le \frac{9\pi}{4}$. $f'(x) > 0$ when $\pi < x < \frac{3\pi}{2}$ and when $\frac{3\pi}{2} < x < 2\pi$. Thus f is increasing on $[\pi, 2\pi]$ and decreasing on each of the intervals $[\pi/4, \pi]$ and $[2\pi, 9\pi/4]$. The critical points of f are at $x = \pi/2$, $x = \pi$, $x = 3\pi/2$ and $x = 2\pi$, where $f'(x) = 0$. Of these, $x = \pi$ gives a local minimum because $f'(x)$ changes sign from negative to positive there; $x = 2\pi$ gives a local maximum because $f'(x)$ changes sign there from positive to negative. The derivative doesn't change sign at either of the other two critical points, so those critical points give neither local maximum nor local minimum.

4. Suppose that $f(2) = 2$, $f(4) = 4$, $f'(2) = 4$, $f'(4) = -2$, $g(2) = 4$, $g(4) = 2$, $g'(2) = -6$, and $g'(4) = -8$.

(a) Find $F(4)$ and $F'(4)$, where $F(x) = \frac{f(x)}{g(x)}$.

(b) Find $G(2)$ and $G'(2)$, where $G(x) = g[2f(x)]$.

(c) Find $H(2)$ and $H'(2)$, where $H(x) = g[f(x^2)]$.

Solution:

(a)

$$F(4) = \frac{f(4)}{g(4)} = \frac{4}{2} = 2;$$
$$F'(4) = \frac{f'(4)g(4) - f(4)g'(4)}{[g(4)]^2} = \frac{(-2)\cdot 2 - 4\cdot(-8)}{2^2} = 7.$$

(b)

$$G(2) = g[2f(2)] = g(4) = 2;$$
$$G'(2) = g'[2f(2)]\cdot 2f'(2) = g'(4)\cdot 8 = -64.$$

(c)

$$\begin{aligned} H(2) &= g[f(4)] = g(4) = 2; \\ H'(x) &= g'[f(x^2)]\cdot f'(x^2)\cdot 2x, \text{ so} \\ H'(2) &= g'[f(2^2)]\cdot f'(2^2)\cdot 2\cdot 2 \\ &= g'(4)\cdot f'(4)\cdot 4 = 64. \end{aligned}$$

5. (a) Show that the point $(3, 2)$ lies on the curve given by the equation

$$x^3 - 5x^2y^3 + 8y^4 + 205 = 0.$$

(b) If x and y are related by the equation, $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$, find the value of y' at $(3, 2)$.

(c) Show how to use the results of parts (a) and (b) of this problem to find an approximate value for y near 2 when $x = 74/25$.

Solution:

(a)

$$(3)^3 - 5(3)^2(2)^3 + 8(2)^4 + 205 = 27 - 360 + 128 + 205 = 0,$$

so the point $(3, 2)$ lies on the curve given by the equation.

(b) Treating y as a function of x and differentiating implicitly, we obtain

$$\begin{aligned} 3x^2 - 10xy^3 - 15x^2y^2y' + 32y^3y' &= 0, \text{ or, putting } x = 3,\ y = 2, \\ -213 - 284y' &= 0, \text{ whence} \\ y' &= -\frac{3}{4} \end{aligned}$$

(c) The linearization of y as a function of x given by the curve $x^3 - 5x^2y^3 + 8y^4 + 205 = 0$ near the point $(3, 2)$ is

$$L(x) = 2 - \frac{3}{4}(x - 3),$$

so the approximate value of y when $x = \frac{74}{25}$ is

$$L\left(\frac{74}{25}\right) = 2 - \frac{3}{4}\left(\frac{74}{25} - 3\right) = 2 - \frac{3}{4}\left(-\frac{1}{25}\right) = \frac{203}{100}.$$

6. Find the points on the hyperbola $4y^2 - x^2 = 1$ whose distance from the point $(5, 0)$ is minimal.

Solution: We are to minimize the distance from a point (x, y) on the curve $4y^2 - x^2 = 1$ to the point $(5, 0)$, and we can do so by minimizing the square of that distance,

$$D = (x - 5)^2 + (y - 0)^2 = (x - 5)^2 + y^2.$$

If (x, y) lies on the curve, then $4y^2 - x^2 = 1$, or

$$y^2 = \frac{1}{4}(1 + x^2).$$

Moreover, x can have arbitrarily large magnitude, so there are no endpoints. Thus,

$$\begin{aligned} D &= (x - 5)^2 + \frac{1}{4}(1 + x^2), \text{ and} \\ \frac{dD}{dx} &= 2(x - 5) + \frac{1}{2}x. \end{aligned}$$

The critical points for D are given by

$$\begin{aligned} 2(x - 5) + \frac{1}{2}x &= 0, \text{ or} \\ x &= 4. \end{aligned}$$

The corresponding points on the curve are where

$$y^2 = \frac{1}{4}(1 + x^2) = \frac{1}{4}(1 + 4^2) = \frac{17}{4}.$$

The minimum distance therefore occurs at two points: $\left(4, \frac{\sqrt{17}}{2}\right)$ and $\left(4, -\frac{\sqrt{17}}{2}\right)$.