Calc II
Exams

Louis A. Talman, Ph.D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

To the Reader

This is a collection of some of the exams I gave in my second semester calculus courses over the twenty-four-year period beginning in 1992 and extending into 2015. I offer it here as a resource for current calculus teachers—and for students, too.

I have little to say here that I haven't already said in the introduction to the companion volume, *Calc I Exams*, that I posted just over a week ago, so I won't again ask you to read several paragraphs. I will remark only that much the same difficulties attended the preparation of this volume, except that I had a better idea of where I was headed and so didn't take as long to complete the task.

And, of course, I offer you the same wishes for success in your pursuit of calculus, in hopes that this volume, too, will help you.

Louis A. Talman
Denver, CO
February 11, 2019

MTH 241, Calculus II, Exam 1 **March 6, 1992**

Instructions: Submit your solutions to the following problems on your own paper; give your reasoning and show enough detail to support your reasong. Your paper is due at 10:55am.

1. Find the following integrals.

   (a) $\displaystyle\int_0^1 xe^x\,dx.$

   (b) $\displaystyle\int_0^1 x\sqrt{1-x^2}\,dx.$

2. What are parametric equations for the polar curve

$$r = \cos\theta?$$

   Give two different integrals that give the area inside this polar curve. Evaluate either of them. [Hint: $2\cos^2\theta = 1+\cos 2\theta$ and $2\sin^2\theta = 1-\cos 2\theta$.]

3. (a) Give the expansion for $\dfrac{1}{1+x^3}$ in powers of x.

   (b) Give the first seven terms of the expansion for $\dfrac{1}{1-x+x^2}$ in powers of x. Can you identify a pattern?

4. Find the volume that is generated when the region between the x-axis and the curve $y = 1+x^2$, with $0 \le x \le 2$ is revolved about the x-axis.

5. The temperature of a pan of water at time t is given by a function $T(t)$. According to Newton's Law of Cooling, the rate of change, $T'(t)$ of the temperature at time t is given by $T'(t) = k[M - T(t)]$ if the pan is placed in a room whose temperature is maintained at M degrees. Find the expression that gives the temperature of a pan of water at time t if temperature is $100°$ initially and $80°$ after 30 minutes.

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1. Find the following integrals.

   (a) $\int_0^1 xe^x\,dx$.

   (b) $\int_0^1 x\sqrt{1-x^2}\,dx$.

   Solution:

   (a) Take $u = x$ and $dv = e^x\,dx$. Then $du = dx$ and $v = e^x$. Thus,

$$\int_0^1 xe^x\,dx = xe^x\Big|_0^1 - \int_0^1 e^x\,dx = e - e^x\Big|_0^1 = 1. \tag{1}$$

   (b) Let $u = 1 - x^2$. Then $du = -2x\,dx$, or $x\,dx = -\frac{1}{2}\,du$. Moreover, $u = 1$ when $x = 0$, and $u = 0$ when $x = 1$. Therefiore,

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int_1^0 u^{1/2}\,du = \frac{1}{2}\int_0^1 u^{1/2}\,du \tag{2}$$

$$= \frac{1}{\not{2}}\cdot\frac{\not{2}}{3}u^{3/2}\Big|_0^1 = \frac{1}{3}. \tag{3}$$

2. What are parametric equations for the polar curve

$$r(t) = \cos\theta?$$

   Give two different integrals that give the area inside this polar curve. Evaluate either of them. [Hint: $2\cos^2\theta = 1 + \cos 2\theta$ and $2\sin^2\theta = 1 - \cos 2\theta$.]

   Solution: If $r = \cos\theta$, then $x = r\cos\theta = \cos^2\theta$ and $y = r\sin\theta = \sin\theta\cos\theta$. So parametric equations for the curve $r = \cos\theta$ are

$$x = \cos^2\theta \tag{4}$$

$$y = \sin\theta\cos\theta. \tag{5}$$

   These can be rewritten

$$x = \frac{1}{2}(1 + \cos 2\theta) \tag{6}$$

$$y = \frac{1}{2}\sin 2\theta \tag{7}$$

or

$$x - \frac{1}{2} = \frac{1}{2}\cos 2\theta \tag{8}$$

$$y = \frac{1}{2}\sin 2\theta. \tag{9}$$

From the latter pair of equations, we see that

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}, \tag{10}$$

and we conclude that the curve is the circle of radius $\frac{1}{2}$ centered at $(\frac{1}{2}, 0)$. Solving (10) for y in terms of x, we find that $y = \pm\sqrt{x - x^2}$.

The area inside the curve can be written as

$$A = 2\int_0^1 \sqrt{x - x^2}\, dx \tag{11}$$

or as

$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} r^2\, d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta \tag{12}$$

$$= \frac{1}{4}\int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\, d\theta \tag{13}$$

$$= \frac{1}{4}\left(t - \frac{1}{2}\sin 2\theta\right)\Big|_{-\pi/2}^{\pi/2} = \frac{\pi}{4}. \tag{14}$$

3. (a) Give the expansion for $\frac{1}{1+x^3}$ in powers of x.

   (b) Give the first seven terms of the expansion for $\frac{1}{1-x+x^2}$ in powers of x. Can you identify a pattern?

Solution:

(a) Using what we know about the geometric series, we see that

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{k=0}^{\infty}(-1)^k x^{3k}. \tag{15}$$

(b)

$$\frac{1}{1-x+x^2} = \frac{1+x}{1+x^3} = \frac{1}{1+x^3} + \frac{x}{1+x^3} \tag{16}$$

$$= \sum_{k=0}^{\infty}(-1)^k x^{3k} + \sum_{k=0}^{\infty}(-1)^k x^{3k+1} \tag{17}$$

$$= (1 - x^3 + x^6 - x^9 + \cdots) + (x - x^4 + x^7 - x^{10} + \cdots) \tag{18}$$

$$= 1 + x - x^3 - x^4 + x^6 + x^7 - x^9 - x^{10} + \cdots. \tag{19}$$

4. Find the volume that is generated when the region between the x-axis and the curve $y = 1+x^2$, with $0 \le x \le 2$ is revolved about the x-axis.

Solution:

$$V = \pi \int_0^2 (1+x^2)^2\,dx = \pi \int_0^2 (1+2x^2+x^4)\,dx \tag{20}$$

$$= \pi \left(x + \frac{2}{3}x^3 + \frac{1}{5}x^5 \right) \Big|_0^2 = \pi \left(2 + \frac{16}{3} + \frac{32}{5} \right) = \frac{206}{15}\pi. \tag{21}$$

5. The temperature of a pan of water at time t is given by a function $T(t)$. According to Newton's Law of Cooling, the rate of change, $T'(t)$ of the temperature at time t is given by $T'(t) = k[M - T(t)]$ if the pan is placed in a room whose temperature is maintained at M degrees. Find the expression that gives the temperature of a pan of water at time t if temperature is $100°$ initially and $80°$ after 30 minutes.

Solution: We must solve the initial problem

$$T' = k(M - T); \tag{22}$$

$$T(0) = 100, \tag{23}$$

and then adjust k so that $T(30) = 80$.

From $T' = k(M - T)$, we find that

$$\int \frac{dT}{M-T} = k\,dt, \tag{24}$$

$$-\ln|M - T| + c = kt, \text{ or} \tag{25}$$

$$Ce^{kt} = \frac{1}{M-T}. \tag{26}$$

We rewrite this as $T = M - Ce^{-kt}$, with a (possibly) different value for the unknown constant C.

Now $T = 100$ when $t = 0$ so $100 = M - C$, or $C = M - 100$. This allows us to write our solution as $T = M + (100 - M)e^{-kt}$. From the fact that $T = 80$ when $t = \frac{1}{2}$ (in hours), we see that $80 = M + (100 - M)e^{-k/2}$, and

$$e^{-k/2} = \frac{80 - M}{100 - M}. \tag{27}$$

This means that

$$k = 2 \ln \frac{100 - M}{80 - M}. \tag{28}$$

Substituting for k in our expression for T and rearranging, we find that

$$T = M + (100 - M) \left(\frac{80 - M}{100 - M} \right)^{2t}, \tag{29}$$

where t is measured in hours. (Note that we have implicitly assumed that $M < 80$. Where?)

MTH 241, Calculus II, Exam 2 April 24, 1992

Instructions: Submit your solutions to the following problems on your own paper; give your reasoning and show enough detail to support your reasoning. Your paper is due at 10:55am.

1. Give the expansions in powers of x for
   (a) $f(x) = \cos x$
   (b) $f(x) = \dfrac{1}{1-x}$
   (c) $f(x) = e^{-x}$
2. Explain where the expansion, in powers of x, of the function
$$f(x) = \frac{x+2}{x^2-4x+5} \tag{1}$$
   converges to $f(x)$.
3. Given a function f, and a positive integer, k, explain how to find a number M which has the property that if p is another function for which
$$|f(x) - p(x)| < M, \tag{2}$$
   for all x for which $a \le x \le b$, then
$$\left| \int_a^b f(x)\,dx - \int_a^b p(x)\,dx \right| < 10^{-k} \tag{3}$$
   Why does the number you have found work?
4. Describe the region of convergence for the series
$$x + \sqrt{2}x^2 + \sqrt{3}x^3 + \cdots \sqrt{n}x^n + \cdots = \sum_{n=1}^{\infty} \sqrt{n}\,x^n. \tag{4}$$
   Be sure to give your reasoning and the calculations that support it.
5. Find the limits:
   (a) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$
   (b) $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^3}$

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1. Give the expansions in powers of x for
   (a) $f(x) = \cos x$
   (b) $f(x) = \dfrac{1}{1-x}$
   (c) $f(x) = e^{-x}$

   Solution:

   (a) The expansion of $\cos x$ in powers of x is

   $$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}. \tag{1}$$

   (b) The expansion of $\dfrac{1}{1-x}$ in powers of x is

   $$1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n. \tag{2}$$

   (c) he expansion of e^{-x} in powers of x is

   $$1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^j}{j!} + \cdots = \sum_{j=0}^{\infty} (-1)^j \frac{x^j}{j!}. \tag{3}$$

2. Explain where the expansion, in powers of x, of the function

   $$f(x) = \frac{x+2}{x^2 - 4x + 5} \tag{4}$$

   converges to $f(x)$.

   Solution: The expansion for a function about x_0 converges inside the interval $(x_0 - R, x_0 + R)$, where R is the distance from x_0 to the nearest singularity of f in the complex plane. For f as given, the only singularities are when the denominator vanishes, or where $0 = x^2 - 4x + 5$, which is at $x = 2 \pm i$. Both of these points lie $|2 \pm i| = \sqrt{5}$ units away from the origin, which is the center of the expansion we are asked to consider, so the expansion converges in the interval $(-\sqrt{5}, \sqrt{5})$. (As discussed in class, end-point behavior need not be determined.)

3. Given a function f, and a positive integer, k, explain how to find a number M which has the property that if p is another function for which

   $$|f(x) - p(x)| < M, \tag{5}$$

   for all x for which $a \le x \le b$, then

   $$\left| \int_a^b f(x)\, dx - \int_a^b p(x)\, dx \right| < 10^{-k} \tag{6}$$

   Why does the number you have found work?

Solution: If the number M has the property that $|fIx) - p(x)| < M$ on $[a, b]$, then we know that

$$\left| \int_a^b f(x)\, dx - \int_a^b p(x)\, dx \right| \leq \int_a^b |f(x) - p(x)|\, dx \tag{7}$$

$$< \int_a^b M\, dx = M(b - a). \tag{8}$$

It therefore will suffice for our purpose to find M so that

$$M(b - a) \leq 10^{-k}, \text{ or} \tag{9}$$

$$M \leq \frac{1}{10^k (b - a)}. \tag{10}$$

We know both k and $(b - 1)$, so we know the right-hand side of (10).

4. Describe the region of convergence for the series

$$x + \sqrt{2}x^2 + \sqrt{3}x^3 + \cdots \sqrt{n}x^n + \cdots = \sum_{n=1}^{\infty} \sqrt{n}\, x^n. \tag{11}$$

Be sure to give your reasoning and the calculations that support it.

Solution: We have

$$\lim_{n\to\infty} \frac{\left|\sqrt{n+1}\, x^{n+1}\right|}{\left|\sqrt{n}\, x^n\right|} = |x| \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} = |x|. \tag{12}$$

We conclude, by the Ratio Test, that the interval $(-1, 1)$ is the required region of convergence. (Discussion of end-point behavior is not required.)

5. Find the limits:

(a) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$

(b) $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^3}$

Solution:

(a) We know that $\sin x \to 0$ as $x \to 0$ and that $x \to 0$ as $x \to 0$ (Duh!). This means that we may attempt l'Hôpital's rule:

$$\lim_{x\to 0} \frac{\sin x}{x} = \frac{\cos x}{1} = 1. \tag{13}$$

(b) The limits in numerator and denominator are both zero, so we may again attempt l'Hôpital's rule:

$$\lim_{x\to 0} \frac{x - \sin x}{x^3} = \lim_{x\to 0} \frac{1 - \cos x}{3x^2} \tag{14}$$

$$= \lim_{x\to 0} \frac{(1 + \cos x)(1 - \cos x)}{3x^2(1 + \cos x)} = \lim_{x\to 0} \frac{1 - \cos^2 x}{3x^2(1 + \cos x)} \tag{15}$$

$$= \frac{1}{6} \lim_{h\to 0} \left(\frac{\sin x}{x}\right)^2 = \frac{1}{6}. \tag{16}$$

MTH 241, Calculus II, Final Exam May 12, 1992

Instructions: Submit your solutions to the following problems on your own paper; give your reasoning and show enough detail to support your reasoning. Your paper is due at 11:55am.

1. Solve: $y'(x) = \dfrac{x^3}{y(x)}$ for $y(x)$, given that $y(0) = 5$.

2. Find the following integrals.

   (a) $\displaystyle\int_0^{\pi} x \sin x \, dx$

   (b) $\displaystyle\int_0^1 \frac{x}{1-x^2} \, dx$

3. (a) Give the expansion of $\dfrac{x}{1+x^3}$ in powers of x.

   (b) The function f has the expansion

$$f(x) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots + (-1)^n \frac{x^{4n}}{(2n)!} + \cdots . \quad (1)$$

   Identify the function f.

4. The base of a certain solid is the disk enclosed by the circle $x^2+y^2 = 4$ in the xy-plane. Each cross-section of the solid by a plane perpendicular to the x-axis is an equilateral triangle whose base lies in the xy-plane. Measure the volume of the solid.

5. (a) Give parametric equations for the polar curve $r = 2 - 2\cos\theta$.

   (b) Find the area enclosed by this curve.

6. Find the expansion, in powers of x, for $f(x) = \arctan x$. Explain where the expansion converges to $f(x)$.

7. Describe the regions where these expansions converge:

   (a)

$$\frac{1}{4} + \frac{x}{5} + \frac{x^2}{6} + \cdots + \frac{x^n}{n+4} + \cdots$$

   (b)

$$\frac{(x-5)}{5} + \frac{(x-5)^2)}{50} + \frac{(x-5)^3}{375} + \cdots + \frac{(x-5)^n}{n5^n} + \cdots$$

8. **Use expansions** to solve the differential equation $y''(x) = x^2 - 2$, with $y(0) = 2$ and $y'(0) = 4$.

MTH 241, Calculus II, Final Exam May 12, 1992

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1. Solve: $y'(x) = \dfrac{x^3}{y(x)}$ for $y(x)$, given that $y(0) = 5$.

   Solution: Choose $x > 0$, let φ be a solution of the given initial value problem, and put $y = \varphi(x)$. Then

$$\varphi'(t) = \frac{t^3}{\varphi(t)}, \text{ so that} \tag{1}$$

$$\varphi(t)\varphi'(t) = t^3. \tag{2}$$

   Thus,

$$\int_0^x \varphi(t)\varphi'(t)\,dt = \int_0^x t^3\,dt, \text{ or} \tag{3}$$

$$\frac{1}{2}\left[\varphi(t)\right]^2\Big|_0^x = \frac{1}{4}t^4\Big|_0^x; \tag{4}$$

$$[\varphi(x)]^2 - 25 = \frac{1}{2}x^4; \tag{5}$$

$$\varphi(x) = \sqrt{\frac{50 + x^4}{2}}, \tag{6}$$

   where we have chosen the positive square root because $\varphi(0) = 5 > 0$.

2. Find the following integrals.

   (a) $\displaystyle\int_0^\pi x \sin x\,dx$

   (b) $\displaystyle\int_0^1 \frac{x}{1 - x^2}\,dx$

   Solution:

   (a) Let us use the method of undetermined coefficients. We seek an antiderivative of the form $Ax\cos x + B\sin x$. [This form is motivated by the observation that $\dfrac{d}{dx}(x\cos x) = \cos x - x\sin x$ contains the object whose antiderivative we seek, while $\dfrac{d}{dx}\sin x = \cos x$, and we can hope that the second term of our proposed antiderivative will supply what we need to cancel out the "extra" part from the first term.] If

$$\int x\sin x\,dx = Ax\cos x + B\sin x, \text{ then} \tag{7}$$

$$x\sin x = A\cos x - Ax\sin x + B\cos x. \tag{8}$$

In order for the cosine terms to cancel each other, we must have $A + B = 0$. For the remaining term to be what we need, we must have $A = -1$, and from what we saw first, this means that $B = 1$. So

$$\int_0^\pi x \sin x \, dx = (\sin x - x \cos x)\Big|_0^\pi = \pi. \tag{9}$$

(b) This integral is improper, because its denominator vanishes at the right endpoint of the interval of integration. Therefore, we must evaluate $\lim_{t\to 0^-} \int_0^t \frac{x\,dx}{1-x^2}$. Let $u = 1 - x^2$. Then $du = -2x\,dx$, or $x\,dx = \frac{1}{2}du$. Now

$$\int \frac{x\,dx}{1-x^2} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u = \frac{1}{2}\ln(1-x^2), \text{ so} \tag{10}$$

$$\lim_{t\to 0^-} \int_0^t \frac{x}{1-x^2} = \frac{1}{2}\lim_{t\to 0^-} \ln(1-x^2)\Big|_0^t = \frac{1}{2}\lim_{t\to 0^-} \ln(1-t^2), \tag{11}$$

which doesn't exist. The improper integral diverges.

3. (a) Give the expansion of $\frac{x}{1+x^3}$ in powers of x.

(b) The function f has the expansion

$$f(x) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots + (-1)^n \frac{x^{4n}}{(2n)!} + \cdots. \tag{12}$$

Identify the function f.

Solution:

(a) We observe first that $1 + x^3 = 1 - (-x^3)$. We know from our analysis of the geometric series that

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots + u^n + \cdots = \sum_{n=0}^{\infty} u^n, \tag{13}$$

$$\tag{14}$$

when $|u| < 1$. Substituting $-x^3$ for u, we find that

$$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \cdots + (-1)^n x^{3n} + \cdots \quad = \sum_{n=0}^{\infty} (-1)^n x^{3n}, \tag{15}$$

when $|-x^3| < 1$, or, equivalently, when $|x| < 1$. Consequently,

$$\frac{x}{1+x^3} = x - x^4 + x^7 - x^{10} + \cdots + (-1)^n x^{3n+1} + \cdots \quad = \sum_{n=0}^{\infty} (-1)^n x^{3n+1}, \tag{16}$$

also when $|x| < 1$.

(b)

$$f(x) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots + (-1)^n \frac{x^{4n}}{(2n)!} + \cdots . \tag{17}$$

$$= 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \cdots + (-1)^n \frac{(x^2)^{2n}}{(2n)!} + \cdots , \tag{18}$$

and we recognize f. It is $f(x) = \cos x^2$.

4. The base of a certain solid is the disk enclosed by the circle $x^2+y^2 = 4$ in the xy-plane. Each cross-section of the solid by a plane perpendicular to the x-axis is an equilateral triangle whose base lies in the xy-plane. Measure the volume of the solid.

Solution: An equilateral triangle whose base is perpendicular to the x-axis at $x = t$ and extends from the bottom to the top of the given disk has endpoints at $y = \pm\sqrt{4 - t^2}$. Hence the length of the base is $2\sqrt{4 - t^2}$. By elementary trigonometry, the altitude of an equilateral triangle with base $2\sqrt{4 - t^2}$ is $\sqrt{3} \cdot \sqrt{4 - t^2}$. Consequently, the area of the cross-section at $x = t$ is $\frac{1}{2} \times$ base $\times$ height $= \sqrt{3}(4 - t^2)$. The volume of the solid is therefore

$$\sqrt{3} \int_{-2}^{2} (4 - t^2)\, dt = \sqrt{3} \left(4t - \frac{t^3}{3} \right) \Big|_{-2}^{2} = \frac{32}{\sqrt{3}}. \tag{19}$$

5. (a) Give parametric equations for the polar curve $r = 2 - 2\cos\theta$.

   (b) Find the area enclosed by this curve.

Solution:

(a) If $r = 2 - 2\cos\theta$, then

$$r\cos\theta = 2\cos\theta - 2\cos^2\theta, \text{ and} \tag{20}$$

$$r\sin\theta = 2\sin\theta - 2\sin\theta\cos\theta. \tag{21}$$

Thus, one parametrization of the curve is

$$x = 2\cos\theta(1 - \cos\theta), \tag{22}$$

$$y = 2\sin\theta(1 - \cos\theta), \tag{23}$$

taking $[0, 2\pi]$ as our parameter interval.

(b) The area enclosed by the curve is

$$\frac{1}{2} \int_0^{2\pi} r^2\, d\theta = \frac{1}{2} \int_0^{2\pi} (4 - 8\cos\theta + 4\cos^2\theta)\, d\theta \tag{24}$$

$$= \frac{1}{2} \int_0^{2\pi} [4 - 8\cos\theta + 4 \cdot \frac{1}{2}(1 + \cos 2\theta)]\, d\theta \tag{25}$$

$$= \int_0^{2\pi} (3 - 4\cos\theta + \cos 2\theta)\, d\theta \tag{26}$$

$$= \left(3\theta - 4\sin\theta + \frac{1}{2}\sin 2\theta \right) \Big|_0^{2\pi} = 6\pi. \tag{27}$$

6. Find the expansion, in powers of x, for $f(x) = \arctan x$. Explain where the expansion converges to $f(x)$.

Solution: If $f(x) = \arctan x$, then

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} \tag{28}$$

$$= 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \tag{29}$$

when $|x| < 1$, by an argument similar to that in Problem 3(a), but with $u = -x^2$. Integrating, we find that

$$f(x) = \int \left(1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots\right) dx \tag{30}$$

$$= c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots, \tag{31}$$

c being an as yet undetermined constant. The integrated series, we know, must converge inside the same interval where the unintegrated series converges, *i.e.* when $|x| < 1$. We determine c by noticing that $f(0) = 0$, and that all of the terms of the series for f, with the exception of c, vanish when $x = 0$. We conclude that $c = 0$, and that

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots \tag{32}$$

when $|x| < 1$.

7. Describe the regions where these expansions converge:

(a)

$$\frac{1}{4} + \frac{x}{5} + \frac{x^2}{6} + \cdots + \frac{x^n}{n+4} + \cdots$$

(b)

$$\frac{(x-5)}{5} + \frac{(x-5)^2)}{50} + \frac{(x-5)^3}{375} + \cdots + \frac{(x-5)^n}{n5^n} + \cdots$$

Solution: (Note: Endpoint analysis is not required.)

(a) We have

$$\lim_{n\to\infty} \left(\frac{|x^{n+1}|}{n+5} \cdot \frac{n+4}{|x^n|}\right) = |x| \lim_{n\to\infty} \frac{n+4}{n+5} = |x|. \tag{33}$$

We conclude by the Ratio Test that $\sum_{n=0}^{\infty} \frac{x^n}{n+4}$ converges when $|x| < 1$, or, equivalently, when x lies in $(-1, 1)$..

(b) We have

$$\lim_{n\to\infty} \left(\frac{|(x-5)^{n+1}|}{(n+1)5^{n+1}} \cdot \frac{n5^n}{|(x-5)^n|}\right) = \frac{|x-5|}{5} \lim_{n\to\infty} \frac{n}{n+1} = \frac{|x-5|}{5}. \tag{34}$$

We conclude by the Ratio Test that $\sum_{n=0}^{\infty} \frac{(x-5)^n}{n5^n}$ converges when $|x-5| < 5$, or, equivalently, when x lies in $(0, 10)$.

8. **Use expansions** to solve the differential equation $y''(x) = x^2 - 2$, with $y(0) = 2$ and $y'(0) = 4$.

Solution: We begin with a solution φ, which we assume to have the form

$$\varphi(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots a_kx^k + \cdots . \tag{35}$$

We must have

$$\varphi'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + (k+1)a_{k+1}x^k + \cdots , \tag{36}$$

and

$$\varphi''(x) = 2a_2 + 3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + 5 \cdot 4a_5x^3 + \cdots + (k+2)(k+1)a_{k+2}x^k + \cdots \tag{37}$$

But $\varphi(0) = 2$, and, in conjunction with (35), this means that $a_0 = 2$. Simliarly, the fact that $\varphi'(0) = 4$ together with (36) mean that $a_1 = 4$. From (37) and the differential equation itself, we see that $a_k = 0$ for all $k \geq 5$. We see as well that $2a_2 = -2$, that $6a_3 = 0$, and that $12a_4 = 1$. Hence $a_2 = -1$, $a_3 = 0$, and $a_4 = 1/12$. Thus, the solution is given by

$$\varphi(x) = 2 + 4x - x^2 + \frac{1}{12}x^4. \tag{38}$$

MTH 241, Calculus II—Exam II **October 21, 1994**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, you must show enough detail to support your conclusions. Your paper is due at 1:55 pm.

1. (a) Give the *form* of the partial fractions decomposition for the rational function:

$$\frac{x^4 - 3x^3 + 11x^2 - x + 12}{(x-1)(x+3)^3(x^2+x+2)(3x^2-x+1)^3}.$$

   Do not try to find the coefficients!

   (b) Use an appropriate trigonometric (or hyperbolic) substitution to transform the integral

$$\int (16 + x^2)^{\frac{3}{2}} dx$$

   into an integral that involves no radicals or fractional powers. *Do not try to evaluate the resulting integral!*

2. Murgatroyd found the formula

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right|$$

   in a table of integrals. "Great," he said. "This means that

$$\int_0^4 \frac{dx}{9-x^2} = \frac{1}{6} \log \left| \frac{3+x}{3-x} \right| \Big|_0^4 = \frac{1}{6}\log 7 - \frac{1}{6}\log 1 = \frac{1}{6}\log 7."$$

   (a) Explain why Murgatroyd's calculation is utter nonsense.

   (b) Give the correct calculation for Murgatroyd's integral.

3. Find the following integrals:

   (a) $\displaystyle\int \sin^2 x \cos^3 x \; dx$

   (b) $\displaystyle\int \frac{3x^2 + 3x + 2}{(x-2)(x^2+2x+2)} dx$

4. Let

$$f(x) = \sqrt{1 + x^4}.$$

Use the plots in Figures 1–5 to determine how many subdivisions one needs in a Simpson's Rule approximation of

$$\int_0^2 f(x)\, dx$$

if one desires a value accurate to within 0.001. Give a full explanation of your reasoning.

5. Find the length of that portion of the curve

$$x^2 + y^2 = 1$$

that lies in the first quadrant and to the right of $x = 1/3$.

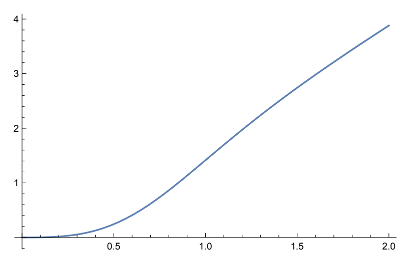

Figure 1: Graph of $f'(x)$

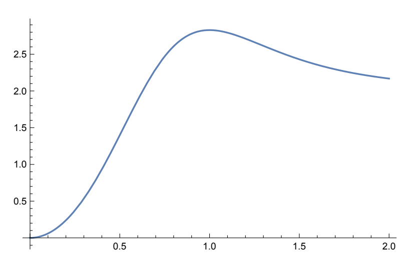

Figure 2: Graph of $f''(x)$

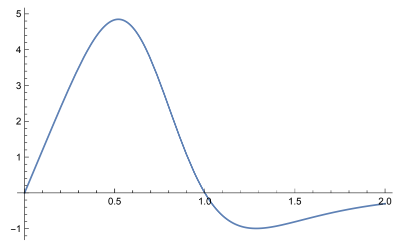

Figure 3: Graph of $f^{(3)}(x)$

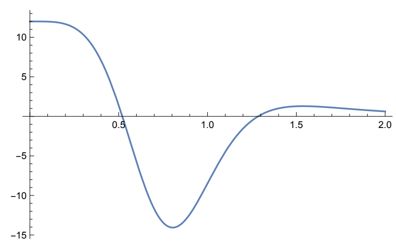

Figure 4: Graph of $f^{(4)}(x)$

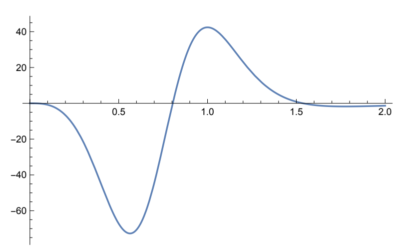

Figure 5: Graph of $f^{(5)}(x)$

MTH 241, Calculus II—Exam II **October 21, 1994**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, you must show enough detail to support your conclusions. Your paper is due at 1:55 pm.

1. (a) Give the *form* of the partial fractions decomposition for the rational function:

$$\frac{x^4 - 3x^3 + 11x^2 - x + 12}{(x-1)(x+3)^3(x^2+x+2)(3x^2-x+1)^3}.$$

Do not try to find the coefficients!

(b) Use an appropriate trigonometric (or hyperbolic) substitution to transform the integral

$$\int (16 + x^2)^{\frac{3}{2}}\, dx$$

into an integral that involves no radicals or fractional powers. *Do not try to evaluate the resulting integral!*

Solution:

(a) The form of the expansion is

$$\frac{A}{x-1} + \frac{Bx+C}{3x^2-x+1} + \sum_{k=1}^{3} \frac{D_k}{(x+3)^k} + \sum_{k=1}^{3} \frac{E_k x + F_k}{(3x^2-x+1)^k}$$

(b) Let $x = 4\sinh u$. Then $dx = 4\cosh u\, du$, and, using the relation $\cosh^2 u - \sinh^2 u = 1$, we find that

$$\int (16 + x^2)^{\frac{3}{2}}\, dx = 4 \int (16 + 16\sinh^2 u)^{3/2} \cosh u\, du \qquad (1)$$

$$= 256 \int \cosh^4 u\, du. \qquad (2)$$

2. Murgatroyd found the formula

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right|$$

in a table of integrals. "Great," he said. "This means that

$$\int_0^4 \frac{dx}{9-x^2} = \frac{1}{6}\log\left|\frac{3+x}{3-x}\right|\Bigg[_0^4 = \frac{1}{6}\log 7 - \frac{1}{6}\log 1 = \frac{1}{6}\log 7."$$

(a) Explain why Murgatroyd's calculation is utter nonsense.

(b) Give the correct calculation for Murgatroyd's integral.

Solution:

(a) Murg has missed the fact that the denominator of the integrand is zero at a point interior to the interval of integration, making this an improper integral.

(b)

$$\int_0^4 \frac{dx}{9-x^2} = \lim_{t\to 3^-}\int_0^t \frac{dx}{9-x^2} + \lim_{s\to 3^+}\int_s^4 \frac{dx}{9-x^2}. \tag{3}$$

But

$$\lim_{t\to 3^-}\int_0^t \frac{dx}{9-x^2} = \lim_{t\to 3^-} \frac{1}{6}\log\left|\frac{3+x}{3-x}\right|\Bigg[_0^t, \tag{4}$$

and the latter limit does not exist. We conclude that the improper integral diverges.

3. Find the following integrals:

(a)

$$\int \sin^2 x \cos^3 x \; dx$$

(b)

$$\int \frac{3x^2+3x+2}{(x-2)(x^2+2x+2)} dx$$

Solution:

(a)

$$\int \sin^2 x \cos^3 x \; dx = \int \sin^2 x(1-\sin^2 x)\cos x \, dx \tag{5}$$

$$= \int (\sin^2 x - \sin^4 x)\cos x \, dx \tag{6}$$

$$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c. \tag{7}$$

(b) We must first find the partial fractions decomposition of the integral, so we write

$$\frac{3x^2+3x+2}{(x-2)(x^2+2x+2)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+2}, \text{ or} \tag{8}$$

$$\frac{3x^2+3x+2}{(x-2)(x^2+2x+2)} = \frac{(A+B)x^2+(2A-2B+C)x+2(A-C)}{(x-2)(x^2+2x+2)} \tag{9}$$

Equation (9) must be true for all values of x, and this can be so only if the coefficients of like powers of x in the numerators of both sides are equal to each other. Thus,

$$A+B = 3 \tag{10}$$

$$2A-2B+C = 3 \tag{11}$$

$$A-C = 1. \tag{12}$$

From (10), we see that $B = 3 - A$; and from (12), we see that $C = A - 1$. Substituting these for B and C in (11) gives

$$2A - 2(3 - A) + (A - 1) = 3, \text{ whence} \tag{13}$$
$$5A - 7 = 3, \text{ or} \tag{14}$$
$$A = 2. \tag{15}$$

Thus,

$$B = 3 - A = 1, \text{ and} \tag{16}$$
$$C = A - 1 = 1. \tag{17}$$

It now follows that

$$\int \frac{3x^2 + 3x + 2}{(x-2)(x^2 + 2x + 2)} dx = \int \frac{2}{x-2}\, dx + \int \frac{x+1}{x^2 + 2x + 2}\, dx \tag{18}$$
$$= 2 \ln |x - 2| + \frac{1}{2} \ln(x^2 + 2x + 2) + c \tag{19}$$

4. Let

$$f(x) = \sqrt{1 + x^4}.$$

Use the plots on the last page of the exam to determine how many subdivisions one needs in a Simpson's Rule approximation of

$$\int_0^2 f(x)\ dx$$

if one desires a value accurate to within 0.001. Give a full explanation of your reasoning.

Solution: We know that the error in a Simpson's Rule approximation to $\int_a^b g(x)\, dx$, with n subdivisions, is at most $\frac{M(b-a)^5}{180n^4}$, where M is any number for which $|g^{(4)}(x)| \leq M$ for all x in the interval $[a, b]$. From Figure 4, we easily see that $|f^{(4)}(x)| \leq 15$ for all x in $[0, 2]$. In our circumstances, therefore, we have

$$\text{Error} \leq \frac{15 \cdot 2^5}{180n^4} = \frac{8}{3n^4}. \tag{20}$$

We want to ensure that error is at most $0.001 = \frac{1}{1000}$, so we must choose an *even* whole number n so that

$$\frac{8}{3n^4} \leq \frac{1}{1000}. \tag{21}$$

Thus it will suffice to have

$$n^4 \geq 2667 > \frac{8000}{3}, \text{ or} \tag{22}$$
$$n \geq 7.2. \tag{23}$$

Because n is required to be an even whole number, we must use at least 8 subdivisions in order to achieve the required accuracy.

5. Find the length of that portion of the curve

$$x^2 + y^2 = 1 \tag{24}$$

that lies in the first quadrant and to the right of $x = 1/3$.

Solution: We must calculuate $\int_{1/3}^{1} \sqrt{1 + (y')^2}\, dx$, and we have, from (24)

$$\begin{aligned}
x^2 + y^2 &= 1, \text{ so that an implicit differentiation gives} && (25)\\
2x + 2yy' &= 0, \text{ or} && (26)\\
y' &= -\frac{x}{y}, \text{ and} && (27)\\
(y')^2 &= \frac{x^2}{y^2}. && (28)
\end{aligned}$$

But, again from (24), $y^2 = 1 - x^2$, so

$$\begin{aligned}
1 + (y')^2 &= 1 + \frac{x^2}{y^2} && (29)\\
&= 1 + \frac{x^2}{1 - x^2} && (30)\\
&= \frac{1 - \cancel{x^2} + \cancel{x^2}}{1 - x^2} = \frac{1}{1 - x^2}. && (31)
\end{aligned}$$

It now follows that the arclength we desire is given by

$$\begin{aligned}
\int_{1/3}^{1} \sqrt{1 + (y')^2}\, dx &= \int_{1/3}^{1} \frac{dx}{\sqrt{1 - x^2}} && (32)\\
&= \arcsin x \Big|_{1/3}^{1} && (33)\\
&= \frac{\pi}{2} - \arcsin \frac{1}{3}. && (34)
\end{aligned}$$

(This is about 1.23096.)

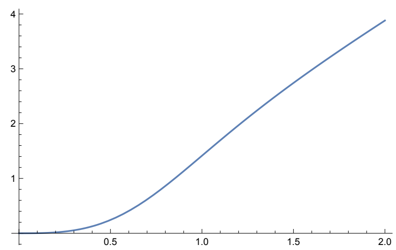

Figure 1: Graph of $f'(x)$

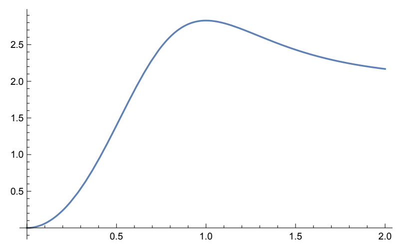

Figure 2: Graph of $f''(x)$

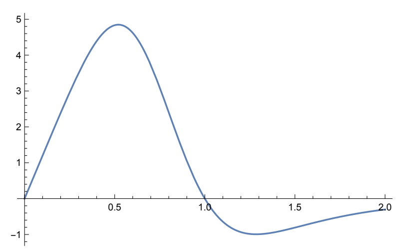

Figure 3: Graph of $f^{(3)}(x)$

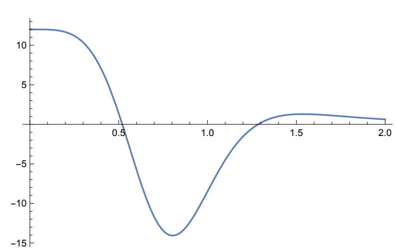

Figure 4: Graph of $f^{(4)}(x)$

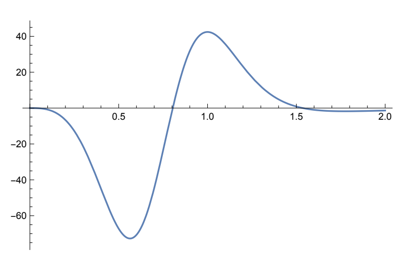

Figure 5: Graph of $f^{(5)}(x)$

MTH 241, Calculus II—Exam I **Feb. 11, 1994**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions*. Your paper is due at 10:50 am.

1. Find $f'(x)$ when:

   (a) $f(x) = \tanh(\cos^2 x)$

   (b) $f(x) = e^{\arctan x}$

2. Find the following limits:

   (a) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$

   (b) $\displaystyle\lim_{x\to 0} \frac{3^x - 2^x}{x}$

3. (a) Give the number e to at least 4 decimal places. Explain why we choose e as the base of the exponential function we use in calculus.

   (b) A scientist has a collection of data points

   $$\{(x_1, y_1),\ (x_2, y_2),\ \ldots,\ (x_n, y_n)\}.$$

   She suspects that there are constants K and p such that each of her points (x_k, y_k) lies on (or, at least, very near) the curve $y = K\,x^p$. To test her hypothesis, she plots the points

   $$\{(\ln x_1, \ln y_1),\ (\ln x_2, \ln y_2),\ \ldots,\ (\ln x_n, \ln y_n)\}$$

   obtained by replacing both coordinates of each of her data points with their natural logarithms. She notes that the points she has plotted all lie on (or very near) the same straight line. Explain why this confirms her hypothesis.

4. A certain object cools at a rate (in degrees Celsius per minute) that is equal to one-tenth of the difference between its own temperature and that of the surrounding air. A room is kept at 21° C and the temperature of the object is 33° C at noon. Use this information to derive an expression for the temperature of the object t minutes after noon.

5. (a) Explain how we can be sure that the hyperbolic tangent function has an inverse function. What are the domain and range of $\tanh^{-1}$?

   (b) Explain how to use what we know about the tanh function to find the derivative of $\tanh^{-1}$.

MTH 241, Calculus II—Exam I,Solutions **Feb. 11, 1994**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions*. Your paper is due at 10:50 am.

1. Find $f'(x)$ when:

(a)

$$f(x) = \tanh(\cos^2 x)$$

Solution:

$$\frac{d}{dx}\tanh(\cos^2 x) = \operatorname{sech}^2(\cos^2 x)\,\frac{d}{dx}(\cos^2 x) =$$

$$\operatorname{sech}^2(\cos^2 x)\left[2\cos x\,\frac{d}{dx}(\cos x)\right] = \operatorname{sech}^2(\cos^2 x)\,[-2\cos x\sin x] =$$

$$-2\cos x\sin x\,\operatorname{sech}^2(\cos^2 x)$$

(b)

$$f(x) = e^{\arctan x}$$

Solution:

$$\frac{d}{dx}e^{\arctan x} = e^{\arctan x}\,\frac{d}{dx}\arctan x = e^{\arctan x}\,\frac{1}{1+x^2}$$

2. Find the following limits:

(a)

$$\lim_{x\to 0}\frac{e^x - 1 - x}{x^2}$$

Solution: The limit in the numerator and the limit in the denominator are both zero, so we may use L'Hôpital's Rule. After we apply the rule, we find that, again, numerator and denominator both have limit zero, and we may apply the rule a second time:

$$\lim_{x\to 0}\frac{e^x - 1 - x}{x^2} = \lim_{x\to 0}\frac{e^x - 1}{2x} = \lim_{x\to 0}\frac{e^x}{2} = \frac{1}{2}$$

(b)

$$\lim_{x\to 0}\frac{3^x - 2^x}{x}$$

Solution: The limit in the numerator and the limit in the denominator are both zero, so we may use L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{3^x - 2^x}{x} = \lim_{x\to 0}\frac{3^x\ln 3 - 2^x\ln 2}{1} = \ln 3 - \ln 2 = \ln\frac{3}{2}$$

3. (a) Give the number e to at least 4 decimal places. Explain why we choose e as the base of the exponential function we use in calculus.

Solution: When we attempt to calculate the derivative of the exponential function $f(x) = a^x$, we find that

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \tag{1}$$

$$= a^x \lim_{h\to 0} \frac{a^h - 1}{h}. \tag{2}$$

When we replace the parameter a with the number e, the resulting value of the limit that appears on the right-hand side of (2) is one, making the derivative of our function as simple as possible.

(b) A scientist has a collection of data points

$$\{(x_1, y_1),\ (x_2, y_2),\ \ldots,\ (x_n, y_n)\}.$$

She suspects that there are constants K and p such that each of her points (x_k, y_k) lies on (or, at least, very near) the curve $y = K\,x^p$. To test her hypothesis, she plots the points

$$\{(\ln x_1, \ln y_1),\ (\ln x_2, \ln y_2),\ \ldots,\ (\ln x_n, \ln y_n)\}$$

obtained by replacing both coordinates of each of her data points with their natural logarithms. She notes that the points she has plotted all lie on (or very near) the same straight line. Explain why this confirms her hypothesis.

Solution: Let $u = \ln x$, $v = \ln y$. Then the statement that the given data points all

lie on the same straight line is equivalent to the existence of two constants, m and b such that the points (u_k, v_k) all satisfy the equation $v = mu + b$. But this latter equation can be rewritten $e^v = e^{mu+b} = e^b \cdot (e^u)^m$. Now $e^v = y$ and $e^u = x$. Thus, putting $K = e^b$ and $p = m$, we can rewrite the relation between u and v yet again as $y = Kx^p$, confirming her hypothesis.

4. A certain object cools at a rate (in degrees Celsius per minute) that is equal to one-tenth of the difference between its own temperature and that of the surrounding air. A room is kept at 21° C and the temperature of the object is 33° C at noon. Use this information to derive an expression for the temperature of the object t minutes after noon.

Solution: If the rate at which the object cools is one-tenth the difference between its own temperature and the ambient temperature, then, writing $T(t)$ for temperature at time t minutes after noon, we have

$$T'(t) = -\frac{1}{10}(T(t) - 21).$$

Thus,

$$\frac{T'(t)}{T(t) - 21} = -\frac{1}{10},$$

so that

$$\int_0^t \frac{T'(s)\,ds}{T(s) - 21} = -\int_0^t \frac{ds}{10},$$

$$\ln(T(t) - 21) - \ln[T(0) - 21] = -\frac{t}{10}.$$

We have been told that $T(0) = 33$, and so we may now write

$$\ln(T(t) - 21) = -\frac{t}{10} + \ln(33 - 21) = -\frac{t}{10} + \ln 12.$$

Applying the exponential function to both sides yields

$$T(t) - 21 = \exp\left[\ln 12 \quad - \frac{t}{10}\right] = \exp\left[\ln 12\right]\ \exp\ \left[-\frac{t}{10}\right].$$

From this we conclude that

$$T(t) = 21 + 12\, e^{-t/10}.$$

5. (a) Explain how we can be sure that the hyperbolic tangent function has an inverse function. What are the domain and range of $\tanh^{-1}$?

Solution: We know that $\frac{d}{dx}[\tanh x] = \text{sech}^2 x > 0$ for all x. It follows that tanh is everywhere increasing and therefore a one-to-one function which must have an inverse. The domain of tanh is the set $\mathbb{R}$ of all real numbers, and the range of tanh is the open interval $(-1, 1)$. Hence, the domain of $\tanh^{-1}$ is the open interval $(-1, 1)$, and the range of $\tanh^{-1}$ is $\mathbb{R}$.

(b) Explain how to use what we know about the tanh function to find the derivative of $\tanh^{-1}$.

Solution: We know that $y = \tanh^{-1} x$ is equivalent to $x = \tanh y$. Differentiating the latter yields:

$$\frac{d}{dx}x = \frac{d}{dx}\tanh y$$

$$1 = \text{sech}^2\ y\ \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\text{sech}^2\ y}$$

But

$$\cosh^2 y - \sinh^2 y = 1,$$

whence

$$\frac{\cosh^2 y}{\cosh^2 y} - \frac{\sinh^2 y}{\cosh^2 y} = \frac{1}{\cosh^2 y},$$

or

$$1 - \tanh^2 y = \text{sech}^2 y.$$

Hence

$$\frac{dy}{dx} = \frac{1}{1 - \tanh^2 y} = \frac{1}{1 - x^2},$$

where the last equality follows from $\tanh y = x$.

Alternate Solution: From $x = \tanh y$ and the definition of the hyperbolic tangent function, we have

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y} - 1}{e^{2y} + 1}, \tag{3}$$

so that

$$\begin{aligned}
xe^{2y} + x &= e^{2y} - 1, \text{ or} && (4)\\
(x-1)\cdot(e^y)^2 + (x+1) &= 0, \text{ or} && (5)\\
(e^y)^2 &= \frac{1+x}{1-x}, \text{ and} && (6)\\
e^y &= \sqrt{\frac{1+x}{1-x}}, && (7)
\end{aligned}$$

where we have chosen the positive square root because e^y can't be negative. Taking logarithms on both sides of (7), we see that

$$\tanh^{-1} x = y = \frac{1}{2}\ln\frac{1+x}{1-x}. \tag{8}$$

Thus,

$$\begin{aligned}
\frac{d}{dx}\tanh^{-1} x &= \frac{1}{2}\cdot\frac{\cancel{1-x}}{1+x}\cdot\frac{(1-x)+(1+x)}{(1-x)^{\cancel{2}}} = \frac{\cancel{2}}{\cancel{2}(1+x)(1-x)} && (9)\\
&= \frac{1}{1-x^2}. && (10)
\end{aligned}$$

MTH 241, Calculus II—Final Exam **Dec. 12, 1994**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions.* Your paper is due at 2:55 pm.

1. Find $f'(x)$ when:

   (a) $f(x) = \sinh(\ln x)$

   (b) $f(x) = \arctan e^x$

2. Find the following limits:

   (a) $\displaystyle\lim_{x\to 0} \frac{e^{2x}-1}{x}$

   (b) $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{x^3}$

3. (a) Give the *form* of the partial fractions decomposition for

   $$\frac{x^4+3x^3-2x+11}{x(x-2)^2(x^2-2x+5)^3}.$$

   Do not try to find the coefficients in the expansion.

   (b) Find

   $$\int \frac{3x^2-x+11}{(x+1)(x^2+4)}\,dx$$

4. How many subdivisions are needed to obtain a Midpoint Rule approximation, accurate to within 0.005, of

   $$\ln 8 = \int_1^8 \frac{dx}{x}?$$

5. Evaluate the improper integrals:

(a) $$\int_{-2}^{1} \frac{dx}{x^{2/3}}$$

(b) $$\int_{1}^{\infty} \frac{x\,dx}{\sqrt{x^2+1}}$$

6. Show how to determine the general term for the Maclaurin series of the function

$$f(x) = \mathrm{Arctan}(3x).$$

What is the radius of convergence for this series? What is the interval of convergence?

7. Find a positive whole number N which has the property that

$$\left| \frac{1}{1-x} - \sum_{k=0}^{n-1} x^k \right| < 0.003$$

for all $n \geq N$ whenever $|x| \leq 0.7$.

8. An engineer needs to estimate values of the integrals

$$I(x) = \int_0^x \ln(1+t^2)\,dt,$$

where $0 \leq x \leq 1/2$. She wants to replace the integrand with the first three non-zero terms of its Maclaurin series:

$$\ln(1+t^2) \cong t^2 - \frac{1}{2}t^4 + \frac{1}{3}t^6,$$

and use the resulting estimates:

$$I(x) \cong \frac{1}{3}x^3 - \frac{1}{10}x^5 + \frac{1}{21}x^7.$$

She needs to be sure that her estimates are accurate to within 0.0005, and she knows that if $0 \leq t \leq 0.5$, then

$$\left| \ln(1+t^2) - (t^2 - \frac{1}{2}t^4 + \frac{1}{3}t^6) \right| \leq 0.0009.$$

Will her approximation scheme produce the required accuracy, or does she need (at least) another term in her Maclaurin series?

MTH 241, Calculus II—Final Exam **Dec. 12, 1994**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions*. Your paper is due at 2:55 pm.

1. Find $f'(x)$ when:

   (a)
$$f(x) = \sinh(\ln x)$$

   (b)
$$f(x) = \arctan e^x$$

   Solution:

   (a) $f'(x) = \dfrac{d}{dx}\sinh(\ln x) = \cosh(\ln x)\cdot\dfrac{d}{dx}\ln x = \dfrac{\cosh(\ln x)}{x}$.

   (b) $f'(x) = \dfrac{d}{dx}\arctan e^x = \dfrac{1}{1+(e^x)^2}\cdot\dfrac{d}{dx}e^x = \dfrac{e^x}{1+e^{2x}}$.

2. Find the following limits:

   (a) $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{x}$

   (b) $\lim\limits_{x\to 0}\dfrac{\sin x - x}{x^3}$

   Solution:

   (a) We know that

$$e^{2x} = 1 + 2x + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} + \frac{2^4x^4}{4!} + \cdots\,. \tag{1}$$

   Hence

$$\lim_{x\to 0}\frac{e^{2x}-1}{x} = \lim_{x\to 0}\frac{2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots}{x} \tag{2}$$

$$= \lim_{x\to 0}\left(2 + 2x + \frac{4}{3}x^2 + \frac{2}{3}x^3 + \cdots\right) = 2. \tag{3}$$

   (b) We know that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\,. \tag{4}$$

   Consequently,

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} = \lim_{x\to 0}\frac{-\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{x^3} \tag{5}$$

$$= \lim_{x\to 0}\left(-\frac{1}{6} + \frac{1}{120}x^2 + \frac{1}{5040}x^4\right) = -\frac{1}{6}. \tag{6}$$

3. (a) Give the *form* of the partial fractions decomposition for

$$\frac{x^4 + 3x^3 - 2x + 11}{x(x-2)^2(x^2 - 2x + 5)^3}.$$

Do not try to find the coefficients in the expansion.

(b) Find

$$\int \frac{3x^2 - x + 11}{(x+1)(x^2+4)}\, dx$$

Solution:

(a) The form of this expansion is

$$\frac{A}{x} + \frac{B}{(x-2)^2} + \frac{C}{x-2} + \frac{Dx+E}{(x^2-2x+5)^3} + \frac{Fx+G}{(x^2-2x+5)^2} + \frac{Hx+I}{x^2-2x+5}.$$

(b) We must find constants A, B,and C so that

$$\frac{3x^2 - x + 11}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}. \tag{7}$$

If this is so, then

$$\frac{A(x^2+4) + (x+1)(Bx+C)}{(x+1)(x^2+4)} = \frac{3x^2 - x + 11}{(x+1)(x^2+4)}, \text{ or} \tag{8}$$

$$\frac{(A+B)x^2 + (B+C)x + (4A+C)}{(x+1)(x^2+4)} = \frac{3x^2 - x + 11}{(x+1)(x^2+4)}. \tag{9}$$

This can be so only if the coefficients of like powers of x in the numerators are equal, or

$$A + B = 3, \tag{10}$$

$$B + C = -1, \text{ and} \tag{11}$$

$$4A + C = 11. \tag{12}$$

Subtracting (11) from (12), we find that $4A - B = 12$. Adding this latter equation to (10), we find that $5A = 15$, whence $A = 3$. Substituting this value for A in (12) gives $C = -1$, and from this value for C we find from (11) that $B = 0$. Thus,

$$\int \frac{3x^2 - x + 11}{(x+1)(x^2+4)}\, dx = \int \frac{3}{x+1}\, dx - \int \frac{1}{x^2+4}\, dx \tag{13}$$

$$= 3\ln|x+1| - \frac{1}{2}\arctan\frac{x}{2} + K, \tag{14}$$

for some undetermined constant K.

4. How many subdivisions are needed to obtain a Midpoint Rule approximation, accurate to within 0.005, of

$$\ln 8 = \int_1^8 \frac{dx}{x}?$$

Solution: If $a < b$, the magnitude of the error in an n-subdivision midpoint approximation to the integral $\int_a^b f(t)\,dt$ is at most $\dfrac{M(b-a)^3}{12n^2}$, where M is any number for which $|f''(x)| \le M$ for all x in $[a,b]$. If $f(x) = \dfrac{1}{x}$, then $f''(x) = \dfrac{2}{x^3}$. Now $f''(x)$ is a decreasing function on the interval $[1,8]$, so $|f''(x)| \le f''(1) = 2$. Hence we may take $M = 2$, and we must therefore find n so that

$$\frac{2(8-1)^3}{12n^2} = \frac{343}{6n^2} \le 0.005 = \frac{1}{200}, \text{ or} \tag{15}$$

$$n^2 \ge \frac{34300}{3} \sim 11434. \tag{16}$$

But n must be a whole number, and the smallest whole number which is larger than $\sqrt{11434} \sim 106.93$ is 107. So we will need $n \ge 107$ in order to insure the desired accuracy.

5. Evaluate the improper integrals:

(a)

$$\int_{-2}^{1} \frac{dx}{x^{2/3}}$$

(b)

$$\int_{1}^{\infty} \frac{x\,dx}{\sqrt{x^2+1}}$$

Solution:

(a) We must evaluate both

$$\lim_{t\to 0^-} \int_{-2}^{t} x^{-2/3}\,dx$$

and

$$\lim_{s\to 0^+} \int_{s}^{1} x^{-2/3}\,dx.$$

We have

$$\lim_{t\to 0^-} \int_{-2}^{t} x^{-2/3}\,dx = 3 \lim_{t\to 0^-} x^{1/3}\Big|_{-2}^{t} = 3 \lim_{t\to 0^-} [t^{1/3} + 2^{1/3}] = 2^{1/3}, \text{ and} \tag{17}$$

$$\lim_{s\to 0^+} \int_{s}^{1} x^{-2/3}\,dx = 3 \lim_{s\to 0^+} x^{1/3}\Big|_{s}^{1} = 3 \lim_{s\to 0^+} [1 - s^{1/3}] = 1 \tag{18}$$

Thus

$$\int_{-2}^{1} \frac{dx}{x^{2/3}} = \lim_{t\to 0^-} \int_{-2}^{t} x^{-2/3}\,dx + \lim_{s\to 0^+} \int_{s}^{1} x^{-2/3}\,dx = 2^{1/3} + 1. \tag{19}$$

(b) We have

$$\int_1^\infty \frac{x\,dx}{\sqrt{x^2+1}} = \lim_{T\to\infty} \int_1^T \frac{x\,dx}{\sqrt{x^2+1}} = \lim_{T\to\infty} \sqrt{x^2+1}\Big|_1^T \tag{20}$$

$$= \lim_{T\to\infty} \left[\sqrt{T^2+1} - \sqrt{2}\right], \tag{21}$$

which doesn't exist. The improper integral $\int_1^\infty \frac{x\,dx}{\sqrt{x^2+1}}$ diverges.

6. Show how to determine the general term for the Maclaurin series of the function

$$f(x) = \text{Arctan}(3x).$$

What is the radius of convergence for this series? What is the interval of convergence?

Solution: From $f(x) = \arctan 3x$, we have

$$f'(x) = \frac{3}{1+9x^2} = 3\frac{1}{1-(-9x^2)} \tag{22}$$

which, by the geometric series,

$$= 3\big[1 + (-9x^2) + (-9x^2)^2 + (-9x^2)^3 + \cdots + (-9x^2)^n + \cdots\big] \tag{23}$$

$$= \sum_{n=0}^{\infty} (-1)^n 3^{2n+1} x^{2n} \tag{24}$$

when $|-9x^2| < 1$, or when $-1 < 3x < 1$—which, in turn, we can rewrite as $-\frac{1}{3} < x < \frac{1}{3}$. We integrate this series to get the Maclaurin series for $f(x)$, which is then

$$f(x) = C + \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{2n+1}. \tag{25}$$

(Here C is a constant we could determine if we needed to, but we don't and won't bother.) Except for (possibly) the endpoints, about which we don't care, the integrated series converges on the same interval, $\left(-\frac{1}{3}, \frac{1}{3}\right)$ as that for f'. We conclude that the general term of the Maclaurin series for $\arctan 3x$ is

$$\frac{(-1)^n 3^{2n+1} x^{2n+1}}{2n+1},$$

and that the radius of convergence is $\frac{1}{3}$.

7. Find a positive whole number N which has the property that

$$\left| \frac{1}{1-x} - \sum_{k=0}^{n-1} x^k \right| < 0.003$$

for all $n \geq N$ whenever $|x| \leq 0.7$.

Solution: We begin by recalling that

$$\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x} \tag{26}$$

and so

$$\left|\frac{1}{1-x} - \sum_{k=0}^{n-1} x^k\right| = \left|\frac{1}{1-x} - \frac{1-x^n}{1-x}\right| = \left|\frac{x^n}{1-x}\right|. \tag{27}$$

Consequently, we can rewrite the inequality we want to establish as

$$\left|\frac{x^n}{1-x}\right| < \frac{3}{1000}. \tag{28}$$

We have been given $|x| \leq 0.7$, which is equivalent to the compound inequality

$$-0.7 \leq -x \leq 0.7. \tag{29}$$

Adding one to each member of this latter inequality preserves the inequalities, and we see that

$$\frac{3}{10} = 0.3 \leq 1 - x \leq 1.7. \tag{30}$$

Thus, the requirement that $|x| \leq 0.7$ guarantees that

$$\frac{1}{1-x} \leq \frac{10}{3}, \tag{31}$$

and this means that

$$\left|\frac{x^n}{1-x}\right| \leq \frac{10}{3}|x|^n \tag{32}$$

Thus, we can ensure that (28) holds if we can be sure that

$$\frac{10}{3}|x|^n < \frac{3}{1000}, \tag{33}$$

or, because $|x| \leq 0.7|$, that

$$\left(\frac{7}{10}\right)^n < \frac{9}{1000}. \tag{34}$$

This latter inequality, in turn, is equivalent to the inequality

$$n \ln \frac{7}{10} < \ln \frac{9}{1000}. \tag{35}$$

Now $0.7 < 1$, so $\ln 0.7 = \ln 7 - \ln 10 < 0$, and this means that we can write (35) as

$$n > \frac{\ln 9 - \ln 1000}{\ln 7 - \ln 10} \sim 13.2. \tag{36}$$

It follows that $N = 14$ has the property that

$$\left| \frac{1}{1-x} - \sum_{k=0}^{n-1} x^k \right| < 0.003$$

for all $n \geq N$ whenever $|x| \leq 0.7$.

8. An engineer needs to estimate values of the integrals

$$I(x) = \int_0^x \ln(1+t^2)\ dt,$$

where $0 \leq x \leq 1/2$. She wants to replace the integrand with the first three non-zero terms of its Maclaurin series:

$$\ln(1+t^2) \cong t^2 - \frac{1}{2}t^4 + \frac{1}{3}t^6,$$

and use the resulting estimates:

$$I(x) \cong \frac{1}{3}x^3 - \frac{1}{10}x^5 + \frac{1}{21}x^7.$$

She needs to be sure that her estimates are accurate to within 0.0005, and she knows that if $0 \leq t \leq 0.5$, then

$$\left| \ln(1+t^2) - (t^2 - \frac{1}{2}t^4 + \frac{1}{3}t^6) \right| \leq 0.0009.$$

Will her approximation scheme produce the required accuracy, or does she need (at least) another term in her Maclaurin series?

Solution: We observe that

$$\left| I(x) - \int_0^x \left(t^2 - \frac{1}{2}t^4 + \frac{1}{3}t^6 \right) dx \right| = \left| \int_0^x \left[\ln(1+t^2) - \left(t^2 - \frac{1}{2}t^4 + \frac{1}{3}t^6 \right) \right] dt \right| \quad (37)$$

$$\leq \int_0^x \left| \ln(1+t^2) - \left(t^2 - \frac{1}{2}t^4 + \frac{1}{3}t^6 \right) \right| dt \quad (38)$$

$$\leq \int_0^x (0.0009)\, dt \quad (39)$$

$$\leq \frac{9}{10000}x \leq \frac{9}{20000} = 0.00045 < 0.0005, \quad (40)$$

the final inequality having x in its left member owing to the fact that $0 \leq x \leq \frac{1}{2}$. Consequently, her approximation scheme meets her requirements.

MTH 2410, Calculus II—Exam I **Oct. 1, 1998**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions.* Your paper is due at 11:55 am.

1. Professor I. M. Brightly put the following problem on a Calc II exam:

   > A bacteria culture starts with 500 bacteria and grows at a rate that is proportional to its size. After 4 hours there are 8000 bacteria. Give an expression for the number of bacteria after t hours.

   When Æthelbert read this, he said to himself "If N is the number of bacteria in the colony at time t, there must be constants C and k such that $N = Ce^{kt}$. Moreover, we know that C must be 500. But then we have to have $8000 = 500e^{4k}$. This means that $k = \ln 2$. Consequently, $N = 500 \cdot 2^t$"

   Each of Æthelbert's sentences expresses a conclusion. Give the reasoning that underlies each of those conclusions.

2. Find y' when

   (a)
   $$y = \tanh^2 x \operatorname{sech} x$$

   (b)
   $$y = x^{\cosh x}$$

3. A surface is generated by revolving the part of the curve $y = \cosh x$ that corresponds to $-1 \le x \le 1$ about the x-axis.

   (a) Write out the integral gives that area of this surface.

   (b) Evaluate the integral you gave for part 3a.

4. Find the limits; explain your reasoning.

   (a)
   $$\lim_{x\to 0} \frac{x + \sin 2x}{x - \sin 2x}$$

   (b)
   $$\lim_{x\to 4} \frac{x^3 - 9x^2 + 24x - 16}{x^3 - 5x^2 - 8x + 48}$$

5. Find

   (a)

$$\int \cos^4 4x \, \sin^3 4x \, dx$$

   (b)

$$\int (\sin 2x - \cos x) \sin x \, dx$$

6. Find

   (a)

$$\int xe^{3x} \, dx$$

   (b)

$$\int x \operatorname{Arctan} x \, dx$$

7. Find

$$\int \frac{5x^2 + 6x - 23}{(x+2)(x-1)(x-3)} \, dx$$

8. Given that $y = f(x)$ satisfies

$$x^3 y' = y^2(x-4), \text{ and} \quad (1)$$

$$f(1) = 1, \quad (2)$$

describe f completely.

MTH 2410, Calculus II—Exam I Solutions **Oct. 1, 1998**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions*. Your paper is due at 11:55 am.

1. Professor I. M. Brightly put the following problem on a Calc II exam:

   > A bacteria culture starts with 500 bacteria and grows at a rate that is proportional to its size. After 4 hours there are 8000 bacteria. Give an expression for the number of bacteria after t hours.

   When Æthelbert read this, he said to himself "If N is the number of bacteria in the colony at time t, there must be constants C and k such that $N = Ce^{kt}$. Moreover, we know that C must be 500. But then we have to have $8000 = 500e^{4k}$. This means that $k = \ln 2$. Consequently, $N = 500 \cdot 2^t$"

   Each of Æthelbert's sentences expresses a conclusion. Give the reasoning that underlies each of those conclusions.

   Solution:

   (a) We are told that $N'(t)$ is proportional to $N(t)$, so there must be a constant k such that $N'(t) = kN(t)$. This means that

   $$\begin{aligned} dN &= kN dt; \\ \frac{dN}{N} &= k dt; \\ \int \frac{dN}{N} &= k \int dt; \\ \ln|N| &= kt + c, \end{aligned}$$

   for some constant c. Taking exponentials on both sides ot the last equation above, we obtain

   $$\begin{aligned} |N| &= e^{kt+c}, \\ &= e^{kt} \cdot e^{c}; \text{ or,} \\ N(t) &= Ce^{kt}, \end{aligned}$$

   where $|C| = e^c$.

   (b) Since $N(t) = Ce^{kt}$, and $N(0)$ is given as 500, we must have

   $$500 = N(0) = Ce^{k \cdot 0} = C,$$

   so that $N(t) = 500e^{kt}$.

   (c) We are also given that $N(4) = 8000$. Thus

   $$8000 = N(4) = 500e^{k \cdot 4} = 500e^{4k}.$$

   (d) Dividing the equation $8000 = 500e^{4k}$ through on both sides by 500 yields $16 = e^{4k}$. Taking natural logs on both sides of this latter equation gives $\ln 16 = 4k$, or $\frac{1}{4} \ln 16 = k$. But $\frac{1}{4} \ln 16 = \ln 16^{1/4} = \ln 2$. Thus, $k = \ln 2$.

(e) We may now write

$$\begin{aligned} N &= 500e^{kt} \\ &= 500e^{t\ln 2} \\ &= 500(e^{\ln 2})^t \\ &= 500(2)^t \\ &= 500 \cdot 2^t. \end{aligned}$$

2. Find y' when

(a) $y = \tanh^2 x \operatorname{sech} x$

Solution: If $y = \tanh^2 x$ sech x, then

$$\begin{aligned} y' &= \left[\frac{d}{dx}\left(\tanh^2 x\right)\right] \cdot \operatorname{sech} x + \tanh^2 x \cdot \left[\frac{d}{dx}\operatorname{sech} x\right] \\ &= \left[2\tanh x \cdot \frac{d}{dx}\tanh x\right] \cdot \operatorname{sech} x + \tanh^2 x \cdot [-\operatorname{sech} x \tanh x] \\ &= 2\tanh x \operatorname{sech}^3 x - \tanh^3 x \operatorname{sech} x. \end{aligned}$$

(b) $y = x^{\cosh x}$

Solution: If $y = x^{\cosh x}$, then

$$\ln y = \ln x^{\cosh x} = \cosh x \ln x.$$

Hence,

$$\begin{aligned} \frac{d}{dx}\ln y &= \frac{d}{dx}(\cosh x \ln x); \\ \frac{y'}{y} &= \left(\frac{d}{dx}\cosh x\right)\ln x + \cosh x\left(\frac{d}{dx}\ln x\right); \\ y' &= y\left[\sinh x \ln x + (\cosh x)\frac{1}{x}\right]; \\ y' &= x^{\cosh x}\left(\sinh x \ln x + \frac{\cosh x}{x}\right). \end{aligned}$$

3. A surface is generated by revolving the part of the curve $y = \cosh x$ that corresponds to $-1 \leq x \leq 1$ about the x-axis.

(a) Write out the integral gives that area of this surface.

Solution: Surface area for a surface formed by revolving a curve $y = f(x)$, $a \leq x \leq b$, about the x-axis given by a integral of the form

$$2\pi \int_a^b y\, ds = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx$$

. Here, $a = -1$, $b = 1$, and $f(x) = \cosh x$, so the desired integral is

$$2\pi \int_{-1}^{1} \cosh x \sqrt{1 + \sinh^2 x}\, dx.$$

(b) Evaluate the integral you gave for part 3a.

Solution: Now $1 + \sinh^2 x = \cosh^2 x$, so

$$\begin{aligned}
2\pi \int_{-1}^{1} \cosh x \sqrt{1 + \sinh^2 x}\, dx &= 2\pi \int_{-1}^{1} \cosh^2 x\, dx \\
&= 2\pi \int_{-1}^{1} \left(\frac{e^x + e^{-x}}{2} \right)^2 dx \\
&= 2\pi \int_{-1}^{1} \left(\frac{e^{2x} + 2 + e^{-2x}}{4} \right) dx \\
&= \frac{\pi}{2} \int_{-1}^{1} (e^{2x} + 2 + e^{-2x})\, dx \\
&= \frac{\pi}{2} (\frac{1}{2} e^{2x} + 2x - \frac{1}{2} e^{-2x}) \Big|_{-1}^{1} \\
&= \frac{\pi}{2} (e^2 + 4 - e^{-2}).
\end{aligned}$$

4. Find the limits; explain your reasoning.

(a) $$\lim_{x \to 0} \frac{x + \sin 2x}{x - \sin 2x}$$

Solution: Because $\lim_{x \to 0} (x \pm \sin 2x) = 0$, we may apply L'Hôpital's Rule:

$$\begin{aligned}
\lim_{x \to 0} \frac{x + \sin 2x}{x - \sin 2x} &= \lim_{x \to 0} \frac{1 + 2\cos 2x}{1 - 2\cos 2x} \\
&= \frac{1 + 2}{1 - 2} \\
&= -3.
\end{aligned}$$

(b) $$\lim_{x \to 4} \frac{x^3 - 9x^2 + 24x - 16}{x^3 - 5x^2 - 8x + 48}$$

Solution: Because

$$\begin{aligned}
\lim_{x \to 4} (x^3 - 9x^2 + 24x - 16) &= 4^3 - 9 \cdot 4^2 + 24 \cdot 4 - 16 \\
&= 64 - 144 + 24 - 16 \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{x \to 4} (x^3 - 5x^2 - 8x + 48) &= 4^3 - 5 \cdot 4^2 - 8 \cdot 4 + 48 \\
&= 64 - 80 - 32 + 48 \\
&= 0
\end{aligned}$$

we may apply L'Hôpital's Rule:

$$\lim_{x \to 4} \frac{x^3 - 9x^2 + 24x - 16}{x^3 - 5x^2 - 8x + 48} = \lim_{x \to 4} \frac{3x^2 - 18x + 24}{3x^2 - 10x - 8}.$$

But

$$\begin{aligned}
\lim_{x \to 4} (3x^2 - 18x + 24) &= 3 \cdot 4^2 - 18 \cdot 4 + 24 \\
&= 48 - 72 + 24 \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to 4}(3x^2-10x-8) &= 3\cdot 4^2-10\cdot 4-8\\ &= 48-40-8\\ &= 0,\end{aligned}$$

so we may apply L"Hôpital's Rule again:

$$\begin{aligned}\lim_{x\to 4}\frac{3x^2-18x+24}{3x^2-10x-8} &= \lim_{x\to 4}\frac{6x-18}{6x-10}\\ &= \frac{24-18}{24-10}\\ &= \frac{6}{14}\\ &= \frac{3}{7}.\end{aligned}$$

Note that L'Hôpital's Rule does not apply a third time, and we *must not* use it.

5. Find

(a) $$\int \cos^4 4x\,\sin^3 4x\,dx$$

Solution: We have:

$$\begin{aligned}\int \cos^4 4x\,\sin^3 4x\,dx &= \int \cos^4 4x\,(\sin^2 4x)\,\sin 4x\,dx\\ &= \int \cos^4 4x\,(1-\cos^2 4x)\,\sin 4x\,dx\\ &= \int (\cos^4 4x-\cos^6 4x)\,\sin 4x\,dx.\end{aligned}$$

Putting $u=\cos 4x$, we have $du=-4\sin 4x\,dx$, or $\sin 4x\,dx=-\frac{1}{4}du$. Hence

$$\begin{aligned}\int (\cos^4 4x-\cos^6 4x)\,\sin 4x\,dx &= -\frac{1}{4}\int (u^4-u^6)\,du\\ &= -\frac{1}{4}\left(\frac{u^5}{5}-\frac{u^7}{7}\right)\\ &= \frac{1}{28}\cos^7 4x-\frac{1}{20}\cos^5 4x.\end{aligned}$$

(b) $$\int (\sin 2x-\cos x)\sin x\,dx$$

Solution: We may write:

$$\begin{aligned}\int (\sin 2x-\cos x)\sin x\,dx &= \int (2\sin x\cos x-\cos x)\sin x\,dx\\ &= \int (2\sin^2 x-\sin x)\cos x\,dx.\end{aligned}$$

If we now put $u=\sin x$, then $\cos x\,dx=du$, and consequently

$$\begin{aligned}\int (2\sin^2 x-\sin x)\cos x\,dx &= \int (2u^2-u)\,du\\ &= \frac{2}{3}u^3-\frac{1}{2}u^2\\ &= \frac{2}{3}\sin^3 x-\frac{1}{2}\sin^2 x.\end{aligned}$$

6. Find

(a) $$\int xe^{3x}\,dx$$

Solution: We let $u = x$; $dv = e^{3x}\,dx$. Then $du = dx$ and $v = \frac{1}{3}e^{3x}$, so

$$\begin{aligned}\int xe^{3x}\,dx &= \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\,dx \\ &= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}.\end{aligned}$$

(b) $$\int x \operatorname{Arctan} x\,dx$$

Solution: Put $u = \operatorname{Arctan} x$; $dv = x\,dx$. Then $du = \dfrac{dx}{1+x^2}$ and $v = \dfrac{1}{2}x^2\,dx$, so

$$\begin{aligned}\int x \operatorname{Arctan} x\,dx &= \frac{1}{2}x^2 \operatorname{Arctan} x - \frac{1}{2}\int \frac{x^2}{x^2+1}\,dx \\ &= \frac{1}{2}x^2 \operatorname{Arctan} x - \frac{1}{2}\int \frac{x^2+1-1}{x^2+1}\,dx \\ &= \frac{1}{2}x^2 \operatorname{Arctan} x - \frac{1}{2}\int \left(1 - \frac{1}{x^2+1}\right)\,dx \\ &= \frac{1}{2}x^2 \operatorname{Arctan} x - \frac{1}{2}x + \frac{1}{2}\operatorname{Arctan} x.\end{aligned}$$

7. Find

$$\int \frac{5x^2+6x-23}{(x+2)(x-1)(x-3)}\,dx$$

Solution: We must decompose the fraction by finding constants A, B, and C so that

$$\begin{aligned}\frac{5x^2+6x-23}{(x+2)(x-1)(x-3)} &= \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{x-3} \\ &= \frac{A(x-1)(x-3) + B(x+2)(x-3) + C(x+2)(x-1)}{(x+2)(x-1)(x-3)}.\end{aligned}$$

Equating numerators, we obtain:

$$5x^2+6x-23 = A(x-1)(x-3) + B(x+2)(x-3) + C(x+2)(x-1).$$

Putting $x = -2$ in this latter equation yields $-15 = 15A$, or $A = -1$. Similarly, setting $x = 1$ gives $-12 = -6B$, whence $B = 2$; and setting $x = 3$ gives $40 = 10C$, so that $C = 4$. Consequently,

$$\begin{aligned}\int \frac{5x^2+6x-23}{(x+2)(x-1)(x-3)}\,dx &= \int \left(\frac{4}{x-3} + \frac{2}{x-1} - \frac{1}{x+2}\right)\,dx \\ &= 4\ln|x-3| + 2\ln|x-1| - \ln|x+2|.\end{aligned}$$

8. Given that $y = f(x)$ satisfies

$$\begin{aligned} x^3 y' &= y^2(x-4), \text{ and} && (1) \\ f(1) &= 1, && (2) \end{aligned}$$

describe f completely.

Solution: We have

$$\begin{aligned} x^3\,dy &= y^2(x-4)\,dx; \\ \frac{dy}{y^2} &= \left(\frac{x-4}{x^3}\right)\,dx; \\ y^{-2}\,dy &= (x^{-2} - 4x^{-3})\,dx; \\ \int y^{-2}\,dy &= \int (x^{-2} - 4x^{-3})\,dx; \\ -y^{-1} &= -x^{-1} - 4\left(\frac{x^{-2}}{-2}\right) + c. \end{aligned}$$

Therefore,

$$\begin{aligned} y^{-1} &= x^{-1} - 2x^{-2} + c; \\ &= \frac{1}{x} - \frac{2}{x^2} + c; \\ &= \frac{x - 2 + cx^2}{x^2}. \end{aligned}$$

Consequently

$$y = \frac{x^2}{x - 2 + cx^2}.$$

But $y = 1$ when $x = 1$, and so

$$\begin{aligned} 1 &= \frac{1}{1 - 2 + c}; \\ 1 &= \frac{1}{c - 1}; \\ c - 1 &= 1; \\ c &= 2. \end{aligned}$$

Therefore

$$f(x) = \frac{x^2}{2x^2 + x - 2}.$$

MTH 2410, Calculus II—Exam II **Nov. 5, 1998**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions*. Your paper is due at 11:55 am.

1. Professor I. M. Brightly gave another calculus test. This time, he asked for the partial fractions decomposition of $\dfrac{1}{x^3-8}$. Brünhilde answered as follows:

   (a) We have $x^3 - 8 = (x-2)(x^2+2x+4)$, so

   (b) I must find constants A, B, and C that make
   $$A(x^2+2x+4) + (Bx+C)(x-2) \quad = \quad 1 \tag{1}$$
   a true statement for all values of x.

   (c) Putting $x = 2$ in equation (1) tells me that $A = \frac{1}{12}$.

   (d) Setting $x = 0$ in equation (1) tells me that
   $$4A - 2C \quad = \quad 1,$$
   whence $C = -\frac{1}{3}$.

   (e) Finally, putting $x = 3$ in equation (1) yields
   $$19A + 3B + C \quad = \quad 1,$$
   and so $B = -\frac{1}{12}$.

   (f) Thus, the required decomposition is
   $$\frac{1}{x^3-8} \quad = \quad \frac{1}{12}\cdot\frac{1}{x-2} - \frac{1}{12}\cdot\frac{x+4}{x^2+2x+4}.$$

   Each of Brünhilde's statements expresses a conclusion. Give the reasoning that underlies each of her conclusions.

2. Find the area bounded by the rays $\theta = 0$, $\theta = \frac{\pi}{2}$, and the polar curve $r = e^{2\theta}$.

3. Give a substitution that is likely to transform each of the following integrals into an integral that can be calculated with reasonable ease. *In this problem, you need not actually carry out any substitutions, and you need not calculate any of the integrals.*

   (a) $$\int \frac{x\,dx}{(4-x^2)^{3/2}}.$$

   (b) $$\int \frac{x^2-2}{\sqrt{4x^2-9}}\,dx$$

   (c) $$\int x\sqrt{25x^2+16}\,dx$$

4. (a) Find the polar equation for the line whose rectangular equation is $y = 2x + 3$. Give your equation in the form $r = f[\theta]$ for suitable f.

   (b) Transform the polar equation $r = 4\sin\theta$ into rectangular coordinates and identify the curve.

5. Use the indicated substitutions to transform the following integrals (simplify the transformed integrands). *You do not need to evaluate the integrals of this problem.*

(a) $\displaystyle\int \frac{\sqrt{x^2-1}}{x^2}\,dx$; substitute $x = \sec\theta$.

(b) $\displaystyle\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$; substitute $v = x^{1/6}$.

6. Decide whether or not the following improper integrals converge; give the values of those that do.

(a) $\displaystyle\int_0^2 \frac{x}{x^2-1}\,dx$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+4}$

7. Let $f[x] = \dfrac{1}{\sqrt[3]{1+x^2}}$.

(a) Calculate the Simpson's Rule approximation for $\displaystyle\int_1^3 f[x]\,dx$ using two subdivisions. Give your answer correct to the third digit to the right of the decimal in your answer.

(b) Unpleasant calculation (or Mathematica) shows that

$$f^{(4)}[x] = \frac{16(55x^4 - 198x^2 + 27)}{81(x^2+1)^{13/3}}.$$

It can be shown from this that if $1 \le x \le 3$, then $|f^{(4)}[x]| \le \frac{3}{2}$. How much error could there be in the approximation you gave in the previous part of this problem?

(c) How many subdivisions will be needed in Simpson's Rule to estimate $\displaystyle\int_1^3 f[x]\,dx$ to within 10^{-6}?

MTH 2410, Calculus II—Exam II **Nov. 5, 1998**

Instructions: Write out complete presentations of your solutions to the following problems on your own paper. If you want full credit, *you must show enough detail to support your conclusions*. Your paper is due at 11:55 am.

1. Professor I. M. Brightly gave another calculus test. This time, he asked for the partial fractions decomposition of $\dfrac{1}{x^3-8}$. Brünhilde answered as follows:

   (a) We have $x^3 - 8 = (x-2)(x^2+2x+4)$, so

   (b) I must find constants A, B, and C that make
   $$A(x^2+2x+4) + (Bx+C)(x-2) \;=\; 1 \qquad (1)$$
   a true statement for all values of x.

   (c) Putting $x = 2$ in equation (1) tells me that $A = \frac{1}{12}$.

   (d) Setting $x = 0$ in equation (1) tells me that
   $$4A - 2C \;=\; 1,$$
   whence $C = -\frac{1}{3}$.

   (e) Finally, putting $x = 3$ in equation (1) yields
   $$19A + 3B + C \;=\; 1,$$
   and so $B = -\frac{1}{12}$.

   (f) Thus, the required decomposition is
   $$\frac{1}{x^3-8} \;=\; \frac{1}{12}\cdot\frac{1}{x-2} - \frac{1}{12}\cdot\frac{x+4}{x^2+2x+4}.$$

   Each of Brünhilde's statements expresses a conclusion. Give the reasoning that underlies each of her conclusions.

   Solution:

   (a) She has used the identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$.

   (b) The desired partial fractions decomposition must have the form
   $$\frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4},$$
   and when we combine these two fractions over a common denominator, the numerator is the quantity that Brünhilde has equated with one—which is the numerator of the fraction she's expanding.

   (c) Equation (1) must be true for all values of x if the expansion is to equal the fraction she began with.

   (d) This follows from reasoning in line just above this one, using both the observation about values of x and the fact that she now knows what A is.

(e) Now she knows both A and C, and substitutes another value for x to obtain B.

(f) Here, she puts it all together.

2. Find the area bounded by the rays $\theta = 0$, $\theta = \frac{\pi}{2}$, and the polar curve $r = e^{2\theta}$.

Solution: The area, A, bounded by the polar curve $r = f(\theta)$, $\alpha \le \theta \le \beta$ is given by

$$A = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2 \, d\theta = \frac{1}{2}\int_0^{\pi/2} e^{4\theta}\, d\theta = \frac{1}{8}e^{4\theta}\Big|_0^{\pi/2} = \frac{1}{8}\left(e^{2\pi} - 1\right). \quad (2)$$

3. Give a substitution that is likely to transform each of the following integrals into an integral that can be calculated with reasonable ease. *In this problem, you need not actually carry out any substitutions, and you need not calculate any of the integrals.*

(a)

$$\int \frac{x\, dx}{(4 - x^2)^{3/2}}.$$

(b)

$$\int \frac{x^2 - 2}{\sqrt{4x^2 - 9}}\, dx$$

(c)

$$\int x\sqrt{25x^2 + 16}\, dx$$

Solution:

(a) Try the substitution $u = 4 - x^2$; $du = -2x\, dx$.

(b) Try the substitution $x = \frac{3}{2}\sec\theta$; $dx = \frac{3}{2}\tan\theta\sec\theta\, d\theta$. Alternately, try the substitution $x = \frac{3}{2}\tanh t$; $dx = \frac{3}{2}\,\text{sech}^2\, t\, dt$.

(c) Try the substitution $x = \frac{4}{5}\tan\theta$; $dx = \frac{4}{5}\sec^2\theta\, d\theta$.

4. (a) Find the polar equation for the line whose rectangular equation is $y = 2x + 3$. Give your equation in the form $r = f(\theta)$ for suitable f.

(b) Transform the polar equation $r = 4\sin\theta$ into rectangular coordinates and identify the curve.

Solution:

(a) The coordinate transormations are $x = r\cos\theta$; $y = r\sin\theta$. Substituting these for x and y in the equation $y = 2x + 3$ gives

$$r\sin\theta = 2r\cos\theta + 3; \text{ or} \quad (3)$$

$$r = \frac{3}{\sin\theta - 2\cos\theta}. \quad (4)$$

(b) If $r = 4\sin\theta$, then multiplying by r introduces no new points (because $r = 0$ already lies on the curve), so that, making use of the identity $r^2 = x^2 + y^2$ we have

$$r^2 = 4r\sin\theta; \tag{5}$$
$$x^2 + y^2 = 4y; \tag{6}$$
$$x^2 + y^2 - 4y + 4 = 4; \tag{7}$$
$$x^2 + (y-2)^2 = 2^2. \tag{8}$$

In rectangular coordinates, the curve can be written as $x^2 + (y-2)^2 = 2^2$, and this shows that the curve is a circle of radius 2 centered at the point with (rectangular) coordinates $(0, 2)$.

5. Use the indicated substitutions to transform the following integrals (simplify the transformed integrands). *You do not need to evaluate the integrals of this problem.*

(a)
$$\int \frac{\sqrt{x^2-1}}{x^2}\,dx;\ \text{substitute } x = \sec\theta.$$

(b)
$$\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}};\ \text{substitute } v = x^{1/6}.$$

Solution:

(a) If $x = \sec\theta$, then $dx = \sec\theta\tan\theta\,d\theta$ so

$$\int \frac{\sqrt{x^2-1}}{x^2}\,dx = \int \frac{\sqrt{\sec^2\theta - 1}}{\sec^{\not 2}\theta}\,\not{\sec\theta}\tan\theta\,d\theta = \int \tan^2\theta\cos\theta\,d\theta. \tag{9}$$

For the curious, here is the complete integration:

$$\int \tan^2\theta\cos\theta\,d\theta = \int (\sec^2\theta - 1)\cos\theta\,d\theta \tag{10}$$
$$= \int (\sec\theta - \cos\theta)\,d\theta \tag{11}$$
$$= \ln|\sec\theta + \tan\theta| - \sin\theta + C \tag{12}$$
$$= \ln\left|x + \sqrt{x^2-1}\right| - \frac{\sqrt{x^2-1}}{x} + C. \tag{13}$$

(b) If $v = x^{1/6}$, then $x = v^6$, so that $dx = 6v^5\,dv$. Thus

$$\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} = \int \frac{6v^5\,dv}{v^3 + v^2} \tag{14}$$

For those who are still curious, this proceeds as follows;

$$\int \frac{6v^5\,dv}{v^3+v^2} = 6\int \frac{v^5\,dv}{v^2(v+1)} = 6\int \frac{v^3\,dv}{v+1} \tag{15}$$
$$= 6\int \frac{(v^3+1)-1}{v+1}dv \tag{16}$$
$$= 6\int \frac{\cancel{(v+1)}(v^2-v+1)}{\cancel{(v+1)}}dv - 6\int \frac{dv}{v+1} \tag{17}$$
$$= 2v^3 - 3v^2 + 6v - 6\ln|v+1| + c \tag{18}$$
$$= 2x^{1/2} - 3x^{1/3} + 6x^{1/6} - 6\ln|x^{1/6}+1| + c. \tag{19}$$

6. Decide whether or not the following improper integrals converge; give the values of those that do.

(a)

$$\int_0^2 \frac{x}{x^2-1}\,dx$$

(b)

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+4}$$

Solution:

(a) We must evaluate the improper integral $\int_0^1 \frac{x\,dx}{x^2-1}$, and, if that one converges, the other improper integral $\int_1^2 \frac{x\,dx}{x^2-1}$, If either diverges, so does the original. If both converge, the original is their sum. Now

$$\int_0^1 \frac{x\,dx}{x^2-1} = \lim_{s\to 1^-}\int_0^s \frac{x\,dx}{x^2-1} \tag{20}$$
$$= \frac{1}{2}\lim_{s\to 1^-} \ln|x^2-1|\Big|_0^s = \frac{1}{2}\lim_{s\to 1^-} \ln|s^2-1|, \tag{21}$$

which does not exist. We conclude that $\int_0^1 \frac{x\,dx}{x^2-1}$ diverges.

(b) Here, we must evaluate the two improper integrals $\int_{-\infty}^0 \frac{dx}{x^2+4}$ and $\int_0^{\infty} \frac{dx}{x^2+4}$. We have

$$\int_{-\infty}^0 \frac{dx}{x^2+4} = \lim_{T\to-\infty}\int_T^0 \frac{dx}{x^2+4} \tag{22}$$
$$= \frac{1}{2}\arctan 0 - \frac{1}{2}\lim_{T\to-\infty}\arctan\frac{T}{2} = \frac{\pi}{4}. \tag{23}$$

while

$$\int_0^\infty \frac{dx}{x^2+4} = \lim_{T\to\infty} \int_0^T \frac{dx}{x^2+4} \tag{24}$$

$$= \frac{1}{2}\lim_{T\to\infty} \arctan\frac{T}{2} - \frac{1}{2}\arctan 0 = \frac{\pi}{4}. \tag{25}$$

We conclude that the improper integral $\int_{-\infty}^{\infty} \frac{dx}{x^2+4}$ converges to the value $\frac{\pi}{2}$.

7. Let $f(x) = \frac{1}{\sqrt[3]{1+x^2}}$.

(a) Calculate the Simpson's Rule approximation for $\int_1^3 f(x)\, dx$ using two subdivisions. Give your answer correct to the third digit to the right of the decimal in your answer.

(b) Unpleasant calculation (or Mathematica) shows that

$$f^{(4)}(x) = \frac{16(55x^4 - 198x^2 + 27)}{81(x^2+1)^{13/3}}.$$

It can be shown from this that if $1 \le x \le 3$, then $|f^{(4)}(x)| \le \frac{3}{2}$. How much error could there be in the approximation you gave in the previous part of this problem?

(c) How many subdivisions will be needed in Simpson's Rule to estimate $\int_1^3 f(x)\, dx$ to within 10^{-6}?

Solution:

(a) The 2-subdivision Simpson's rule approximation for $\int_1^3 f(x)\, dx$ is

$$\frac{1}{3}[f(1) + 4f(2) + f(3)] = \frac{1}{3}\left[\frac{1}{\sqrt[3]{1+1^2}} + \frac{4}{\sqrt[3]{1+2^2}} + \frac{1}{\sqrt[3]{1+3^2}}\right] \tag{26}$$

$$= \frac{1}{3}\left[\frac{1}{\sqrt[3]{2}} + \frac{4}{\sqrt[3]{5}} + \frac{1}{\sqrt[3]{10}}\right] \sim 1.199. \tag{27}$$

(b) The error in an n-subdivision Simpson's rule approximation to $\int_a^b g(t)\, dt$ is at most $\frac{M(b-a)^5}{180n^4}$, where M is any number that exceeds $|g^{(4)}(x)|$ at every point x of $[a, b]$. Consequently, the error in the approximation we have just calculated is at most

$$\frac{3}{2} \cdot \frac{(3-1)^5}{180 \cdot 2^4} = \frac{1}{60} \sim 0.0167. \tag{28}$$

(c) If the error in an n-subdivision Simpson's rule approximation to this integral is to have error that we can be sure doesn't exceed 10^{-6}, then n must satisfy

$$\frac{3}{2}\frac{(3-1)^5}{180 \cdot n^4} \leq \frac{1}{1000000}, \text{ or} \tag{29}$$

$$n^4 \geq \frac{800000}{3} \sim 266,667. \tag{30}$$

Now $22^4 = 234,256$, and $23^4 = 279,841$, so the smallest *even* integer[1] that meets the requirement is $n = 24$.

[1] Simpson's rule requires an even number of subdivisions.

MTH 2410, Calculus II—Exam I **Feb. 21, 2002**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Your work is due at 1:50 pm. You may keep this copy of the exam.

1. Find the limits:

   (a) $$\lim_{x \to 0^+} x \ln x$$

   (b) $$\lim_{x \to 0} \frac{x - \sin x}{x^3}$$

2. Show how to use Newton's Method, together with your calculator, to approximate the root of the equation $x^4 + x - 4 = 0$ that lies in the interval $[1, 2]$. Use $x_0 = 1$ as your initial guess and give both x_1 and x_2 as fractions whose numerators and denominators are whole numbers.

3. Use an appropriate substitution to evaluate the definite integral $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$.

4. Use integration by parts to find $\displaystyle\int x \cos(3x)\, dx$.

5. Find

   (a) $$\int_0^2 x^2(1 + 2x^3)^3\, dx$$

   (b) $$\int_0^1 \tan^{-1} x\, dx$$

6. Find $\displaystyle\int \frac{8x - 1}{2x^2 + x - 1}\, dx$.

7. Use Simpson's Rule with four subdivisions to find an approximate value of $\ln 2 = \displaystyle\int_1^2 \frac{dx}{x}$. Show what calculations you must perform and give your answer correct to five digits to the right of the decimal point. How does your answer compare to the value your calculator gives for $\ln 2$?

8. Is the integral $\displaystyle\int_0^\infty x e^{-x^2}\, dx$ convergent or divergent? If it is convergent, find its value.

9. Find the area of the region enclosed by the curves $y = 6x^2 - 64x + 140$ and $y = 2x - 4$.

10. Find $\displaystyle\int \frac{2x + 5}{x^2 + 4x + 8}\, dx$.

MTH 2410, Calculus II—Exam I Solutions **Feb. 21, 2002**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Your work is due at 1:50 pm. You may keep this copy of the exam.

1. Find the limits:

   (a)
   $$\lim_{x\to 0^+} x\ln x$$

   Solution: First note that
   $$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)},$$
   and that both numerator and denominator of the latter become infinite as $x \to 0^+$, so that we may apply L'Hôpital's Rule to it. Doing so, we obtain
   $$\begin{aligned} \lim_{x\to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} &= \lim_{x\to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} \\ &= \lim_{x\to 0^+} -x \\ &= 0. \end{aligned}$$

   (b)
   $$\lim_{x\to 0} \frac{x-\sin x}{x^3}$$

   Solution: The limit in the numerator and the denominator are both zero so we may apply L'Hôpital's Rule.
   $$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \lim_{x\to 0} \frac{1-\cos x}{3x^2}.$$
   L'Hôpital's Rule applies to this problem, too, because both numerator and denominator go to zero as x goes to zero. We get
   $$\lim_{x\to 0} \frac{1-\cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x}.$$
   Using L'Hôpital's Rule, which is applicable yet again, we get
   $$\begin{aligned} \lim_{x\to 0} \frac{\sin x}{6x} &= \lim_{x\to 0} \frac{\cos x}{6} \\ &= \frac{1}{6}. \end{aligned}$$

2. Show how to use Newton's Method, together with your calculator, to approximate the root of the equation $x^4 + x - 4 = 0$ that lies in the interval $[1, 2]$. Use $x_0 = 1$ as your initial guess and give both x_1 and x_2 as fractions whose numerators and denominators are whole numbers.

Solution: In order to find approximate solutions for an equation $f(x) = 0$ by Newton's Method, we begin with an initial guess x_0 for the solution and then refine the guess iteratively using the relation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

In this case, $f(x) = x^4 + x - 4$, so $f'(x) = 4x^3 + 1$. Therefore,

$$\begin{aligned} x - \frac{f(x)}{f'(x)} &= x - \frac{x^4 + x - 4}{4x^3 + 1} \\ &= \frac{4x^4 + x - x^4 - x + 4}{4x^3 + 1} \\ &= \frac{3x^4 + 4}{4x^3 + 1}. \end{aligned}$$

Taking $x_0 = 1$, we obtain

$$\begin{aligned} x_1 &= \frac{3(1)^4 + 4}{4(1)^3 + 1} = \frac{7}{5} \\ x_2 &= \frac{3\left(\frac{7}{5}\right)^4 + 4}{4\left(\frac{7}{5}\right)^3 + 1} \\ &= \frac{3(7)^4 + 4(5)^4}{4 \cdot 5(7)^3 + (5^4)} = \frac{9703}{7485}. \end{aligned}$$

3. Use an appropriate substitution to evaluate the definite integral $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$.

Solution: We let $u = \ln x$. Then $du = \dfrac{dx}{x}$. Also, if $x = e$, then $u = 1$, while if $x = e^4$, then $u = 4$. Thus,

$$\begin{aligned} \int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} &= \int_1^4 \frac{du}{\sqrt{u}} \\ &= \int_1^4 u^{-1/2}\, du \\ &= 2u^{1/2}\Big|_1^4 \\ &= 2\sqrt{4} - 2\sqrt{1} = 2. \end{aligned}$$

4. Use integration by parts to find $\int x\cos(3x)\,dx$.

Solution: We take $u = x$; $dv = \cos(3x)\,dx$. Then $du = dx$ and we may take $v = \frac{1}{3}\sin(3x)$. Therefore

$$\begin{aligned}\int x\cos(3x)\,dx &= x \cdot \left[\frac{1}{3}\sin(3x)\right] - \frac{1}{3}\int \sin(3x)\,dx \\ &= \frac{x}{3}\sin(3x) + \frac{1}{9}\cos(3x) + c.\end{aligned}$$

5. Find

(a)

$$\int_0^2 x^2(1+2x^3)^3\,dx$$

Solution: Let $u = 1 + 2x^3$. Then $du = 6x^2\,dx$. When $x = 0$, $u = 1$; and when $x = 2$, $u = 17$. Hence

$$\begin{aligned}\int_0^2 x^2(1+2x^3)^3\,dx &= \frac{1}{6}\int_1^{17} u^3\,du \\ &= \left.\frac{u^4}{24}\right|_1^{17} \\ &= 3480.\end{aligned}$$

(b)

$$\int_0^1 \tan^{-1} x\,dx$$

Solution: Take $u = \tan^{-1} x$ and $dv = dx$. Then $du = \dfrac{dx}{1+x^2}$, and we may take $v = x$. Thus

$$\int_0^1 \tan^{-1} x\,dx = x\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x\,dx}{1+x^2} \tag{1}$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln(1+x^2)\Big|_0^1 = \frac{\pi}{4} - \ln\sqrt{2}. \tag{2}$$

6. Find $\int \dfrac{8x-1}{2x^2+x-1}\,dx$.

Solution: First note that $2x^2 + x - 1 = (2x-1)(x+1)$. We will find A and B so that

$$\begin{aligned} \frac{8x-1}{(2x-1)(x+1)} &= \frac{A}{2x-1} + \frac{B}{x+1} \\ &= \frac{Ax + A + 2Bx - B}{(2x-1)(x+1)} \\ &= \frac{(A+2B)x + (A-B)}{(2x-1)(x+1)}. \end{aligned}$$

If this is to be so for all x, we must have $A + 2B = 8$, while $A - B = -1$. Subtracting the latter equation from the former yields $3B = 9$, whence $B = 3$. Then $A - 3 = -1$, so that $A = 2$. Hence

$$\begin{aligned} \int \frac{8x-1}{2x^2+x-1}\,dx &= \int \left[\frac{2}{2x-1} + \frac{3}{x+1}\right] dx \\ &= \ln|2x-1| + 3\ln|x+1| + c, \end{aligned}$$

Where we have used the substitution $u = 2x - 1$, $du = 2\,dx$, to find $\int \frac{2}{2x-1}\,dx$ and the substitution $v = x + 1$, $dv = dx$, to find $\int \frac{3}{x+1}\,dx$.

7. Use Simpson's Rule with four subdivisions to find an approximate value of $\ln 2 = \displaystyle\int_1^2 \frac{dx}{x}$. Show what calculations you must perform and give your answer correct to five digits to the right of the decimal point. How does your answer compare to the value your calculator gives for $\ln 2$?

   Solution: The Simpson's Rule Approximation S_4 for $\int_1^2 f(x)\,dx$ with four subdivisions is

$$\begin{aligned} S_4 &= \frac{1}{3}\left((f[1] + 4f\left[\frac{5}{4}\right] + f\left[\frac{3}{2}\right]\right)\cdot\frac{1}{4} + \frac{1}{3}\left((f\left[\frac{3}{2}\right] + 4f\left[\frac{7}{4}\right] + f\,[2]\right)\cdot\frac{1}{4} \\ &= \frac{1}{12}\left[\left(1 + 4\cdot\frac{4}{5} + \frac{2}{3}\right) + \left(\frac{2}{3} + 4\cdot\frac{4}{7} + \frac{1}{2}\right)\right] \\ &= \frac{1}{12}\cdot\frac{1747}{210} \\ &= \frac{1747}{2520} = 0.69325 \end{aligned}$$

   The calculator value of $\ln 2$ is 0.6931471806.

8. Is the integral $\displaystyle\int_0^\infty xe^{-x^2}\,dx$ convergent or divergent? If it is convergent, find its value.

   Solution: The integral converges if $\lim_{T\to\infty}\int_0^T xe^{-x^2}\,dx$ exists. Substituting $u = -x^2$; $du = -2x\,dx$, we find that $\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u$, so that $\int xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2}$. Thus,

$$\begin{aligned}\lim_{T\to\infty}\int_0^T xe^{-x^2}\,dx &= \lim_{T\to\infty} -\frac{1}{2}e^{-T^2}\Big|_0^T \\ &= -\frac{1}{2}\lim_{T\to\infty}\left(e^{-T^2}-1\right) \\ &= \frac{1}{2}.\end{aligned}$$

The integral converges; its value is $\frac{1}{2}$.

9. Find the area of the region enclosed by the curves $y = 6x^2 - 64x + 140$ and $y = 2x - 4$.

Solution: We first find the intersection points by solving $6x^2 - 64x + 140 = 2x - 4$. This is equivalent to

$$\begin{aligned}6x^2 - 66x + 144 &= 0, \text{ or} \\ x^2 - 11x + 24 &= 0.\end{aligned}$$

Factoring, we find that $(x-3)(x-8) = 0$, so that the solutions are $x = 3$ and $x = 8$. The curve $y = 6x^2 - 64x + 140$ is a parabola opening upward, while the curve $y = 2x - 4$ is a straight line. The area they enclose between $x = 3$ and $x = 8$ is therefore above the parabola and below the line. We want

$$\begin{aligned}\int_3^8 [(2x-4) - (6x^2 - 64x + 140)]\,dx &= -\int_3^8 (6x^2 - 66x + 144)\,dx \\ &= (-2x^3 + 33x^2 - 144x)\Big|_3^8 \\ &= -64 - (-189) \;=\; 125\end{aligned}$$

10. Find $\displaystyle\int \frac{2x+5}{x^2+4x+8}\,dx$.

Solution: If we take $u = x^2 + 4x + 8$, then $du = (2x+4)\,dx$, so

$$\begin{aligned}\int \frac{2x+5}{x^2+4x+8}\,dx &= \int \frac{2x+4}{x^2+4x+8}\,dx + \int \frac{1}{x^2+4x+8}\,dx \\ &= \int \frac{du}{u} + \int \frac{1}{x^2+4x+8}\,dx \\ &= \ln|u| + \int \frac{1}{x^2+4x+8}\,dx \\ &= \ln|x^2+4x+8| + \int \frac{1}{x^2+4x+8}\,dx.\end{aligned}$$

It remains to find $\int \frac{1}{x^2+4x+8}\,dx$. We have:

$$\begin{aligned}\int \frac{1}{x^2+4x+8}\,dx &= \int \frac{1}{(x^2+4x+4)+4}\,dx \\ &= \int \frac{1}{(x+2)^2+4}\,dx \\ &= \frac{1}{4}\int \frac{dx}{\left(\frac{x+2}{2}\right)^2+1}.\end{aligned}$$

Put $v = \frac{x+2}{2}$. Then $dv = \frac{1}{2}\,dx$, so

$$\begin{aligned}\frac{1}{4}\int \frac{dx}{\left(\frac{x+2}{2}\right)^2+1} &= \frac{1}{2}\int \frac{du}{u^2+1} \\ &= \frac{1}{2}\tan^{-1} u \\ &= \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right)+c.\end{aligned}$$

Thus,

$$\int \frac{2x+5}{x^2+4x+8}\,dx = \ln|x^2+4x+8| + \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right)+c.$$

MTH 2410, Calculus II—Exam II **Apr. 16, 2002**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Your work is due at 1:50 pm. You may keep this copy of the exam.

1. Find the volume generated when the region bounded by the curve $y = \sqrt{x}$, the x-axis, and the lines $x = 4$, $x = 9$ is rotated about the x-axis.

2. A function $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = y^4 - 6y^3 + 5y^2.$$

   (a) What are the constant solutions of the equation?

   (b) For what values of y is y increasing? Why?

   (c) For what values of y is y decreasing? Why?

3. The base of a solid is the triangle cut from the first quadrant by the line $4x+3y = 12$. Every cross-section of the solid perpendicular to the y-axis is a semi-circle. Find the volume of the solid.

4. Find the volume generated when the region bounded by the curve $y = e^{x^2}$, the x-axis, and the lines $x = 1$ and $x = \sqrt{2}$ is rotated about the y-axis.

5. A colony of bacteria contained 600 organisms when $t = 0$. When $t = 24$ hr, there were $38,400$ organisms in the colony. Give an expression for $N(t)$ for the number of organisms in the colony at time t under the assumption that there are no external restraints on its growth. How many organisms were there in the colony at time $t = 10$ hr?

6. Find the centroid of the region in the first quadrant cut off by the line

$$\frac{x}{a} + \frac{y}{b} = 1,$$

   if a and b are both positive constants.

7. Find $y(x)$ given that $e^{-x}y' = \sqrt{1-y^2}$ with $y(0) = 0$. What is the exact value of $y(x)$ when $x = \frac{\pi}{4} + 1$?

8. Use Euler's Method with a step-size of 1/10 to find an approximate value for $y(13/10)$ if $y(1) = 1$ and

$$\frac{dy}{dt} = \frac{1}{y+t}.$$

   Show all of the intermediate y values you obtain. Either give your answers as fractions of integers or maintain a minimum of five digits accuracy to the right of the decimal point.

MTH 2410, Calculus II—Exam II Solutions **Apr. 16, 2002**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Your work is due at 1:50 pm. You may keep this copy of the exam.

1. Find the volume generated when the region bounded by the curve $y = \sqrt{x}$, the x-axis, and the lines $x = 4$, $x = 9$ is rotated about the x-axis.

   Solution: A typical cross-section of this solid perpendicular to the x-axis for $4 \leq x \leq 9$ is a disk of radius $\sqrt{x}$ and having area $A(x) = \pi\left(\sqrt{x}\right)^2 = \pi x$. Hence the required volume is $\int_4^9 A(x)\,dx = \pi\int_4^9 x\,dx = \left(\frac{\pi}{2}\right)x^2\Big|_4^9 = \frac{65\pi}{2}$.

2. A function $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = y^4 - 6y^3 + 5y^2.$$

   (a) What are the constant solutions of the equation?
   (b) For what values of y is y increasing? Why?
   (c) For what values of y is y decreasing? Why?

   Solution:

   (a) We have $y^4 - 6y^3 + 5y^2 = y^2(y-1)(y-5)$, which is zero only when $y = 0$, $y = 1$, or $y = 5$. Consequently the constant solutions for the differential equation are the constant functions $y(x) \equiv 0$, $y(x) \equiv 1$, and $y(x) \equiv 5$.
   (b) If $y(x)$ is to be increasing, then we must have $y'(x) > 0$. But $y'(x)$ can be positive only if there are an even number of negative factors in the product $y^2(y-1)(y-5)$. This can be so only if $y < 0$, if $0 < y < 1$, or if $5 < y$. Thus, y is increasing if $y < 0$, if $0 < y < 1$, or if $5 < y$.
   (c) If $y(x)$ is to be decreasing, then we must have $y'(x) < 0$. But $y'(x)$ can be negative only if there are an odd number of negative factors in the product $y^2(y-1)(y-5)$. This can be so only if $1 < y < 5$. Thus, y is decreasing if $1 < y < 5$.

3. The base of a solid is the triangle cut from the first quadrant by the line $4x+3y = 12$. Every cross-section of the solid perpendicular to the y-axis is a semi-circle. Find the volume of the solid.

   Solution: The equation $4x+3y = 12$ is eqivalent to $x = 3-3y/4$. The semi-circle with base at distance y from the x-axis therefore has base of length $3 - 3y/4$, and so has area $A(x) = (\pi/2)[(3-3y/4)/2]^2$. The required volume is therefore

$$\begin{aligned}\int_0^4 A(y)\,dy &= \frac{9\pi}{8}\int_0^4\left(1-\frac{y}{4}\right)^2 dy \\ &= \frac{3\pi}{2}\left(1-\frac{y}{4}\right)^3\Bigg|_0^4 = \frac{3\pi}{2}.\end{aligned}$$

4. Find the volume generated when the region bounded by the curve $y = e^{x^2}$, the x-axis, and the lines $x = 1$ and $x = \sqrt{2}$ is rotated about the y-axis.

Solution: Using the method of shells, we find that the required volume is given by

$$\begin{aligned} 2\pi \int_1^{\sqrt{2}} x e^{x^2}\, dx &= \pi e^{x^2} \Big|_1^{\sqrt{2}} \\ &= \pi(e^2 - e). \end{aligned}$$

5. A colony of bacteria contained 600 organisms when $t = 0$. When $t = 24$ hr, there were $38,400$ organisms in the colony. Give an expression for $N(t)$ for the number of organisms in the colony at time t under the assumption that there are no external restraints on its growth. How many organisms were there in the colony at time $t = 10$ hr?

Solution: We know that under conditions of unrestrained growth, the size N of a bacterial population is give by $N(t) = N_0 e^{kt}$, so in this case we must have $N(t) = 600e^{kt}$ for a certain constant k. Because $N(24) = 38400$, we must have $38400 = 600e^{24k}$, or $e^{24k} = 64$. Thus, $k = (\ln 64)/24$, and

$$\begin{aligned} N(t) &= 600e^{t(\ln 64)/24} \\ &= 600 \cdot 64^{t/24}. \end{aligned}$$

This means that $N(10) = 600 \cdot 64^{5/12}$, which is approximately 3400 organisms.

6. Find the centroid of the region in the first quadrant cut off by the line

$$\frac{x}{a} + \frac{y}{b} = 1,$$

if a and b are both positive constants.

Solution: We begin by noting that the region in question is a right triangle whose base extends from $x = 0$ to $x = a$ on the x-axis, and whose vertical leg extends from the origin to $y = b$ along the y-axis. Solving the equation $(x/a) + (y/b) = 1$ for y, we obtain $y = b - (b/a)x$. Thus

$$\begin{aligned} \overline{x} &= \frac{\int_0^a x\left(b - \frac{b}{a}x\right) dx}{\int_0^a \left(b - \frac{b}{a}x\right) dx} \\ &= \frac{\int_0^a \left(bx - \frac{b}{a}x^2\right) dx}{\int_0^a \left(b - \frac{b}{a}x\right) dx} \end{aligned}$$

$$= \frac{\left(\frac{bx^2}{2} - \frac{bx^3}{3a}\right)\Big|_0^a}{\left(bx - \frac{bx^2}{2a}\right)\Big|_0^a}$$

$$= \frac{\left(\frac{a^2b}{6}\right)}{\left(\frac{ab}{2}\right)} = \frac{a}{3}.$$

Solving for x, on the other hand, yields $x = a - (a/b)y$, so that

$$\overline{y} = \frac{\int_0^b y\left(a - \frac{a}{b}y\right)\, dy}{\left(\frac{ab}{2}\right)}$$

$$= \frac{\left(\frac{ab^2}{6}\right)}{\left(\frac{ab}{2}\right)} = \frac{b}{3}.$$

The coordinates of the centroid are therefore $\left(\frac{a}{3}, \frac{b}{3}\right)$.

7. Find $y(x)$ given that $e^{-x}y' = \sqrt{1 - y^2}$ with $y(0) = 0$. What is the exact value of $y(x)$ when $x = \frac{\pi}{4} + 1$?

Solution: The given differential equation can be rewritten as

$$\frac{dy}{\sqrt{1 - y^2}} = e^x \, dx,$$

and, integrating, we obtain $\arcsin y = e^x + c$, or $y = \sin(e^x + c)$. From the fact that $y = 0$ when $x = 0$, we now obtain $0 = y(0) = \sin(e^0 + c) = \sin(1 + c)$, so that $c = -1$. Thus, $y(x) = \sin(e^x - 1)$. From this, we obtain $y = \sin\left[e^{1+\pi/4} - 1\right]$ when $x = \frac{\pi}{4} + 1$

8. Use Euler's Method with a step-size of $1/10$ to find an approximate value for $y(13/10)$ if $y(1) = 1$ and

$$\frac{dy}{dt} = \frac{1}{y + t}.$$

Show all of the intermediate y values you obtain. Either give your answers as fractions of integers or maintain a minimum of five digits accuracy to the right of the decimal point.

Solution: Putting $t_0 = 1$, $y_0 = 1$, and using the relations

$$\begin{aligned} t_k &= t_{k-1} + \frac{1}{10} \\ y_k &= y_{k-1} + \frac{1}{y_{k-1} + t_{k-1}} \cdot \frac{1}{10} \end{aligned}$$

when $k = 1, 2, 3$, we obtain:

k	t_k	y_k
0	1.00000	1.00000
1	$\frac{11}{10} = 1.10000$	$\frac{21}{20} = 1.05000$
2	$\frac{6}{5} = 1.20000$	$\frac{943}{860} \sim 1.09651$
3	$\frac{13}{10} = 1.30000$	$\frac{387277}{339700} \sim 1.14006$

MTH 2410, Calculus II—Final Exam **May 14, 2002**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Your work is due at 1:50 pm. You may keep this copy of the exam.

1. Let $\{a_n\}$ be the sequence given by $a_n = \dfrac{\ln(1+e^{n/17})}{3n}$. Determine whether $\{a_n\}$ converges or diverges. If it converges, find the limit.

2. Evaluate:

   (a) $\displaystyle\int_0^3 \frac{dx}{\sqrt{x}}$

   (b) $\displaystyle\int_0^3 \frac{dx}{x\sqrt{x}}$

3. Determine whether the series
$$\sum_{n=1}^{\infty} \frac{5}{2+3^n}$$
is convergent or divergent. Explain your conclusion.

4. Show how to use Newton's Method, together with your calculator, to approximate the root of the equation $x^4 + 8x - 8 = 0$ that lies in the interval $[0, 1]$. Use $x_0 = 1$ as your initial guess and give both x_1 and x_2 as fractions whose numerators and denominators are whole numbers.

5. Find the radius of convergence for the series $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{(x+2)^k}{k\,2^k}$. Give the interior of the interval of convergence. Justify your conclusions.

6. The base of a solid is the triangle cut from the first quadrant by the line $4x+3y = 12$. Every cross-section of the solid perpendicular to the y-axis is a square. Find the volume of the solid.

7. Find $y(x)$ given that $y' = y(1-x)$ with $y(0) = 2$. What is the exact value of $y(x)$ when $x = 1$?

8. Show how to determine the general term for the Taylor expansion, centered at $x = 0$, for the function $f(x) = \text{Arctan}(3x)$. What is the radius of convergence for this series? Give the interior of the interval of convergence.

MTH 2410, Calculus II—Final Exam **May 14, 2002**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Your work is due at 1:50 pm. You may keep this copy of the exam.

1. Let $\{a_n\}$ be the sequence given by $a_n = \dfrac{\ln(1+e^{n/17})}{3n}$. Determine whether $\{a_n\}$ converges or diverges. If it converges, find the limit.

   Solution: Note that both numerator and denominator approach infinity as $n \to \infty$. We may apply L'Hôpital's Rule:

$$\begin{aligned} \lim_{n\to\infty} \frac{\ln(1+e^{n/17})}{3n} &= \lim_{n\to\infty} \frac{\left(\frac{1}{17}\frac{e^{n/17}}{1+e^{n/17}}\right)}{3} \\ &= \lim_{n\to\infty} \frac{1}{51(e^{-n/17}+1)} \\ &= \frac{1}{51}. \end{aligned}$$

2. Evaluate:

   (a) $\displaystyle\int_0^3 \frac{dx}{\sqrt{x}}$

   (b) $\displaystyle\int_0^3 \frac{dx}{x\sqrt{x}}$

   Solution: Both integrals are improper. For (a) have

$$\begin{aligned} \int_0^3 \frac{dx}{\sqrt{x}} &= \lim_{T\to 0^+} \int_T^3 x^{-1/2}dx \\ &= \lim_{T\to 0^+} 2\sqrt{3} - 2\sqrt{T} \; = \; 2\sqrt{3}. \end{aligned}$$

   For part (b), we have

$$\begin{aligned} \int_0^3 \frac{dx}{x\sqrt{x}} &= \lim_{T\to 0^+} \int_T^3 x^{-3/2}dx \\ &= \lim_{t\to 0^+} -\frac{2}{\sqrt{3}} + \frac{2}{\sqrt{T}}, \end{aligned}$$

   and, because the latter limit does not exist, the integral diverges.

3. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{5}{2+3^n}$$

   is convergent or divergent. Explain your conclusion.

   Solution: We surely have $3^n < 2 + 3^n$ for every whole number n. Hence, $5/(2+3^n) < 5/3^n$ for every whole number n. Now $\sum_{n=1}^{\infty} 5/3^n$ is a convergent geometric series. Consequently, $\sum_{n=1}^{\infty} 5/(2+3^n)$ converges by the Ratio Test.

4. Show how to use Newton's Method, together with your calculator, to approximate the root of the equation $x^4 + 8x - 8 = 0$ that lies in the interval $[0, 1]$. Use $x_0 = 1$ as your initial guess and give both x_1 and x_2 as fractions whose numerators and denominators are whole numbers.

Solution: We have, according to Newton's Method,

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^4 + 8x_n - 8}{4x_n^3 + 8} \\ &= \frac{3x_n^4 + 8}{4x_n^3 + 8}. \end{aligned}$$

Thus

$$x_1 = \frac{3 \cdot 1^4 + 8}{4 \cdot 1^3 + 8} = \frac{11}{12},$$

and

$$x_2 = \frac{3 \cdot \left(\frac{11}{12}\right)^4 + 8}{4\left(\frac{11}{12}\right)^3 + 8} = \frac{69937}{76592}.$$

5. Find the radius of convergence for the series $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{(x+2)^k}{k\,2^k}$. Give the interior of the interval of convergence. Justify your conclusions.

Solution: We let

$$g(x) = \sum_{k=1}^{\infty} (-1)^k \frac{(x+2)^k}{k2^k}$$

where the series converges. Then

$$g'(x) = \sum_{k=1}^{\infty} (-1)^k \frac{(x+2)^{k-1}}{2^k},$$

and the latter series has the same radius of convergence as the first. But then

$$\begin{aligned} g'(x) &= -\frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(x+2)^{k-1}}{2^{k-1}} \\ &= -\frac{1}{2} \sum_{k=1}^{\infty} \left[-\frac{(x+2)}{2} \right]^{k-1} \end{aligned}$$

is given by a geometric series with common ratio $-(x+2)/2$. The derived series therefore converges when $|-(x+2)/2| < 2$, or when $|x+2| < 2$. It follows that the radius of convergence for both series is 2 and that the common interior of their intervals of convergence is $(-4, 0)$.

6. The base of a solid is the triangle cut from the first quadrant by the line $4x+3y=12$. Every cross-section of the solid perpendicular to the y-axis is a square. Find the volume of the solid.

Solution: The cross-section at y is a square whose side is x, where $4x+3y=12$. This means that $x = 3-(3/4)y$. Hence,

$$\begin{aligned} A &= \int_0^4 x^2\,dy \\ &= \int_0^4 \left(3-\frac{3}{4}\right)^2\,dy \\ &= \frac{1}{3}\left(-\frac{4}{3}\right)\left(3-\frac{3}{4}y\right)^3\Big|_0^4 = 12. \end{aligned}$$

7. Find $y(x)$ given that $y' = y(1-x)$ with $y(0)=2$. What is the exact value of $y(x)$ when $x=1$?

Solution: From $y' = y(1-x)$, we obtain:

$$\begin{aligned} \frac{dy}{dx} &= y(1-x) \\ \frac{dy}{y} &= (1-x)\,dx \\ \ln|y| &= x-\frac{x^2}{2}+c. \end{aligned}$$

Hence,

$$y = Ce^{x-x^2/2}.$$

But $y(0)=2$, so $2 = Ce^{0-0^2/2} = C$. Hence, $y = 2e^{x-x^2/2}$, and $y(1) = 2\sqrt{e}$.

8. Show how to determine the general term for the Taylor expansion, centered at $x=0$, for the function $f(x) = \text{Arctan}(3x)$. What is the radius of convergence for this series? Give the interior of the interval of convergence.

Solution: Differentiating, we obtain

$$\begin{aligned} \frac{d}{dx}\arctan(3x) &= \frac{3}{1+9x^2} \\ &= 3\cdot\frac{1}{1+9x^2} \\ &= 3\frac{1}{1-(-9x^2)}. \end{aligned}$$

But the latter is the sum of a certain geometric series when $|-9x^2|<1$, or, equivalently, when $|x|<1/3$. Thus, for such x,

$$\frac{d}{dx}\arctan(3x) = 3(1-9x^2+81x^4-729x^6+\cdots)$$

$$
\begin{aligned}
&= 3\sum_{k=0}^{\infty}(-1)^k(9x^2)^k \\
&= 3\sum_{k=0}^{\infty}(-1)^k 3^{2k} x^{2k}
\end{aligned}
$$

It follows that when $|x| < 1/3$ there is a constant c for which we may write:

$$
\begin{aligned}
\arctan(3x) &= c + 3\sum_{k=0}^{\infty}(-1)^k 3^{2k}\frac{x^{2k+1}}{2k+1} \\
&= c + \sum_{k=0}^{\infty}(-1)^k\frac{3^{2k+1}x^{2k+1}}{2k+1}.
\end{aligned}
$$

Now we note that when $x = 0$, the left side is 0 while the right side is c, so that $c = 0$. Hence,

$$
\begin{aligned}
\arctan(3x) &= \sum_{k=0}^{\infty}(-1)^k\frac{3^{2k+1}}{2k+1}x^{2k+1} \\
&= 3x - \frac{27}{3}x^3 + \frac{243}{5}x^5 - \frac{2187}{7}x^7 + \cdots,
\end{aligned}
$$

and the series on the right has radius of convergence $1/3$. The interior of the interval of convergence is $(-1/3, 1/3)$.

MTH 2410, Calculus II—Exam I **Feb. 12, 2003**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Your work is due at 2:20 pm. You may keep this copy of the exam.

1. Find the limits:

   (a) $\displaystyle\lim_{x\to 0^+} x \ln x$

   (b) $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^3}$

2. Show how to use Newton's Method, together with your calculator, to approximate the root of the equation $x^4 + x - 4 = 0$ that lies in the interval $[1, 2]$. Use $x_0 = 1$ as your initial guess and give both x_1 and x_2 as fractions whose numerators and denominators are whole numbers.

3. Use an appropriate substitution to evaluate the definite integral $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$.

4. Use integration by parts to find $\displaystyle\int x \cos(3x)\, dx$.

5. Find

   (a) $\displaystyle\int_0^2 x^2(1 + 2x^3)^3\, dx$

   (b) $\displaystyle\int_0^1 \tan^{-1} x\, dx$

6. Find $\displaystyle\int \frac{8x - 1}{2x^2 + x - 1}\, dx$. [Hint: $2x^2 + x - 1 = (2x - 1)(x + 1)$.]

7. Use Simpson's Rule with four subdivisions to find an approximate value of $\ln 3 = \displaystyle\int_1^3 \frac{dx}{x}$. Show what calculations you must perform and give your answer correct to five digits to the right of the decimal point. How does your answer compare to the value your calculator gives for $\ln 3$?

8. Find $\displaystyle\int \frac{2x + 5}{x^2 + 4x + 8}\, dx$. [Hint: $2x + 5 = (2x + 4) + 1$.]

MTH 2410, Calculus II—Exam I **Feb. 12, 2003**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Your work is due at 2:20 pm. You may keep this copy of the exam.

1. Find the limits:

(a)
$$\lim_{x\to 0^+} x \ln x$$

(b)
$$\lim_{x\to 0} \frac{x - \sin x}{x^3}$$

Solution:

(a) We begin by noticing that

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)}, \tag{1}$$

and, because both numerator and denominator of the latter fraction become unbounded as $x \to 0^+$, we may attempt a l'Hôpital's Rule solution:

$$\lim_{x\to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} = \lim_{x\to 0^+} \frac{\left(\frac{1}{x}\right)}{-\left(\frac{1}{x^2}\right)} = -\lim_{x\to 0^+} \frac{x^{\not 2}}{\not x} = 0. \tag{2}$$

(b) Numerator and denominator both approach zero as x approaches zero, so we may attempt a l'Hôpital's Rule solution:

$$\lim_{x\to 0} \frac{x - \sin x}{x^3} = \lim_{x\to 0} \frac{1 - \cos x}{3x^2}, \tag{3}$$

provided the latter limit exists. But, again, numerator and denominator both approach zero, and we can make another attempt at l'Hôpital's Rule.

$$\lim_{x\to 0} \frac{x - \sin x}{x^3} = \lim_{x\to 0} \frac{1 - \cos x}{3x^2} \tag{4}$$

$$= \lim_{x\to 0} \frac{\sin x}{6x} = \frac{1}{6} \lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{6}, \tag{5}$$

because we know that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1. \tag{6}$$

2. Show how to use Newton's Method, together with your calculator, to approximate the root of the equation $x^4 + x - 4 = 0$ that lies in the interval $[1, 2]$. Use $x_0 = 1$ as your initial guess and give both x_1 and x_2 as fractions whose numerators and denominators are whole numbers.

Solution: We take $x_0 = 1$, $f(x) = x^4 + x - 1$, and we apply the Newton's Method iteration

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \tag{7}$$

$$= x_k - \frac{x_k^4 + x_k - 1}{4x_k^3 + 1}. \tag{8}$$

We find that

$$x_1 = x_0 - \frac{x_0^4 + x_0 - 1}{4x_0 + 1} = 1 - \frac{1}{5} = \frac{4}{5}; \tag{9}$$

$$x_2 = x_1 - \frac{x_1^4 + x_1 - 1}{4x_1 + 1} = \frac{4}{5} - \frac{131}{1905} = \frac{1393}{1905}. \tag{10}$$

3. Use an appropriate substitution to evaluate the definite integral $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$.

Solution: We let $u = \ln x$. Then $du = \dfrac{dx}{x}$, $u = 1$ when $x = e$, and $u = 4$ when $x = e^4$. Thus

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_1^4 \frac{du}{\sqrt{u}} = 2\sqrt{u}\Big|_1^4 = 2\sqrt{4} - 2\sqrt{1} = 2. \tag{11}$$

4. Use integration by parts to find $\displaystyle\int x\cos(3x)\,dx$.

Solution: We let $u = x$, $dv = \cos 3x\,dx$. This gives $du = dx$ and we may take $v = \frac{1}{3}\sin 3x$. Then $\int u\,dv = uv - \int v\,du$ becomes

$$\int x\cos(3x)\,dx = \frac{1}{3}x\sin 3x - \frac{1}{3}\int \sin 3x\,dx \tag{12}$$

$$= \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C. \tag{13}$$

5. Find

(a)

$$\int_0^2 x^2(1 + 2x^3)^3\,dx$$

(b)

$$\int_0^1 \tan^{-1} x\, dx$$

Solution:

(a) Let $u = 1 + 2x^3$. Then $du = 6x^2\, dx$—so that $x^2\, dx = \frac{1}{6}\, du$. Also, $u = 1$ when $x = 0$, and $u = 17$ when $x = 2$. Thus

$$\int_0^2 x^2(1+2x^3)^3\, dx = \frac{1}{6}\int_1^{17} u^3\, du \tag{14}$$

$$= \left.\frac{u^4}{24}\right|_1^{17} = \frac{17^4}{24} - \frac{1}{24} = 3480. \tag{15}$$

(b) Let $u = \tan^{-1} x$, $dv = dx$. Then $du = \dfrac{dx}{1+x^2}$, and we may take $v = x$. Integrating by parts, we obtain

$$\int_0^1 \tan^{-1} x\, dx = x \tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x\, dx}{1+x^2} \tag{16}$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln(1+x^2)\Big|_0^1 \tag{17}$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2. \tag{18}$$

6. Find $\displaystyle\int \frac{8x-1}{2x^2+x-1}\, dx$.

Solution: We must find constants A and B so that

$$\frac{8x-1}{2x^2+x-1} = \frac{8x-1}{(x+1)(2x-1)} = \frac{A}{x+1} + \frac{B}{2x-2} \tag{19}$$

$$= \frac{(A+2B)x + (-2A+B)}{(x+1)(2x-1)}. \tag{20}$$

For this to be so, we must have $A + 2B = 8$ and $-2A + B = -1$. Adding twice the first of these two equations to the second, we find that $5B = 15$, or $B = 3$. Substituting this latter value for B in either equation then gives $A = 2$. Thus

$$\int \frac{8x-1}{2x^2+x-1}\, dx = \int \left[\frac{8x-1}{(x+1)(2x-1)}\right] dx \tag{21}$$

$$= \int \left[\frac{3}{x+1} + \frac{2}{2x-1}\right] dx \tag{22}$$

$$= 3\ln|x+1| + \ln|2x-1| + C \tag{23}$$

7. Use Simpson's Rule with four subdivisions to find an approximate value of $\ln 2 = \int_1^2 \frac{dx}{x}$. Show what calculations you must perform and give your answer correct to five digits to the right of the decimal point. How does your answer compare to the value your calculator gives for $\ln 2$?

Solution: Using Simpson's Rule with four subdivisions, we put $x_k = 1 + \frac{k}{4}$, where $k = 0, 1, \ldots, 4$. Then

$$\int_1^2 \frac{dx}{x} \sim \frac{1}{3}\left[\frac{1}{x_0} + \frac{4}{x_1} + \frac{2}{x_2} + \frac{4}{x_3} + \frac{1}{x_4}\right] \cdot \frac{2-1}{4} \tag{24}$$

$$\sim \frac{1}{12}\left[1 + \frac{16}{5} + \frac{4}{3} + \frac{16}{7} + \frac{1}{2}\right] = \frac{1747}{2520} \sim 0.693254. \tag{25}$$

In fact, $\ln 2 \sim 0.6931472$, so the Simpson's Rule calculation is correct to three decimal places..

8. Is the integral $\int_0^\infty xe^{-x^2}\,dx$ convergent or divergent? If it is convergent, find its value.

Solution:

$$\int_0^\infty xe^{-x^2}\,dx = \lim_{T\to\infty} \int_0^T xe^{-x^2}\,dx \tag{26}$$

$$= -\frac{1}{2}\lim_{T\to\infty} e^{-x^2}\Big|_0^T \tag{27}$$

$$= -\frac{1}{2}\lim_{T\to\infty}\left[e^{-T^2} - 1\right] = \frac{1}{2} \tag{28}$$

9. Find the area of the region enclosed by the curves $y = 6x^2 - 64x + 140$ and $y = 2x - 4$.

Solution: The two curves intersect at the points where

$$6x^2 - 64x + 140 = 2x - 4; \tag{29}$$

$$6x^2 - 66x + 144 = 0; \tag{30}$$

$$x^2 - 11x + 24 = 0; \tag{31}$$

$$(x-8)(x-3) = 0, \tag{32}$$

or where $x = 3$ and where $x = 8$. The graph of the quadratic is a polynomial opening upward, while the other equation graphs as a straight line. Consequently, we need the area below the line $y = 2x - 4$, above the parabola $y = 6x^2 - 64x + 140$, and between the vertical lines $x = 3$ and $x = 8$. So we must compute

$$\int_3^8 (-6x^2 + 66x - 144)\,dx = -6\int_3^8 (x^2 - 11x + 24)\,dx \tag{33}$$

$$= -6\left(\frac{1}{3}x^3 - \frac{11}{2}x^2 + 24x\right)\Bigg|_3^8 \tag{34}$$

$$= -6\left(\frac{512}{3} - \frac{704}{2} + 192\right) + 6\left(9 - \frac{99}{2} + 72\right) \tag{35}$$

$$= 125. \tag{36}$$

10. Find $\displaystyle\int \frac{2x+5}{x^2+4x+8}\,dx$.

Solution:

$$\int \frac{2x+5}{x^2+4x+8}\,dx = \int \frac{(2x+4)+1}{x^2+4x+8}\,dx \tag{37}$$

$$= \int \frac{2x+4}{x^2+4x+8}\,dx + \int \frac{1}{(x+2)^2+4}\,dx \tag{38}$$

$$= \int \frac{d(x^2+4x+8)}{x^2+4x+8} + \int \frac{d(x+2)}{(x+2)^2+4} \tag{39}$$

$$= \ln(x^2+4x+8) + \frac{1}{2}\arctan\frac{x+2}{2} + C. \tag{40}$$

MTH 2410, Calculus II—Exam III **Apr. 16, 2003**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Your work is due at 2:20 pm. You may keep this copy of the exam.

1. Show that the function f given by

$$f(x) \quad = \quad Ce^{3x} - x - \frac{1}{3},$$

   where C is an arbitrary constant, is a solution of the differential equation $y' - 3y = 3x$. What value should C have if $y(0) = 1$?

2. A function $y = y(t)$ satisfies the differential equation

$$\frac{dy}{dt} \quad = \quad y^4 - 5y^2 + 4.$$

   (a) What are the constant solutions of the equation?

   (b) For what values of y is y decreasing? Why?

   (c) For what values of y is y increasing? Why?

3. A colony of bacteria contained 600 organisms when $t = 0$. When $t = 24$ hr, there were $38,400$ organisms in the colony. Derive an expression for $N(t)$, the number of organisms in the colony at time t, under the assumption of unconstrained growth. How many organisms were there in the colony at time $t = 10$ hr?

4. Find $y(x)$ given that $e^{-x}y' = \sqrt{1 - y^2}$ and $y(0) = 0$. What is the exact value of $y(x)$ when $x = \ln\left(1 + \frac{\pi}{4}\right)$?

5. Use Euler's Method with a step-size of 1/10 to find an approximate value for $y(13/10)$ if $y(1) = 1$ and

$$\frac{dy}{dt} = \frac{1}{t + y}.$$

   Show all of your intermediate calculations; either give your answers as fractions of integers or maintain an accuracy of no fewer than five digits to the right of the decimal.

6. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n}$ converges or diverges. Give all of your reasoning.

7. Show how to use the Integral Test to determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges or diverges.

8. For what values of x does the series

$$\sum_{k=1}^{\infty} \frac{1}{2^{k-1}} \left(\frac{x+1}{x-1}\right)^k \quad = \quad \left(\frac{x+1}{x-1}\right) + \frac{1}{2}\left(\frac{x+1}{x-1}\right)^2 + \frac{1}{2^2}\left(\frac{x+1}{x-1}\right)^3 + \frac{1}{2^3}\left(\frac{x+1}{x-1}\right)^4 + \cdots$$

   converge?

MTH 2410, Calculus II—Exam III **Apr. 16, 2003**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Your work is due at 2:20 pm. You may keep this copy of the exam.

1. Show that the function f given by

$$f(x) = Ce^{3x} - x - \frac{1}{3},$$

where C is an arbitrary constant, is a solution of the differential equation $y' - 3y = 3x$. What value should C have if $y(0) = 1$?

Solution: If f is as given, then

$$f'(x) = 3Ce^{3x} - 1,$$

so that

$$\begin{aligned} f'(x) - 3f'(x) &= (3Ce^{3x} - 1) - 3\left(Ce^{3x} - x - \frac{1}{3}\right) \\ &= 3Ce^{3x} - 1 - 3Ce^{3x} + 3x - 1 \\ &= 3x, \end{aligned}$$

and we see that f satisfies the given differential equation.

In order to have $y(0) = 1$, we must have

$$\begin{aligned} 1 &= f(0) \\ &= Ce^{3\cdot 0} - 0 - \frac{1}{3} \\ &= C - \frac{1}{3}, \end{aligned}$$

whence $C = \dfrac{4}{3}$.

2. A function $y = y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = y^4 - 5y^2 + 4.$$

(a) What are the constant solutions of the equation?

Solution: We have $y' = 0$ when $y^4 - 5y^2 + 4 = 0$. But $y^4 - 5y^2 + 4 = (y^2 - 4)(y^2 - 1)$, and the latter is zero only when $y = \pm 2$ or $y = \pm 1$. The constant solutions are thus $y = \pm 2$ and $y = \pm 1$.

(b) For what values of y is y decreasing? Why?

Solution: We know that the sign of y' is the same of the sign of $(y^2 - 4)(y^2 - 1)$. The latter is positive when both factors are positive and when both factors are negative; it is negative when one factor is positive but the other is negative. Both factors are negative only if $y^2 < 1$, or equivalently $-1 < y < 1$. Both factors are positive only if $4 < y^2$, or equivalently, $y < -2$ or $2 < y$. On the other hand, one factor is positive and the other negative when $1 < y^2 < 4$, which is equivalent to the condition that either $-2 < y < -1$ or $1 < y < 2$. Thus y is decreasing when either $-2 < y < -1$ or $1 < y < 2$.

(c) For what values of y is y increasing? Why?

Solution: By the analysis in the preceding part of this problem, y is increasing when either of the three conditions $y < -2$, $-1 < y < 1$, $2 < y$ holds.

3. A colony of bacteria contained 600 organisms when $t = 0$. When $t = 24$ hr, there were 38,400 organisms in the colony. Derive an expression for N, the number of organisms in the colony at time t, under the assumption of unconstrained growth. How many organisms were there in the colony at time $t = 10$ hr?

Solution: Under the condition that growth is unconstrained, we have $\dfrac{dN}{dt} = kN$, or $\dfrac{dN}{N} = k\,dt$. Therefore

$$\int_{600}^{N} \frac{dn}{n} = k\int_0^t d\tau,$$

or $\ln N - \ln 600 = kt$. This is equivalent to $N = 600e^{kt}$. But $n = 38400$ when $t = 24$, so $38400 = 600e^{24k}$. Thus $e^{24k} = 64$, or $24k = \ln 64 = \ln 2^6 = 6\ln 2$. This means that $k = \dfrac{1}{4}\ln 2 = \ln 2^{1/4}$. Consequently, $N = 600e^{t\ln 2^{1/4}} = 600e^{\ln 2^{t/4}} = 600 \cdot 2^{t/4}$.

When $t = 10$, this yields $N(10) = 600 \cdot 2^{5/2} = 2400\sqrt{2}$. This is about 3400.

4. Find $y(x)$ given that $e^{-x}y' = \sqrt{1-y^2}$ and $y(0) = 0$. What is the exact value of $y(x)$ when $x = \ln\left(1 + \frac{\pi}{4}\right)$?

Solution: The equation $e^{-x}y' = \sqrt{1-y^2}$ separates as $dy/\sqrt{1-y^2} = e^x\,dx$, so

$$\begin{aligned} \int_0^y \frac{dv}{\sqrt{1-v^2}} &= \int_0^x e^u\,du, \text{ or} \\ \arcsin y &= e^x - 1. \end{aligned}$$

The latter equation can be rewritten as $y = \sin(e^x - 1)$, which is therefore the solution to the given initial value problem. When $x = \ln\left(1 + \frac{\pi}{4}\right)$, we have

$$\begin{aligned} y &= \sin\left(e^{\ln\left(1+\frac{\pi}{4}\right)} - 1\right) \\ &= \sin\left(1 + \frac{\pi}{4} - 1\right) \\ &= \frac{\sqrt{2}}{2}. \end{aligned}$$

5. Use Euler's Method with a step-size of $1/10$ to find an approximate value for $y(13/10)$ if $y(1) = 1$ and

$$\frac{dy}{dt} = \frac{1}{t+y}.$$

Show all of your intermediate calculations; either give your answers as fractions of integers or maintain an accuracy of no fewer than five digits to the right of the decimal.

Solution: In order to apply Euler's Method to this differential equation with step size $1/10$, we put

$$\begin{aligned} h &= \frac{1}{10}; \\ F(t,y) &= \frac{1}{t+y}. \end{aligned}$$

We need y_3, where

$$\begin{aligned} t_0 &= 1; \\ y_0 &= 1 \end{aligned}$$

and, when $k \geq 1$,

$$\begin{aligned} t_k &= t_{k-1} + h; \\ y_k &= y_{k-1} + hF(t_{k-1}, y_{k-1}). \end{aligned}$$

Thus

$$\begin{aligned} t_1 &= \frac{11}{10}; \\ y_1 &= 1 + \frac{1}{10} \cdot \frac{1}{\frac{11}{10}+1} &= \frac{21}{20}; \\ t_2 &= \frac{12}{10}; \\ y_2 &= \frac{21}{20} + \frac{1}{10} \cdot \frac{1}{\frac{12}{10}+\frac{21}{20}} &= \frac{943}{860}; \\ t_3 &= \frac{13}{10}; \\ y_3 &= \frac{943}{860} + \frac{1}{10} \cdot \frac{1}{\frac{13}{10}+\frac{943}{860}} &= \frac{387277}{339700}. \end{aligned}$$

6. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n}$ converges or diverges. Give all of your reasoning.

Solution: For every positive integer n, we have $n2^n < n2^n \cdot 2 = n2^{n+1} < (n+1)2^{n+1}$. Therefore $\dfrac{1}{n2^n} > \dfrac{1}{(n+1)2^{n+1}} > 0$, and the terms of the sequence $\dfrac{1}{n2^n}$, $n = 1, 2, 3, \ldots$, are decreasing. Moreover, $\displaystyle\lim_{n\to\infty} \frac{1}{n2^n} = 0$. By the Alternating Series Test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n}$ converges.

7. Show how to use the Integral Test to determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges or diverges.

Solution: Let $f(x) = \frac{1}{x(\ln x)^2} = [x(\ln x)^2]^{-1}$. Then

$$\begin{aligned} f'(x) &= -[x(\ln x)^2]^{-2}[(\ln x)^2 + x(\ln x)(1/x)] \\ &= -\frac{2 + \ln x}{x^2(\ln x)^3)}, \end{aligned}$$

which is negative for all $x > 1$. Consequently, f is a decreasing function on the interval $[2, \infty)$. We also have, using the substitution $u = \ln x$; $du = dx/x$,

$$\begin{aligned} \int_2^T \frac{dx}{x(\ln x)^2} &= \int_{\ln 2}^{\ln T} u^{-2}\, du \\ &= -\frac{1}{u}\Big|_{\ln 2}^{\ln T} \\ &= \frac{1}{\ln 2} - \frac{1}{\ln T}, \end{aligned}$$

and this goes to $(\ln 2)^{-1}$ as $T \to \infty$. Thus, the improper integral $\int_2^{\infty} \frac{dx}{x(\ln x)^2}$ converges. By the Integral Test, the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges as well.

8. For what values of x does the series

$$\sum_{k=1}^{\infty} \frac{1}{2^{k-1}} \left(\frac{x+1}{x-1}\right)^k = \left(\frac{x+1}{x-1}\right) + \frac{1}{2}\left(\frac{x+1}{x-1}\right)^2 + \frac{1}{2^2}\left(\frac{x+1}{x-1}\right)^3 + \frac{1}{2^3}\left(\frac{x+1}{x-1}\right)^4 + \cdots$$

converge?

Solution: The series is geometric, with common ratio $\frac{1}{2}\left(\frac{x+1}{x-1}\right)$, and therefore converges when, and only when,

$$\left|\frac{1}{2}\left(\frac{x+1}{x-1}\right)\right| < 1.$$

In order to solve this inequality, we first note that it is equivalent to the compound inequality

$$-2 < \frac{x+1}{x-1} < 2.$$

In order to solve the first of these two inequalities, we add two to the first two members to obtain

$$\begin{aligned} 0 &< \frac{x+1}{x-1} + 2, \text{ or} \\ 0 &< \frac{3x-1}{x-1} \end{aligned}$$

This inequality holds when numerator and denominator are both positive or when the numerator and denominator are both negative, i.e., when $x < 1/3$ or $x > 1$. The second of the two inequalities in the compound inequality is

$$\begin{aligned} \frac{x+1}{x-1} - 2 &< 0, \text{ or} \\ \frac{3-x}{x-1} &< 0. \end{aligned}$$

Here, numerator and denominator must have opposite signs, and, solving through an analysis similar to what we have just done, we find that $x < 1$ or $x > 3$. The series therefore converges when x meets *both* of the conditions

(a) $x < 1/3$ or $x > 1$, and

(b) $x < 1$ or $x > 3$

simultaneously. Equivalently, the series converges when $x > 3$ or $x < 1/3$.

MTH 2410, Calculus II—Final Exam **May 12, 2003**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Your work is due at 2:25 pm. You may keep this copy of the exam.

1. Find the limits:

   (a) $\lim_{x \to 0^+} x \ln x$

   (b) $\lim_{x \to 0} \dfrac{\tan x - x}{x - \sin x}$

2. Determine whether the improper integral $\int_0^\infty x e^{-x^2}\, dx$ converges or diverges. If it converges, find its value.

3. Show how to use the Integral Test to determine whether the series $\sum_{n=1}^{\infty} \dfrac{1}{n(\ln n)^2}$ converges or diverges.

4. Show how to use Newton's Method, together with your calculator, to approximate the root of the equation $x^4 + 8x - 8 = 0$ that lies in the interval $[0, 1]$. Use $x_0 = 1$ as your initial guess and give both x_1 and x_2 as fractions whose numerators and denominators are whole numbers.

5. Find the radius of convergence for the series $\sum_{k=1}^{\infty} (-1)^k \dfrac{(x+2)^k}{k\, 2^k}$. Give the interior of the interval of convergence. Justify your conclusions.

6. The base of a solid is the triangle cut from the first quadrant by the line $4x + 3y = 12$. Every cross-section of the solid perpendicular to the y-axis is a square. Find the volume of the solid.

7. Find $y(x)$ given that $y' = y(1 - x)$ with $y(0) = 2$. What is the exact value of $y(x)$ when $x = 1$?

8. Find a whole number N which has the property that

$$\left| \sum_{k=0}^{n-1} x^k - \frac{1}{1-x} \right| \quad < \quad \frac{1}{20}$$

   whenever $n \geq N$ and $-\dfrac{7}{8} \leq x \leq \dfrac{7}{8}$.

MTH 2410, Calculus II—Final Exam **May 12, 2003**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Your work is due at 2:25 pm. You may keep this copy of the exam.

1. Find the limits:

   (a) $\lim_{x\to 0^+} x\ln x$

   (b) $\lim_{x\to 0} \frac{\tan x - x}{x - \sin x}$

   Solution:

   (a) We begin by noting that if we write

   $$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} \tag{1}$$

   then we may try to use l'Hôpital's rule on the fraction because both its numerator and its denominator become unbounded as $x \to 0^+$. Thus,

   $$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} \tag{2}$$

   $$= -\lim_{x\to 0^+} \frac{x^2}{x} = -\lim_{x\to 0^+} x = 0. \tag{3}$$

   (b) We may again attempt a l'hôpital's rule solution, because the limit in the numerator and the limit in the denominator are both 0. We find that

   $$\lim_{x\to 0} \frac{\tan x - x}{x - \sin x} = \lim_{x\to 0} \frac{\sec^2 x - 1}{1 - \cos x} \tag{4}$$

   $$= \lim_{x\to 0} \frac{\sec^2 x(1 - \cos^2 x)}{1 - \cos x} \tag{5}$$

   $$= \left(\lim_{x\to 0} \sec^2 x\right) \cdot \left(\lim_{x\to 0} \frac{\cancel{(1-\cos x)}(1+\cos x)}{\cancel{(1-\cos x)}}\right) = 2. \tag{6}$$

2. Determine whether the improper integral $\int_0^\infty xe^{-x^2}\,dx$ converges or diverges. If it converges, find its value.

Solution:

$$\begin{aligned}\int_0^\infty xe^{-x^2}\,dx &= \lim_{T\to\infty}\int_0^T xe^{-x^2}\,dx &(7)\\ &= -\frac{1}{2}\lim_{T\to\infty}\left[\int_0^T\left(-2xe^{-x^2}\right)\,dx\right] &(8)\\ &= -\frac{1}{2}\lim_{T\to\infty} e^{-x^2}\Big|_0^T &(9)\\ &= \frac{1}{2}-\frac{1}{2}\lim_{T\to\infty} e^{-T^2} = \frac{1}{2}. &(10)\end{aligned}$$

3. Show how to use the Integral Test to determine whether the series $\sum_{n=1}^{\infty}\frac{1}{n(\ln n)^2}$ converges or diverges.

Solution: Let $f(x) = \frac{1}{x(\ln x)^2} = [x(\ln x)^2]^{-1}$. Then

$$\begin{aligned} f'(x) &= -[x(\ln x)^2]^{-2}[(\ln x)^2 + x(\ln x)(1/x)]\\ &= -\frac{2+\ln x}{x^2(\ln x)^3)},\end{aligned}$$

which is negative for all $x > 1$. Consequently, f is a decreasing function on the interval $[2,\infty)$. We also have, using the substitution $u = \ln x$; $du = dx/x$,

$$\begin{aligned}\int_2^T \frac{dx}{x(\ln x)^2} &= \int_{\ln 2}^{\ln T} u^{-2}\,du\\ &= -\frac{1}{u}\Big|_{\ln 2}^{\ln T}\\ &= \frac{1}{\ln 2}-\frac{1}{\ln T},\end{aligned}$$

and this goes to $(\ln 2)^{-1}$ as $T\to\infty$. Thus, the improper integral $\int_2^\infty \frac{dx}{x(\ln x)^2}$ converges. By the Integral Test, the series $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ converges as well.

4. Show how to use Newton's Method, together with your calculator, to approximate the root of the equation $x^4 + 8x - 8 = 0$ that lies in the interval $[0,1]$. Use $x_0 = 1$ as your initial guess and give both x_1 and x_2 as fractions whose numerators and denominators are whole numbers.

Solution: We set $x_0 = 1$ and $f(x) = x^4+8x-8$. Then we employ the Newton interation scheme

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^4+8x_k-8}{4x_k^3+8}. \qquad (11)$$

This gives

$$x_2 = 1 - \frac{1+8-8}{4+8} = \frac{11}{12}; \tag{12}$$

$$x_3 = \frac{11}{12} - \frac{\left(\frac{11}{12}\right)^4 + 8\left(\frac{11}{12}\right) - 8}{4\left(\frac{11}{12}\right)^3 + 8} = \frac{69937}{76592}. \tag{13}$$

5. Find the radius of convergence for the series $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{(x+2)^k}{k\,2^k}$. Give the interior of the interval of convergence. Justify your conclusions.

   Solution:

$$\lim_{k\to\infty} \frac{|a_{k+1}x^{k+1}|}{|a_k x^k|} = \lim_{k\to\infty}\left[\frac{|x+2|^{k+1}}{(k+1)2^{k+1}} \cdot \frac{k2^k}{|x+2|^k}\right] \tag{14}$$

$$= \frac{|x+2|}{2} \lim_{k\to\infty} \frac{k}{k+1} \tag{15}$$

$$= \frac{|x+2|}{2} \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}} = \frac{|x+2|}{2}. \tag{16}$$

   This limit is less than one when $-4 < x < 0$, one when $x = -4$ and when $x = 0$, and larger than one when $x < -4$ and when $x > 0$. By the Ratio Test, the series converges when $-4 < x < 0$ and diverges when x lies in $(-\infty, -4) \cup (0, \infty)$. The required interior is therefore $(-4, 0)$.

6. The base of a solid is the triangle cut from the first quadrant by the line $4x+3y = 12$. Every cross-section of the solid perpendicular to the y-axis is a square. Find the volume of the solid.

   Solution: If we assign the value t to y, then the corresponding value of x is

$$x = 3 - \frac{3}{4}t, \tag{17}$$

   so the base of the cross-section at $y = t$ has length $3 - \frac{3}{4}t$ and area $A(t) = \left(4 - \frac{3}{4}t\right)^2$. The solid extends from $y = 0$ to $y = 4$. Consequently, the volume of the solid is

$$\int_0^4 \left(3 - \frac{3}{4}t\right)^2 dt = \int_0^4 \left(9 - \frac{9}{2}t + \frac{9}{16}t^2\right) dt \tag{18}$$

$$= \left(9t - \frac{9}{4}t^2 + \frac{3}{16}t^3\right)\Bigg|_0^4 = 12. \tag{19}$$

7. Find $y(x)$ given that $y' = y(1-x)$ with $y(0) = 2$. What is the exact value of $y(x)$ when $x = 1$?

   Solution: We must have

$$\frac{y'(x)}{y(x)} = 1 - x, \text{ so} \tag{20}$$

$$\int_0^x \frac{y'(t)}{y(t)}\, dt = \int_0^x (1-t)\, dt. \tag{21}$$

Now $y(0) = 2 > 0$, so when x is close to 0, $y(x) > 0$. For such x, we have

$$\ln y(x) - \ln y(0) = -\frac{1}{2}(1-t)^2\Big|_0^x, \text{ or} \tag{22}$$

$$\ln y(x) = \ln 2 + \frac{1}{2} - \frac{1}{2}(1-x)^2. \tag{23}$$

Thus,

$$y(x) = 2\exp\left(x - \frac{1}{2}x^2\right), \tag{24}$$

where, for ease of reading, we have employed the convention $e^u = \exp(u)$. From this expression, it now follows that $y(1) = 2\sqrt{e}$.

8. Find a whole number N which has the property that

$$\left|\sum_{k=0}^{n-1} x^k - \frac{1}{1-x}\right| \quad < \quad \frac{1}{20}$$

whenever $n \geq N$ and $-\dfrac{7}{8} \leq x \leq \dfrac{7}{8}$.

Solution: We know that

$$\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x} \tag{25}$$

as long as $x \neq -1$, so

$$\sum_{k=0}^{n-1} x^k - \frac{1-x^n}{1-x} = 1 - \frac{1-x^n}{1-x} = \frac{x^n}{1-x}. \tag{26}$$

Consequently, we would like to choose N so that when $n \geq N$ and x is in the given interval we will have

$$\left|\frac{x^n}{1-x}\right| \leq \frac{1}{20}. \tag{27}$$

When $-\frac{7}{8} < x < \frac{7}{8}$, we note, that $-\frac{7}{8} < -x < \frac{7}{8}$, and, adding one to all three members of this inequality, that

$$\frac{1}{8} = 1 - \frac{7}{8} < 1 - x < 1 + \frac{7}{8} = \frac{15}{8}. \tag{28}$$

Therefore,

$$0 < \frac{8}{15} < \frac{1}{1-x} < \frac{8}{1}. \tag{29}$$

Thus, $\frac{1}{1-x}$ is larger that $\frac{8}{15}$, which means—among other things—that $\frac{1}{|1-x|} = \frac{1}{1-x}$, and this latter fraction is smaller than 8. Hence,

$$\left|\frac{x^n}{1-x}\right| = \frac{|x|^n}{1-x} < 8|x|^n < \frac{7^n}{8^{n-1}}. \tag{30}$$

On account of this latter inequality, we need only be able to guarantee that

$$7\left(\frac{7}{8}\right)^{n-1} < \frac{1}{20}, \text{ or} \tag{31}$$

$$\left(\frac{7}{8}\right)^{n-1} < \frac{1}{140} \tag{32}$$

when n is large enough. This will be so precisely when

$$(n-1)\ln\frac{7}{8} < -\ln 140, \tag{33}$$

or, because $\ln\frac{7}{8} = \ln 7 - \ln 8$ is negative, when

$$n > 1 + \frac{\ln 140}{\ln 8 - \ln 7} \sim 37.01. \tag{34}$$

Now N must be a whole number larger than 37.01, so we may take $N = 38$.

MTH 2140, Calculus II, Exam 1 **September 14, 2005**

Instructions: Write out, *on your own paper*, complete solutions of the following problems. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your paper is due at 1:50 pm.

1. Show how to evaluate the following limits:

   (a) $\lim_{x\to 5} \dfrac{x^2 - 5x}{x^2 - 4x - 5}$

   (b) $\lim_{x\to\infty} \dfrac{\ln x}{x^3}$

2. Apply Newton's Method to find an approximation to the root in $[1, 2]$ of the equation $x^3 + x - 4 = 0$. Take $x_1 = 1.5$ and give x_2 and x_3. Maintain at least 8 correct digits in all of your calculations.

3. Show how to evaluate $\displaystyle\int_1^2 \frac{(\ln x)^2}{x}\,dx$ and give the value.

4. Show how to use the substitution $x = a\sin\theta$ to transform the definite integral $\displaystyle\int_0^a \sqrt{a^2 - x^2}\,dx$ into another definite integral. Then evaluate the latter definite integral.

5. Show how to evaluate $\displaystyle\int x\,e^{2x}\,dx$ and give the value.

6. Show how to use the method of partial fractions to evaluate $\displaystyle\int \frac{3x\,dx}{(x+2)(x-1)}$ and give the value.

7. Show how to evaluate $\displaystyle\int_0^9 \frac{dx}{\sqrt{x}}$ and give the value.

8. Show how the substitution $3x - 4 = u^3$ reduces the integral

$$\int \frac{x\,dx}{\sqrt[3]{3x-4}}$$

   to an integral of a polynomial. Then use what you have shown to evaluate the original integral.

MTH 2140, Calculus II, Exam 1 **September 14, 2005**

Instructions: Write out, *on your own paper*, complete solutions of the following problems. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your paper is due at 1:50 pm.

1. Show how to evaluate the following limits:

   (a) $\lim_{x\to 5} \frac{x^2 - 5x}{x^2 - 4x - 5}$

   Solution:

$$\begin{aligned}\lim_{x\to 5} \frac{x^2 - 5x}{x^2 - 4x - 5} &= \lim_{x\to 5} \frac{x(x-5)}{(x+1)(x-5)} \\ &= \lim_{x\to 5} \frac{x}{x+1} = \frac{5}{6}.\end{aligned}$$

   One may also use L'Hôpital's Rule. Of course, this requires that one first explain how one knows that the rule is applicable.

   (b) $\lim_{x\to\infty} \frac{\ln x}{x^3}$

   Solution: The limit in the numerator is infinite, and so is the limit in the denominator. We may therefore attempt L'Hôpital's Rule:

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln x}{x^3} &= \lim_{x\to\infty} \frac{(1/x)}{3x^2} \\ &= \lim_{x\to\infty} \frac{1}{3x^3} = 0.\end{aligned}$$

2. Apply Newton's Method to find an approximation to the root in $[1, 2]$ of the equation $x^3 + x - 4$. Take $x_1 = 1.5$ and give x_2 and x_3. Maintain at least 8 correct digits in all of your calculations.

   Solution: We take $x_1 = 1.5$, and

$$\begin{aligned}x_{k+1} &= x_k - \frac{f(x_k)}{f'(x_k)} \\ &= x_k - \frac{x_k^3 + x_k - 4}{3x_k^2 + 1}.\end{aligned}$$

Thus,

$$\begin{aligned} x_2 &= 1.5 - \frac{(1.5)^3 + 1.5 - 4}{3(1.5)^2 + 1} \\ &= 1.3870967742, \text{ while} \\ x_3 &= 1.3870967742 - \frac{(1.3870967742)^3 + 1.3870967742 - 4}{3(1.3870967742)^2 + 1} \\ &= 1.3788389476. \end{aligned}$$

3. Show how to evaluate $\displaystyle\int_1^2 \frac{(\ln x)^2}{x}\, dx$.

Solution: Let $u = \ln x$. Then $du = \dfrac{dx}{x}$, while $x = 1$ implies $u = 0$ and $x = 2$ implies $u = \ln 2$. Hence

$$\begin{aligned} \int_1^2 \frac{(\ln x)^2}{x}\, dx &= \int_0^{\ln 2} u^2\, du \\ &= \frac{1}{3}u^3 \bigg|_0^{\ln 2} = \frac{(\ln 2)^3}{3}. \end{aligned}$$

4. Show how to use the substitution $x = a \sin\theta$ to transform the definite integral $\displaystyle\int_0^a \sqrt{a^2 - x^2}\, dx$ into another definite integral. Then evaluate the latter definite integral.

Solution: If $x = a\sin\theta$, then $dx = a\cos\theta\, d\theta$. Also, $\theta = 0$ when $x = 0$, and $\theta = \pi/2$ when $x = a$. Thus

$$\begin{aligned} \int_0^a \sqrt{a^2 - x^2}\, dx &= \int_0^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta} \cdot a\cos\theta\, d\theta \\ &= \int_0^{\pi/2} \sqrt{a^2(1 - \sin^2\theta)} \cdot a\cos\theta\, d\theta \\ &= a^2 \int_0^{\pi/2} \cos^2\theta\, d\theta \\ &= \frac{a^2}{2} \int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta \\ &= \frac{a^2}{2} \left(\theta + \frac{1}{2}\sin 2\theta\right) \bigg|_0^{\pi/2} \\ &= \frac{a^2}{2} \left[\left(\frac{\pi}{2} + 0\right) - (0 + 0)\right] = \frac{\pi a^2}{4}. \end{aligned}$$

5. Show how to evaluate $\displaystyle\int x\,e^{2x}\,dx$.

Solution: Let us integrate by parts, taking $u = x$ and $dv = e^{2x}\,dx$. Then $du = dx$, and $v = e^{2x}/2$. thus:

$$\begin{aligned}\int xe^{2x}\,dx &= \int u\,dv \\ &= uv - \int v\,du \\ &= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx \\ &= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C.\end{aligned}$$

6. Show how to use the method of partial fractions to evaluate $\displaystyle\int \frac{3x\,dx}{(x+2)(x-1)}$.

Solution: We need to expand the integrand $\dfrac{3x}{(x+2)(x-1)}$ in the form

$$\begin{aligned}\frac{3x}{(x+2)(x-1)} &= \frac{A}{x+2} + \frac{B}{x-1} \\ &= \frac{A(x-1)+B(x+2)}{(x+2)(x-1)} \\ &= \frac{(A+B)x+(-A+2B)}{(x+2)(x-1)}.\end{aligned}$$

Consequently,

$$\begin{aligned}A+B &= 3; \\ -A+2B &= 0.\end{aligned}$$

Adding these latter equations gives $3B = 3$, whence $B = 1$. Putting $B = 1$ into the first of the latter displayed pair of equations, we find that $A = 2$. Hence,

$$\begin{aligned}\int \frac{3x\,dx}{(x+2)(x-1)} &= \int \left(\frac{2}{x+2} + \frac{1}{x-1}\right) dx \\ &= 2\int \frac{dx}{x+2} + \int \frac{dx}{x-1} \\ &= 2\ln|x+2| + \ln|x-1| + C \\ &= \ln\left|K(x+2)^2(x-1)\right|.\end{aligned}$$

7. Evaluate $\displaystyle\int_0^9 \frac{dx}{\sqrt{x}}$.

Solution: Note first that the integral is improper. Hence,

$$\int_0^9 \frac{dx}{\sqrt{x}} = \lim_{T\to 0^+} \int_T^9 \frac{dx}{\sqrt{x}}$$

To evaluate the integral on the right, we make the substitution $u = \sqrt{x}$. Then $du = \dfrac{dx}{2\sqrt{x}}$, or $\dfrac{dx}{\sqrt{x}} = 2\,du$. Also, $u = 3$ when $x = 9$, and $u = \sqrt{T}$ when $x = T$. Consequently,

$$\begin{aligned}
\int_0^9 \frac{dx}{\sqrt{x}} &= \lim_{T\to 0^+} \int_T^9 \frac{dx}{\sqrt{x}} \\
&= 2 \lim_{T\to 0^+} \int_{\sqrt{T}}^3 du \\
&= 2 \lim_{T\to 0^+} u\Big|_{\sqrt{T}}^3 \\
&= 2 \lim_{T\to 0^+} (3 - \sqrt{T}) \ = \ 6.
\end{aligned}$$

8. Show how the substitution $3x - 4 = u^3$ reduces the integral

$$\int \frac{x\,dx}{\sqrt[3]{3x-4}}$$

to an integral of a polynomial. Then use what you have shown to evaluate the original integral.

Solution: If we put $3x - 4 = u^3$, then $x = (u^3 + 4)/3$ and $dx = u^2\,du$. Thus,

$$\begin{aligned}
\int \frac{x\,dx}{\sqrt[3]{3x-4}} &= \int \frac{[(u^3+4)/3]\cdot u^2\,du}{\sqrt[3]{(u^3)}} \\
&= \frac{1}{3}\int \frac{u^5 + 4u^2}{u}\,du \\
&= \frac{1}{3}\int (u^4 + 4u)\,du \\
&= \frac{1}{3}\left(\frac{1}{5}u^5 + 2u^2\right) + C \\
&= \frac{1}{15}(3x-4)^{5/3} + \frac{2}{3}(3x-4)^{2/3} + C.
\end{aligned}$$

MTH 2140, Calculus II, Exam 2 **October 12, 2005**

Instructions: Write out, *on your own paper*, complete solutions of the following problems. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your paper is due at 1:50 pm.

1. Find the average value of the function f given by

$$f(x) \quad = \quad x \sin x$$

   on the interval $[0, \pi]$.

2. The region bounded by the curve $y = x^2$ and the line $y = 4$ is revolved about the y-axis. Find the volume of the solid generated in this fashion. (See Figure 1.)

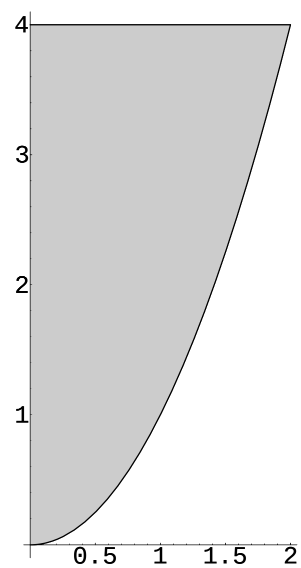

Figure 1: Problem 2

3. A spring whose natural length is one foot stretches 1.2 inches when subjected to a 5-pound load. Find the work done when the spring length of the spring is increased from 15 inches to 20 inches.

4. A curve is given parametrically by

$$\begin{aligned} x(t) &= \cos^3 t; \\ y(t) &= \sin^3 t. \end{aligned}$$

Find the length of that part of the curve which corresponds to $0 \le t \le \pi/2$.

5. Use the Trapezoidal Rule with 3 subdivisions to approximate the integral

$$\int_1^4 \frac{1}{1+x^3}\,dx.$$

Report your approximation either as a fraction whose numerator and denominator are both integers or to at least five significant digits.

6. Brünhilde is working an engineering problem, and she needs an approximate value for the integral

$$\int_{-2}^{2} \frac{1-2x}{\sqrt{1+x^4}}\,dx.$$

She has found that the Simpson's Rule approximation for the integral using 10 subdivisions is, to six significant digits, 2.71538. But she doesn't know how many of those digits to trust. Make appropriate use of the six figures below to estimate potential error in her approximation. Explain your reasoning.

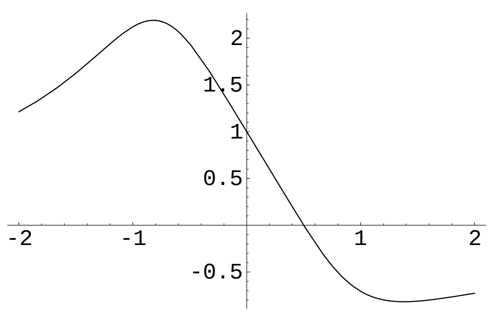

Figure 2: Graph of $f(x)$

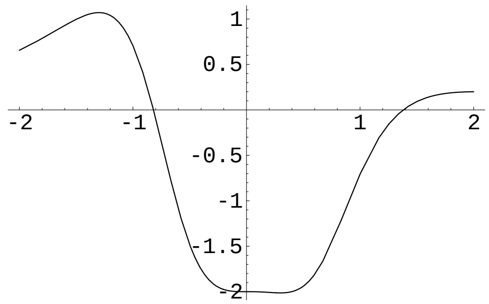

Figure 3: Graph of $f'(x)$

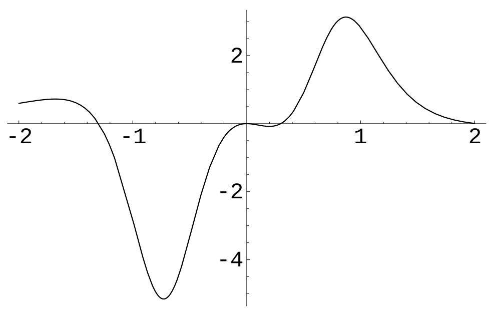

Figure 4: Graph of $f''(x)$

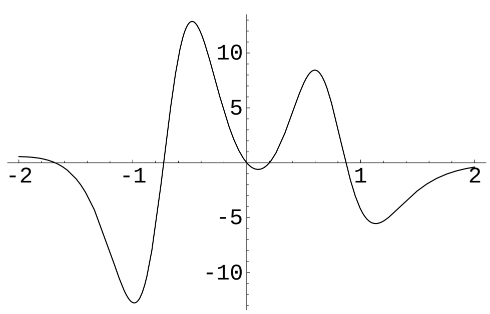

Figure 5: Graph of $f^{(3)}(x)$

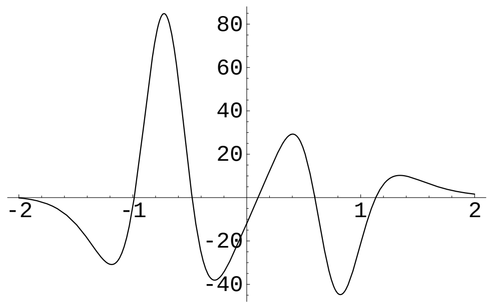

Figure 6: Graph of $f^{(4)}(x)$

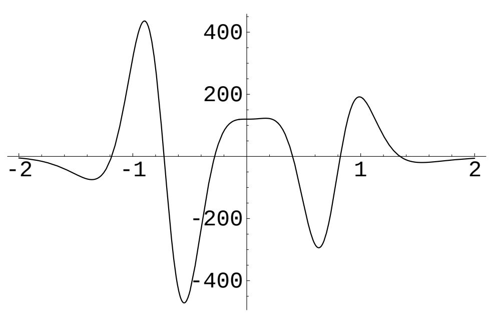

Figure 7: Graph of $f^{(5)}(x)$

MTH 2140, Calculus II, Exam 2 **October 12, 2005**

Instructions: Write out, *on your own paper*, complete solutions of the following problems. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your paper is due at 1:50 pm.

1. Find the average value of the function f given by

$$f(x) \quad = \quad x \sin x$$

on the interval $[0, \pi]$.

Solution: The average value is

$$\frac{1}{\pi} \int_0^{\pi} x \sin x \, dx.$$

We integrate by parts, taking $u = x$ and $dv = \sin x \, dx$, whence $du = dx$ and $v = -\cos x$. Thus

$$\begin{aligned} \frac{1}{\pi} \int_0^{\pi} x \sin x \, dx &= \frac{1}{\pi} \left[-x \cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x \, dx \right] \\ &= \frac{1}{\pi} \left[\pi + \sin x \Big|_0^{\pi} \right] \;=\; 1. \end{aligned}$$

2. The region bounded by the curve $y = x^2$ and the line $y = 4$ is revolved about the y-axis. Find the volume of the solid generated in this fashion. (See Figure 1.)

Solution 1 (Disks): Cross-sections of the solid perpendicular to the y-axis are disks. Because $y = x^2$, we have $x = \sqrt{y}$, so the cross-section at height y has radius $\sqrt{y}$. The vertical extent of the region is from $y = 0$ to $y = 4$. Hence, volume V is given by

$$\begin{aligned} V &= \pi \int_0^4 (\sqrt{y})^2 \, dy \\ &= \pi \int_0^4 y \, dy \\ &= \frac{\pi}{2} y^2 \Big|_0^4 \;=\; 8\pi. \end{aligned}$$

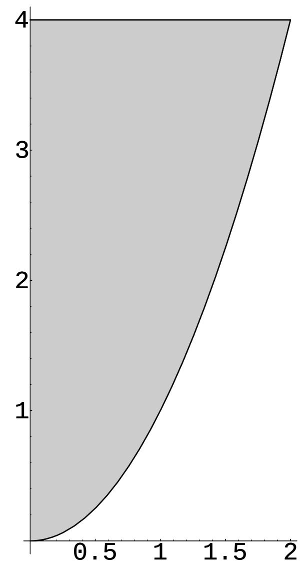

Figure 1: Problem 2

Solution 2 (Cylindrical Shells): Using the method of cylindrical shells leads to

$$
\begin{aligned}
V &= 2\pi \int_0^2 x(4 - x^2)\, dx \\
&= 2\pi \left(2x^2 - \frac{x^4}{4}\right)\Big|_0^2 = 8\pi.
\end{aligned}
$$

3. A spring whose natural length is one foot stretches 1.2 inches when subjected to a 5-pound load. Find the work done when the spring length of the spring is increased from 15 inches to 20 inches.

   Solution: 1.2 inches is 0.1 feet. Hooke's Law for this spring thus requires that $5 = 0.1k$, or $k = 50$. Taking conversion of inches to feet and an equilibrium position of 1 foot into account, we find that the work required to stretch the spring from a length of 15 inches to a length of 20 inches is

$$
\begin{aligned}
W &= 50 \int_{1/4}^{2/3} x\, dx \\
&= 25x^2\Big|_{1/4}^{2/3} = \frac{1375}{144} \text{ ft-lb.}
\end{aligned}
$$

4. A curve is given parametrically by

$$\begin{aligned} x(t) &= \cos^3 t; \\ y(t) &= \sin^3 t. \end{aligned}$$

Find the length of that part of the curve which corresponds to $0 \leq t \leq \pi/2$.

Solution: We have $x'(t) = -3\cos^2 t \sin t$ and $y'(t) = 3\sin^2 t \cos t$. Hence

$$\begin{aligned} s &= \int_0^{\pi/2} \sqrt{[x'(t)]^2 + (y'(t)]^2}\, dt \\ &= \int_0^{\pi/2} \sqrt{9\cos^4 t \sin^2 t + 9\cos^2 t \sin^4 t}\, dt \\ &= \int_0^{\pi/2} \sqrt{9\cos^2 t \sin^2 t\,(\cos^2 t + \sin^2 t)}\, dt \\ &= 3\int_0^{\pi/2} \sin t \cos t\, dt \\ &= \frac{3}{2}\sin^2 t\bigg|_0^{\pi/2} = \frac{3}{2}. \end{aligned}$$

5. Use the Trapezoidal Rule with 3 subdivisions to approximate the integral

$$\int_1^4 \frac{1}{1+x^3}\, dx.$$

Report your approximation either as a fraction whose numerator and denominator are both integers or to at least five significant digits.

Solution: According to the Trapezoidal Rule,

$$\begin{aligned} \int_1^4 f(x)\, dx &\sim \frac{1}{2}[f(1) + 2f(2) + 2f(3) + f(4)] \\ &= \frac{1}{2}\left(\frac{1}{1+1} + \frac{2}{1+8} + \frac{2}{1+27} + \frac{1}{1+64}\right) \\ &= \frac{1}{2}\left(\frac{3313}{4095}\right) = \frac{3313}{8190}. \end{aligned}$$

6. Brünhilde is working an engineering problem, and she needs an approximate value for the integral

$$\int_{-2}^{2} \frac{1-2x}{\sqrt{1+x^4}}\, dx.$$

She has found that the Simpson's Rule approximation for the integral using 10 subdivisions is, to six significant digits, 2.71538. But she doesn't know how many of those digits to trust. Make appropriate use of the six figures below to estimate potential error in her approximation. Explain your reasoning.

Solution: Error in Simpson's Rule is bounded in magnitude by $\dfrac{M(b-a)^5}{180n^4}$, where n is the number of subdivisions and M is any number for which $|f^{(4)}(x)| \leq M$ for all x that lie in $[a, b]$. From Figure 6 we see that we may take $M = 90$ in this case. Thus error in Brünhilde's approximation with $n = 10$ is at most

$$\begin{aligned}\frac{M(b-a)^5}{180n^4} &= \frac{90[2-(-2)]^5}{180(10)^4} \\ &= \frac{32}{625} = 0.0512.\end{aligned}$$

Brünie had better not trust any of the digits to the right of the "7".

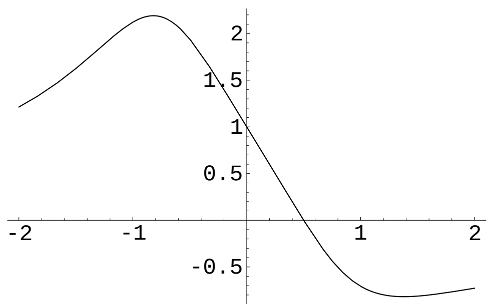

Figure 2: Graph of $f(x)$

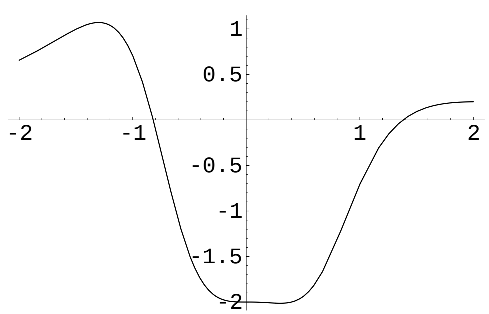

Figure 3: Graph of $f'(x)$

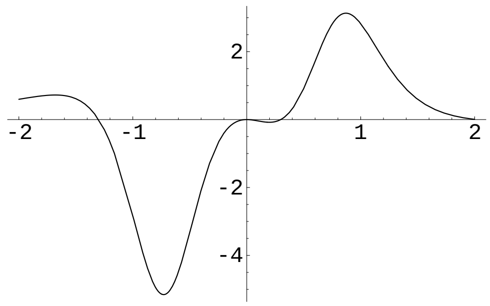

Figure 4: Graph of $f''(x)$

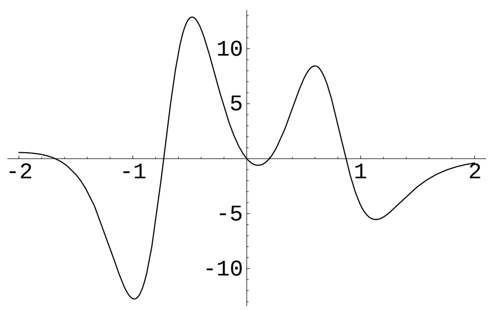

Figure 5: Graph of $f^{(3)}(x)$

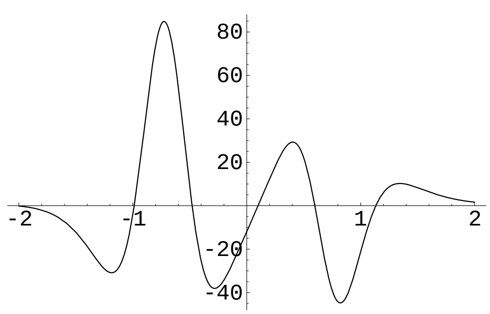

Figure 6: Graph of $f^{(4)}(x)$

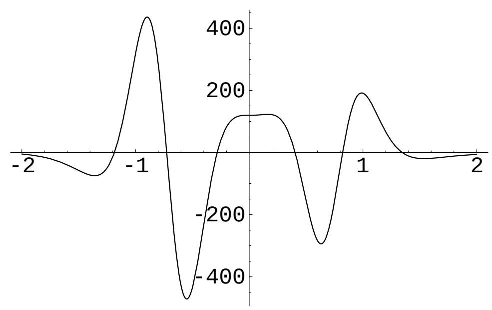

Figure 7: Graph of $f^{(5)}(x)$

MTH 2410, Calculus II, Exam 3 **November 9, 2005**

Instructions: Write out, *on your own paper*, complete solutions of the following problems. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your paper is due at 1:50 pm.

1. Let $y(x)$ be the solution of the initial value problem

$$\begin{aligned} \frac{dy}{dx} &= 2x + y; \\ y(0) &= 1. \end{aligned}$$

   Use Euler's Method with step size 1/10 to find an approximate value for $y(0.2)$. **Show the calculations that support your answer.**

2. Find the area enclosed by the polar curve $r^2 = \cos\theta$. (See Figure 1.)

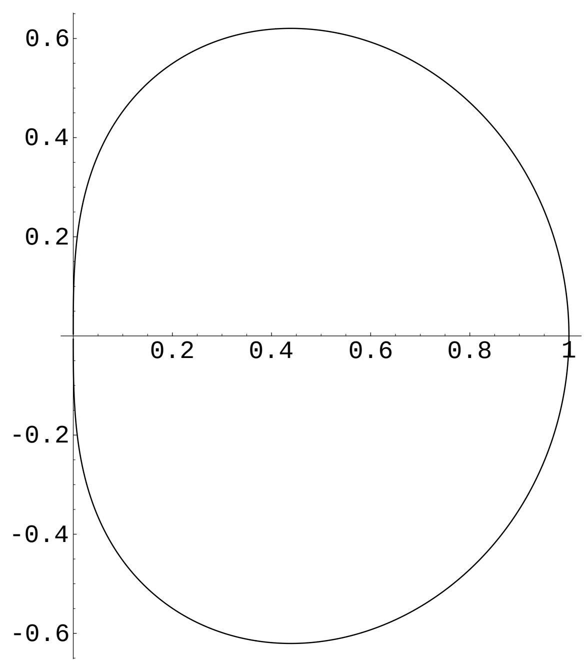

Figure 1: $r^2 = \cos\theta$

3. Determine whether each of the following sequences is divergent or convergent. If convergent, give the limit; if divergent, explain how you know.

   (a) $a_n = \dfrac{\ln(n^2)}{n}$

   (b) $a_n = \dfrac{n}{1+\sqrt{n}}$

   (c) $a_n = \ln(n+1) - \ln n$

4. Determine whether each of the following series is divergent or convergent. **Explain.**

   (a) $\displaystyle\sum_{n=1}^{\infty} 3^n$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n + 4^n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+2}}{5^n}$

5. Find the solution of the initial value problem:

$$\begin{aligned} \frac{dy}{dx} &= 1 + y^2; \\ y(1) &= 0. \end{aligned}$$

6. A tank contains 30 kg of salt dissolved in 7500 L of water. Brine that contains 0.03 kg of salt per liter enters the tank at a rate of 25 L/min. The solution in the tank is kept thoroughly mixed and drains from the tank at the same rate that the brine enters. How much salt remains in the tank after half an hour?

MTH 2410, Calculus II, Exam 3 **November 9, 2005**

Instructions: Write out, *on your own paper*, complete solutions of the following problems. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your paper is due at 1:50 pm.

1. Let $y(x)$ be the solution of the initial value problem

$$\begin{aligned} \frac{dy}{dx} &= 2x + y; \\ y(0) &= 1. \end{aligned}$$

Use Euler's Method with step size 1/10 to find an approximate value for $y(0.2)$. **Show the calculations that support your answer.**

Solution: For this initial value problem, Euler's Method requires us to calculate

$$\begin{aligned} x_0 &= 0; \\ y_0 &= 1; \\ x_k &= x_{k-1} + 0.1, \; k \;=\; 1, 2; \\ y_k &= y_{k-1} + (0.1) \cdot (2x_{k-1} + y_{k-1}), \; k \;=\; 1, 2. \end{aligned}$$

Thus

$$\begin{aligned} x_1 &= 0 + 0.1 \;=\; 0.1; \\ y_1 &= 1 + (0.1)(2 \cdot 0 + 1) \;=\; 1.1; \\ x_2 &= 0.1 + 0.1 \;=\; 0.2; \\ y_2 &= 1.1 + (0.1)[(2) \cdot (0.1) + (1.1)] \;=\; 1.23. \end{aligned}$$

2. Find the area enclosed by the polar curve $r^2 = \cos\theta$. (See figure.)

Solution: The entire curve is swept out as θ ranges over the interval $[-\pi/2, \pi/2]$, so

$$\begin{aligned} A &= \frac{1}{2}\int_{-\pi/2}^{\pi/2} r^2 \, d\theta \\ &= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta \\ &= \frac{1}{2}\sin\theta \Big|_{-\pi/2}^{\pi/2} \\ &= \frac{1}{2}\left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{-\pi}{2}\right)\right] \;=\; 1. \end{aligned}$$

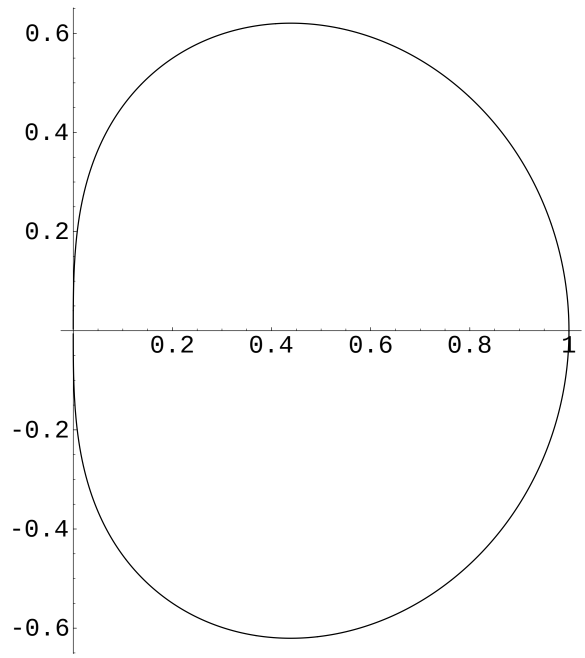

Figure 1: $r^2 = \cos\theta$

3. Determine whether each of the following sequences is divergent or convergent. If convergent, give the limit; if divergent, explain how you know.

(a) $a_n = \dfrac{\ln(n^2)}{n}$

(b) $a_n = \dfrac{n}{1+\sqrt{n}}$

(c) $a_n = \ln(n+1) - \ln n$

Solution:

(a) Numerator and denominator both grow without bound, so L'Hôpital's Rule applies:

$$\begin{aligned} \lim_{n\to\infty} \frac{\ln(n^2)}{n} &= \lim_{n\to\infty} \frac{(2n/n^2)}{1} \\ &= \lim_{n\to\infty} \frac{2}{n} \;=\; 0. \end{aligned}$$

This sequence is convergent.

(b) We have

$$\begin{aligned} \lim_{n\to\infty} \frac{n}{1+\sqrt{n}} &= \lim_{n\to\infty} \frac{1}{(1/n)+(1/\sqrt{n})} \\ &= \infty, \end{aligned}$$

because $(1/n)+(1/\sqrt{n}) \to 0$ as $n \to \infty$, while the 1 in the numerator of the main fraction is fixed. This sequence diverges.

(c) We have

$$\begin{aligned} \lim_{n\to\infty} [\ln(n+1) - \ln n] &= \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) \\ &= \lim_{n\to\infty} \ln\left(\frac{1+(1/n)}{1}\right) \\ &= \ln \lim_{n\to\infty} \left(\frac{1+(1/n)}{1}\right) \\ &= \ln 1 \;=\; 0. \end{aligned}$$

This sequence converges.

4. Determine whether each of the following series is divergent or convergent. **Explain.**

(a) $\displaystyle\sum_{n=1}^{\infty} 3^n$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n + 4^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+2}}{5^n}$

Solution:

(a) $\lim_{n\to\infty} 3^n = \infty \neq 0$. Consequently, the series fails the gateway test for convergence, and so must diverge.

(b)

$$\lim_{n\to\infty} \frac{5^n}{3^n + 4^n} = \lim_{n\to\infty} \frac{(5/4)^n}{(3/4)^n + 1} = \infty \neq 0,$$

because $(5/4)^n \to \infty$ while $(3/4)^n \to 0$ as $n \to \infty$. So this series also fails the gateway test for convergence. It must therefore diverge.

(c)

$$\sum_{n=1}^{\infty} \frac{3^{n+2}}{5^n} = \frac{27}{5} \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$$

has the form $a\sum_{n=0}^{\infty} r^n$ with $r = 3/5$—so that $|r| < 1$. Consequently, this series is a convergent geometric series.

5. Find the solution of the initial value problem:

$$\begin{aligned} \frac{dy}{dx} &= 1 + y^2; \\ y(1) &= 0. \end{aligned}$$

Solution: We write

$$\begin{aligned} \frac{dy}{1+y^2} &= dx; \text{ whence} \\ \int \frac{dy}{1+y^2} &= \int dx; \\ \arctan y &= x + c; \\ y &= \tan(x + c). \end{aligned}$$

But $y(1) = 0$, so $0 = \tan(1 + c)$. This means that $c = -1$, and the solution is $y(x) = \tan(x - 1)$.

6. A tank contains 30 kg of salt dissolved in 7500 L of water. Brine that contains 0.03 kg of salt per liter enters the tank at a rate of 25 L/min. The solution in the tank is kept thoroughly mixed and drains from the tank at the same rate that the brine enters. How much salt remains in the tank after half an hour?

Solution: Let $S(t)$ denote the amount of salt, in kg, in the tank t minutes after the brine begins entering the tank. Salt enters the tank at the rate of $(0.03) \cdot (25) = 0.75$ kg/min. The concentration of salt in the tank at time t is $S(t)/7500$ kg/L. Hence salt leaves the tank at the rate of $[S(t)/7500] \cdot (25) = S(t)/300$ kg/min. Therefore

$$\begin{aligned}
\frac{dS}{dt} &= 0.75 - \frac{S}{300}, \text{ or} \\
300\frac{dS}{dt} &= 225 - S; \\
\frac{300\,dS}{225 - S} &= dt; \\
300 \int \frac{dS}{225 - S} &= \int dt; \\
-300 \ln|225 - S| &= t + c; \\
\ln|225 - S| &= -\frac{t}{300} + c; \\
225 - S &= Ce^{-t/300} \\
S &= 225 - Ce^{-t/300}.
\end{aligned}$$

But $S(0) = 30$ is given, so

$$\begin{aligned}
30 &= S(0) \\
&= 225 - Ce^0 \\
&= 225 - C,
\end{aligned}$$

so that $C = 195$. Thus $S(t) = 225 - 195e^{-t/300}$. Finally, half an hour after the brine starts flowing, we have $S(30) = 225 - 195e^{-30/300} = 225 - 195e^{-1/10} = 48.56$ kg.

MTH 2410, Calculus II, Final Exam **December 12, 2005**

Instructions: Write out, *on your own paper*, complete solutions of the following problems. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your paper is due at 1:50 pm.

1. Find the radius of convergence for each of the series

   (a) $\displaystyle\sum_{k=0}^{\infty} \frac{2^k x^k}{k^2}$

   (b) $\displaystyle\sum_{k=0}^{\infty} \frac{x^{3k+1}}{(3k+1)!}$

2. The region bounded by the curve $y = x^2$ and the line $y = 4$ is revolved about the y-axis. (See Figure 1.) Find the volume of the solid generated in this fashion.

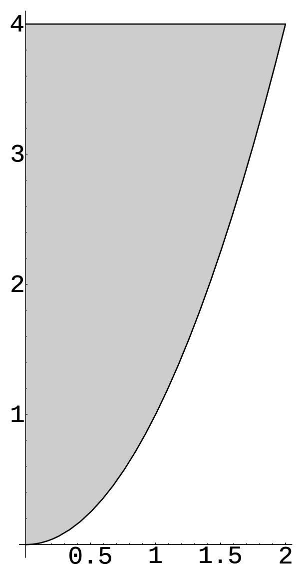

Figure 1: Problem 2

3. Show how the substitution $3x - 4 = u^3$ reduces the integral

$$\int \frac{x\,dx}{\sqrt[3]{3x-4}}$$

to an integral of a polynomial. Then use what you have shown to evaluate the original integral.

4. Show how to use the Integral Test to determine whether or not the series

$$\sum_{k=1}^{\infty} ke^{-k}$$

converges.

5. Determine which of the following series converge and which diverge. Give your reasoning.

   (a) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^n$

   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{3k^2+3}{k^3+3k}$

6. Find the Maclaurin series (*i. e.*, the Taylor series centered at $x = 0$) for the function $f(x) = \ln(1 - x^2)$. What is its radius of convergence?

7. Find the solution of the initial value problem:

$$\begin{aligned} y\frac{dy}{dx} &= 1 + y^2; \\ y(0) &= -1. \end{aligned}$$

8. Does the integral

$$\int_2^{\infty} \frac{dx}{x(\ln x)^{3/2}}$$

converge? diverge? Support your conclusion.

MTH 2410, Calculus II, Final Exam **December 12, 2005**

Instructions: Write out, *on your own paper*, complete solutions of the following problems. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your paper is due at 1:50 pm.

1. Find the radius of convergence for each of the series

   (a) $\displaystyle\sum_{k=0}^{\infty} \frac{2^k x^k}{k^2}$

   (b) $\displaystyle\sum_{k=0}^{\infty} \frac{x^{3k+1}}{(3k+1)!}$

   Solution:

   (a)

$$\lim_{k\to\infty} \frac{\left|\frac{2^{k+1}x^{k+1}}{(k+1)^2}\right|}{\left|\frac{2^k x^k}{k^2}\right|} = \lim_{k\to\infty} \frac{2^{k+1}|x|^{k+1}k^2}{2^k|x|^k(k+1)^2}$$
$$= 2|x| \lim_{k\to\infty} \frac{k^2}{k^2+1}$$
$$= 2|x| \lim_{k\to\infty} \frac{1}{1+k^{-2}}$$
$$= 2|x|.$$

   This is less than 1 when $|x| < 1/2$, so, by the Ratio Test, the radius of convergence for this series is $1/2$.

   (b)

$$\lim_{k\to\infty} \frac{\left|\frac{x^{3k+4}}{(3k+4)!}\right|}{\left|\frac{x^{3k+1}}{(3k+1)!}\right|} = \lim_{k\to\infty} \frac{|x|^{3k+4}(3k+1)!}{|x|^{3k+1}(3k+4)!}$$
$$= |x|^3 \lim_{k\to\infty} \frac{1}{(3k+4)(3k+3)(3k+2)}$$
$$= 0.$$

   By the Ratio Test, this series converges for all values of x, and its radius of convergence is ∞.

2. The region bounded by the curve $y = x^2$ and the line $y = 4$ is revolved about the y-axis. (See Figure 1.) Find the volume of the solid generated in this fashion.

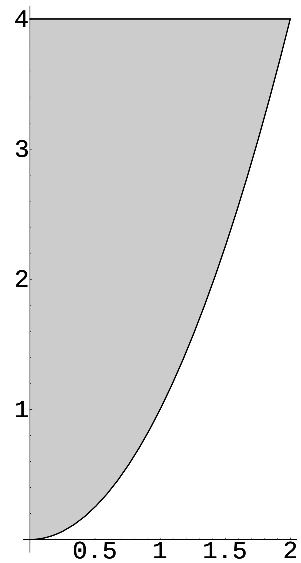

Figure 1: Problem 2

Solution: If $y = x^2$, then $x = \sqrt{y}$. Hence the volume of the solid given is

$$\begin{aligned} \pi \int_0^4 (\sqrt{y})^2 \, dy &= \pi \int_0^4 y \, dy \\ &= \left. \frac{\pi y^2}{2} \right|_0^4 = 8\pi. \end{aligned}$$

3. Show how the substitution $3x - 4 = u^3$ reduces the integral

$$\int \frac{x \, dx}{\sqrt[3]{3x - 4}}$$

to an integral of a polynomial. Then use what you have shown to evaluate the original integral.

Solution: If $u^3 = 3x - 4$, then $x = (u^3 + 4)/3$ and $dx = u^2\,du$. Hence,

$$\begin{aligned}
\int \frac{x\,dx}{\sqrt[3]{3x-4}} &= \int \frac{[(u^3+4)/3](u^2\,du)}{u} \\
&= \frac{1}{3}\int (u^4 + 4u)\,du \\
&= \frac{1}{15}u^5 + \frac{2}{3}u^2 + c \;=\; \frac{1}{15}(3x-4)^{5/3} + \frac{2}{3}(3x-4)^{2/3} + c.
\end{aligned}$$

4. Show how to use the Integral Test to determine whether or not the series

$$\sum_{k=1}^{\infty} ke^{-k}$$

converges.

Solution: Let $f(x) = xe^{-x}$. Then $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$. Now $e^{-x} > 0$ for all x, and $(1-x) < 0$ for all $x > 1$. Hence f is a positive decreasing function on $[1, \infty)$. By the Integral Test, the series $\sum_{k=1}^{\infty} ke^{-k}$ converges iff the improper integral $\int_1^{\infty} xe^{-x}\,dx$ converges. Taking $u = x$ and $dv = e^{-x}\,dx$, we have $du = dx$ and $v = -e^{-x}$. Therefore

$$\begin{aligned}
\int_1^{\infty} xe^{-x}\,dx &= \lim_{T\to\infty} \int_1^{T} xe^{-x}\,dx \\
&= \lim_{T\to\infty} \left[-xe^{-x}\Big|_0^T + \int_0^T e^{-x}\,dx \right] \\
&= \lim_{T\to\infty} (Te^{-T}) - \lim_{T\to\infty} e^{-x}\Big|_0^T \\
&= \lim_{T\to\infty} (Te^{-T}) - \lim_{T\to\infty} e^{-T} + 1.
\end{aligned}$$

Now $\lim_{T\to\infty} e^{-T} = 0$ and

$$\begin{aligned}
\lim_{T\to\infty} Te^{-T} &= \lim_{T\to\infty} \frac{T}{e^T} \\
&= \lim_{T\to\infty} \frac{1}{e^T} \;=\; 0 \text{ by L'Hôpital's rule.}
\end{aligned}$$

The integral therefore converges and so does the series.

5. Determine which of the following series converge and which diverge. Give your reasoning.

(a) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^n$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{3k^2+3}{k^3+3k}$

Solutions:

(a) This is a geometric series with common ratio $3/5 < 1$. It therefore converges.

(b) The harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. Moreover,

$$\lim_{k\to\infty} \frac{\left(\dfrac{3k^2+3}{k^3+3k}\right)}{\left(\dfrac{1}{k}\right)} = \lim_{k\to\infty} \frac{3k^3+3k}{k^3+3k} = \lim_{k\to\infty} \frac{3+3k^{-2}}{1+3k^{-2}} = 3.$$

By the Limit Comparison Test, the series $\displaystyle\sum_{k=1}^{\infty} \frac{3k^2+3}{k^3+3k}$ diverges.

6. Find the Maclaurin series (*i. e.*, the Taylor series centered at $x = 0$) for the function $f(x) = \ln(1 - x^2)$. What is its radius of convergence?

Solution: Let $\varphi(u) = \ln(1-u)$. Then when $|u| < 1$ we have

$$\varphi'(u) = -\frac{1}{(1-u)} = -1 - u - u^2 - u^3 - \cdots .$$

Thus if $|u| < 1$ we have

$$\varphi(u) = c - u - \frac{u^2}{2} - \frac{u^3}{3} - \frac{u^4}{4} - \cdots$$

and, because $\varphi(0) = 1$, we must have $c = 1$. Putting $u = x^2$, we have

$$\ln(1-x^2) = 1 - x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \frac{x^8}{4} - \cdots ,$$

for which the radius of convergence is 1.

7. Find the solution of the initial value problem:

$$\begin{aligned} y\frac{dy}{dx} &= 1+y^2; \\ y(0) &= -1. \end{aligned}$$

Solution: Separating variables and integrating, we obtain

$$\begin{aligned} \frac{y\,dy}{1+y^2} &= dx; \\ \int \frac{y\,dy}{1+y^2} &= \int dx; \\ \frac{1}{2}\ln(1+y^2) &= x+c; \\ \ln(1+y^2) &= 2x+c; \\ 1+y^2 &= Ce^{2x}; \\ y^2 &= Ce^{2x}-1; \\ y &= \pm\sqrt{Ce^{2x}-1}. \end{aligned}$$

But we are given $y=-1$ when $x=0$, so $-1=\pm\sqrt{C-1}$, whence $C=2$ and we must take the minus sign. The solution to the initial value problem is therefore $y=-\sqrt{2e^{2x}-1}$.

8. Does the integral

$$\int_2^\infty \frac{dx}{x(\ln x)^{3/2}}$$

converge? diverge?

Solution: We have

$$\int_2^\infty \frac{dx}{x(\ln x)^{3/2}} = \lim_{T\to\infty}\int_2^T \frac{dx}{x(\ln x)^{3/2}}.$$

If we take $u=\ln x$, then $du=dx/x$. Also, $x=2 \Rightarrow u=\ln 2$ and $x=T \Rightarrow u=\ln T$. Thus

$$\begin{aligned} \lim_{T\to\infty}\int_2^T \frac{dx}{x(\ln x)^{3/2}} &= \lim_{T\to\infty}\int_{\ln 2}^{\ln T}\frac{du}{u^{3/2}} \\ &= \lim_{T\to\infty}\left(\frac{-2}{u^{1/2}}\right)\Bigg|_{\ln 2}^{\ln T} \\ &= -\lim_{T\to\infty}\left(\frac{2}{(\ln T)^{1/2}}\right)+\left(\frac{2}{(\ln 2)^{1/2}}\right) = \frac{2}{\sqrt{\ln 2}}. \end{aligned}$$

The integral converges.

MTH 2410, Calculus II—Exam I **Feb. 7, 2006**

Instructions: Write out, *on your own paper*, complete solutions of following problems. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your work is due at 2:50 pm. You may keep this copy of the exam.

1. Find the limits:

   (a) $$\lim_{x\to 0^+} x \ln x$$

   (b) $$\lim_{x\to 0} \frac{x - \sin x}{x^3}$$

2. Use an appropriate substitution to evaluate the definite integral $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$.

3. Use integration by parts to find $\displaystyle\int x\cos(3x)\,dx$.

4. Find

   (a) $$\int_0^4 x\sqrt{9+x^2}\,dx$$

   (b) $$\int_0^1 \tan^{-1} x\,dx$$

5. Find $\displaystyle\int \frac{8x-1}{(2x-1)(x+1)}\,dx$.

6. Use Simpson's Rule with four subdivisions to find an approximate value of $\displaystyle\int_1^3 \frac{dx}{x}$. Show what calculations you must perform and give your answer correct to five digits to the right of the decimal point.

7. Is the integral $\displaystyle\int_0^\infty xe^{-x^2}\,dx$ convergent or divergent? If it is convergent, find its value.

8. Find $\displaystyle\int \frac{2x+5}{x^2+4x+8}\,dx$.

MTH 2410, Calculus II—Exam I **Feb. 7, 2006**

Instructions: Write out, *on your own paper*, complete solutions of following problems. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your work is due at 2:50 pm. You may keep this copy of the exam.

1. Find the limits:

   (a)
   $$\lim_{x \to 0^+} x \ln x$$

   (b)
   $$\lim_{x \to 0} \frac{x - \sin x}{x^3}$$

   Solution:

   (a) $\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} (\ln x)/(x^{-1})$, where both numerator and denominator become infinite as $x \to 0^+$. We may therefore attempt L'Hôpital's Rule. We get:

   $$\begin{aligned} \lim_{x \to 0^+} x \ln x &= \lim_{x \to 0^+} \frac{\ln x}{x^{-1}} \\ &= \lim_{x \to 0^+} \frac{x^{-1}}{-x^{-2}} \\ &= \lim_{x \to 0^+} -x \;=\; 0. \end{aligned}$$

   (b) $\lim_{x \to 0}(x - \sin x) = 0$ and $\lim_{x \to 0} x^3 = 0$. We may therefore attempt L'Hôpital's Rule. This leads to:

   $$\begin{aligned} \lim_{x \to 0} \frac{x - \sin x}{x^3} &= \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \\ &= \lim_{x \to 0} \frac{1 - \cos^2 x}{3x^2(1 + \cos x)} \\ &= \frac{1}{3} \lim_{x \to 0} \frac{\sin^2 x}{x^2(1 + \cos x)} \\ &= \frac{1}{3} \cdot \lim_{x \to 0} \left(\frac{\sin x}{x} \right)^2 \cdot \lim_{x \to 0} \left(\frac{1}{1 + \cos x} \right) \\ &= \frac{1}{3} \cdot 1 \cdot \frac{1}{2} \;=\; \frac{1}{6}. \end{aligned}$$

2. Use an appropriate substitution to evaluate the definite integral $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$.

Solution: Let $u = \ln x$. Then $du = dx/x$. Moreover, $u = 1$ when $x = e$ and $u = 4$ when $x = e^4$. So

$$\begin{aligned} \int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} &= \int_1^4 u^{-1/2} du \\ &= 2u^{1/2}\Big|_1^4 \\ &= 2\sqrt{4} - 2\sqrt{1} \;=\; 2. \end{aligned}$$

3. Use integration by parts to find $\displaystyle\int x\cos(3x)\,dx$.

Solution: Let $u = x$; $dv = \cos(3x)\,dx$. then $du = dx$ and $v = (1/3)\sin(3x)$. Thus

$$\begin{aligned} \int x\cos(3x)\,dx &= x \cdot \frac{1}{3}\sin(3x) - \frac{1}{3}\int \sin(3x)\,dx \\ &= \frac{x}{3}\sin(3x) + \frac{1}{9}\cos(3x) + C \end{aligned}$$

4. Find

(a)

$$\int_0^4 x\sqrt{9+x^2}\,dx$$

(b)

$$\int_0^1 \tan^{-1} x\,dx$$

Solution:

(a) Let $u = 9 + x^2$. Then $du = 2x\,dx$, or $dx = du/2$. Moreover, when $x = 0$, $u = 9$, and when $x = 4$, $u = 25$. Hence

$$\begin{aligned} \int_0^4 x\sqrt{9+x^2}\,dx &= \frac{1}{2}\int_9^{25} u^{1/2}\,du \\ &= \frac{1}{3}u^{3/2}\Big|_9^{25} \\ &= \frac{1}{3}(125-27) \;=\; \frac{98}{3} \end{aligned}$$

(b) Let $u = \tan^{-1} x$, $dv = dx$. Then $du = \dfrac{dx}{1+x^2}$, and we may take $v = x$. Integrating by parts, we have

$$\int_0^1 \tan^{-1} x\,dx = x\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x\,dx}{1+x^2} \tag{1}$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln(1+x^2)\Big|_0^1 = \frac{\pi}{4} - \ln\sqrt{2}. \tag{2}$$

5. Find $\displaystyle\int \frac{8x-1}{(2x-1)(x+1)}\,dx$.

Solution: Let A and B be such that

$$\begin{aligned}\frac{8x-1}{(2x-1)(x+1)} &= \frac{A}{2x-1} + \frac{B}{x+1} \\ &= \frac{A(x+1)+B(2x-1)}{(2x-1)(x+1)} \\ &= \frac{(A+2B)x+(A-B)}{(2x-1)(x+1)}.\end{aligned}$$

Thus $A + 2B = 8$, while $A - B = -1$. Subtracting the second of these latter equations from the first, we find that $3B = 9$, or $B = 3$. Then $A - 3 = -1$, so $A = 2$. Thus

$$\begin{aligned}\int \frac{8x-1}{(2x-1)(x+1)}\,dx &= \int \frac{2}{2x-1}\,dx + \int \frac{3}{x+1}\,dx \\ &= \ln|2x-1| + 3\ln|x+1| + C\end{aligned}$$

6. Use Simpson's Rule with four subdivisions to find an approximate value of $\displaystyle\int_1^3 \frac{dx}{x}$. Show what calculations you must perform and give your answer correct to five digits to the right of the decimal point.

Solution: We have $f(x) = 1/x$. The points $x_0 = 1$, $x_1 = 5/2$, $x_2 = 2$, $x_3 = 7/2$, and $x_4 = 3$ divide the interval $[1, 3]$ into four equal parts. Taking $h = 1/2$, Simpson's Rule then gives for the integral the approximate value

$$\begin{aligned}\frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)] &= \frac{1}{6}\left[1 + \frac{8}{3} + 1 + \frac{8}{5} + \frac{1}{3}\right] \\ &= \frac{11}{10} \;=\; 1.10000.\end{aligned}$$

7. Is the integral $\displaystyle\int_0^\infty xe^{-x^2}\,dx$ convergent or divergent? If it is convergent, find its value.

Solution: The integrand is continuous on $(0, \infty)$, so we must consider

$$\lim_{T\to\infty} \int_0^T xe^{-x^2}\, dx.$$

Let $u = x^2$. Then $du = 2x\, dx$, or $x\, dx = du/2$. Also, $u = 0$ when $x = 0$, and $u = T^2$ when $x = T$. Thus

$$\begin{aligned}
\lim_{T\to\infty} \int_0^T xe^{-x^2}\, dx &= \lim_{T\to\infty} \frac{1}{2} \int_0^{T^2} e^{-u}\, du \\
&= \frac{1}{2} \lim_{T\to\infty} \left(-e^{-u} \right)\Big|_0^{T^2} \\
&= \frac{1}{2} \left[\lim_{T\to 0} \left(-e^{-T^2} + 1 \right) \right] \\
&= \frac{1}{2}.
\end{aligned}$$

The improper integral converges to the value $1/2$.

8. Find $\displaystyle\int \frac{2x+5}{x^2+4x+8}\, dx$.

Solution: We have

$$\begin{aligned}
\int \frac{2x+5}{x^2+4x+8}\, dx &= \int \frac{(2x+4)+1}{x^2+4x+8}\, dx \\
&= \int \frac{2x+4}{x^2+4x+8}\, dx + \int \frac{dx}{(x^2+4x+4)+4} \\
&= \int \frac{2x+4}{x^2+4x+8}\, dx + \int \frac{dx}{(x+2)^2+4} \\
&= \ln|x^2+4x+8| + \frac{1}{2} \arctan\left(\frac{x+2}{2} \right) + C.
\end{aligned}$$

MTH 2410, Calculus II—Exam II **Mar. 9, 2006**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Do not use decimal approximations unless the nature of a problem requires them. Your work is due at 2:50 pm. You may keep this copy of the exam.

1. Find the volume generated when the region bounded by the curve $y = \sqrt{x}$, the x-axis, and the lines $x = 4$, $x = 9$ is rotated about the line $y = -1$.

2. Find the area of the region that lies inside of the curve $r = (2\sqrt{2} - 1)\cos\theta$ but outside of the curve $r = 2 - \cos\theta$ (See Figure 1.)

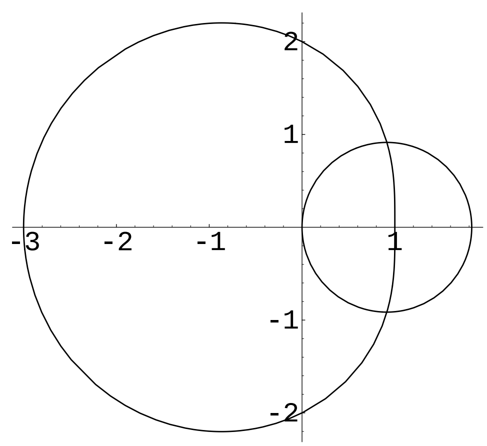

Figure 1: Problem 2

3. The base of a solid is the triangle cut from the first quadrant by the line $4x + 3y = 12$. Every cross-section of the solid perpendicular to the y-axis is a semi-circle. Find the volume of the solid.

4. Use Euler's Method with a step-size of 1/10 to find an approximate value for $y(13/10)$ if $y(1) = 1$ and

$$\frac{dy}{dx} = x + y.$$

Show all of the intermediate y values you obtain. Either give your answers as fractions of integers or maintain a minimum of five digits accuracy to the right of the decimal point.

5. Find $y(x)$ given that $e^{-x}y' = \sqrt{1-y^2}$ with $y(0) = 0$. What is the exact value of $y(x)$ when $x = \ln\left(1 - \frac{\pi}{6}\right)$?

6. Use the accompanying figures to determine how many subdivisions will be required to obtain a Midpoint Rule approximation, accurate to within 10^{-5} for

$$\int_0^2 f(x)\,dx,$$

where $f(x) = \cos x^3$. Be sure to explain all of your reasoning. (*Do not* try to find the Midpoint Rule approximation in question.)

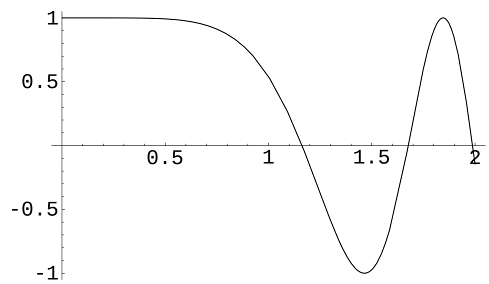

Figure 2: Problem 6, Graph of $f(x)$

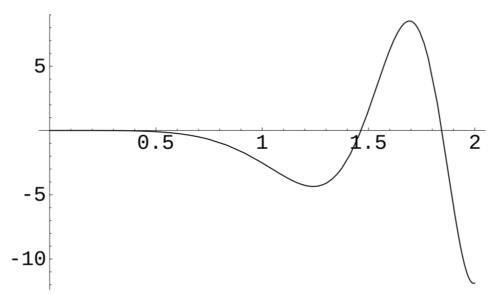

Figure 3: Problem 6, Graph of $f'(x)$

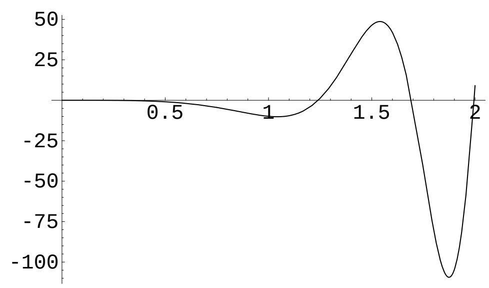

Figure 4: Problem 6, Graph of $f''(x)$

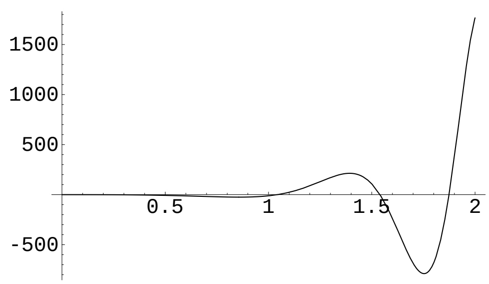

Figure 5: Problem 6, Graph of $f^{(3)}(x)$

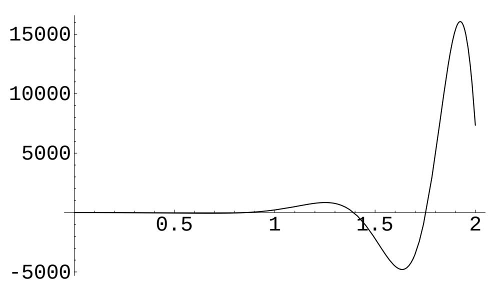

Figure 6: Problem 6, Graph of $f^{(4)}(x)$

MTH 2410, Calculus II—Exam II **Mar. 9, 2006**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Do not use decimal approximations unless the nature of a problem requires them. Your work is due at 2:50 pm. You may keep this copy of the exam.

1. Find the volume generated when the region bounded by the curve $y = \sqrt{x}$, the x-axis, and the lines $x = 4$, $x = 9$ is rotated about the line $y = -1$.

   Solution: A plane perpendicular to the x-axis intersects this solid in a washer whose outer radius is $\sqrt{x} + 1$ and whose inner radius is 1. The area of such a washer is

$$\begin{aligned} A(x) &= \pi[(\sqrt{x}+1)^2 - 1^2] \\ &= \pi(x + 2\sqrt{x}). \end{aligned}$$

   The solid extends from $x = 4$ to $x = 9$, so the volume is

$$\begin{aligned} \pi \int_4^9 (x + 2\sqrt{x})\, dx &= \pi \left(\frac{x^2}{2} + \frac{4x^{3/2}}{3} \right) \Bigg|_4^9 \\ &= \pi \left(\frac{81}{2} + \frac{108}{3} \right) - \pi \left(\frac{16}{2} + \frac{32}{3} \right) \\ &= \frac{347}{6}\pi. \end{aligned}$$

2. Find the area of the region that lies inside of the curve $r = (2\sqrt{2} - 1)\cos\theta$ but outside of the curve $r = 2 - \cos\theta$ (See Figure 1.)

   Solution: In order to locate the intersection points of the two curves, we must solve the equation

$$(2\sqrt{2} - 1)\cos\theta = 2 - \cos\theta.$$

   Adding $\cos\theta$ to both sides, we find that $2\sqrt{2}\cos\theta = 2$, or that $\cos\theta = 1/\sqrt{2}$. Hence $\theta = \pm\pi/4$. Our area is therefore given by

$$\frac{1}{2} \int_{-\pi/4}^{\pi/4} [(2\sqrt{2} - 1)^2 \cos^2\theta - (2 - \cos\theta)^2]\, d\theta$$

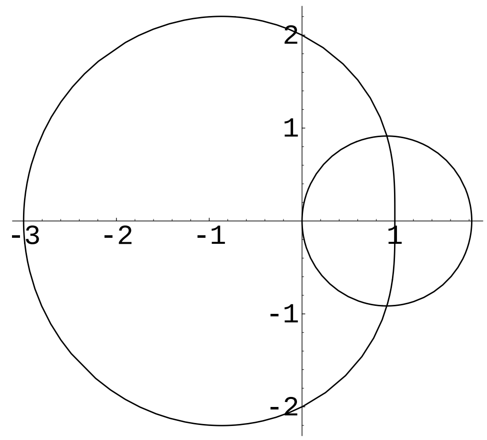

Figure 1: Problem 2

$$
\begin{aligned}
&= \int_{-\pi/4}^{\pi/4} [(4-2\sqrt{2})\cos^2\theta + 2\cos\theta - 2]\, d\theta \\
&= \int_{-\pi/4}^{\pi/4} [(2-\sqrt{2})\cos 2\theta + 2\cos\theta - \sqrt{2}]\, d\theta \\
&= \left(\frac{2-\sqrt{2}}{2} \sin 2\theta + 2\sin\theta - \sqrt{2}\,\theta \right) \Bigg|_{-\pi/4}^{\pi/4} \\
&= \frac{4 + 2\sqrt{2} - \sqrt{2}\,\pi}{2}.
\end{aligned}
$$

3. The base of a solid is the triangle cut from the first quadrant by the line $4x + 3y = 12$. Every cross-section of the solid perpendicular to the y-axis is a semi-circle. Find the volume of the solid.

   Solution: The base of the cross-section of the solid made by the plane $y = t$ is a semi-circle with a diameter whose left end-point is at $x = 0$ and whose right end-point is at $x = 3 - 3t/4$. Hence the radius of the semi-circle is $\frac{1}{2}(3 - 3t/4)$, and $A(y) = \frac{\pi}{8}(3 - 3t/4)^2$. The solid extends from $y = 0$, to $y = 4$. The required volume is therefore

$$
\begin{aligned}
\frac{\pi}{8} \int_0^4 \left(3 - \frac{3t}{4}\right)^2 dt &= -\frac{\pi}{8} \cdot \frac{4}{3} \left(3 - \frac{3t}{4}\right)^3 \Bigg|_0^4 \\
&= \frac{3\pi}{2}
\end{aligned}
$$

4. Use Euler's Method with a step-size of $1/10$ to find an approximate value for $y(13/10)$ if $y(1) = 1$ and

$$\frac{dy}{dx} = x + y.$$

Show all of the intermediate y values you obtain. Either give your answers as fractions of integers or maintain a minimum of five digits accuracy to the right of the decimal point.

Solution: Euler's Method with $h = 1/10$ for this initial value problem requires that

$$\begin{aligned} x_0 &= 1; \\ y_0 &= 1; \\ x_k &= x_{k-1} + \frac{1}{10}, \text{ when } k > 0; \\ y_k &= y_{k-1} + \frac{1}{10}(x_{k-1} + y_{k-1}), \text{ when } k > 0. \end{aligned}$$

Thus, we find that

$$\begin{aligned} x_1 &= 1 + \frac{1}{10} = \frac{11}{10} \\ y_1 &= 1 + \frac{1}{10}(1+1) = \frac{6}{5} \\ x_2 &= \frac{11}{10} + \frac{1}{10} = \frac{6}{5} \\ y_2 &= \frac{6}{5} + \frac{1}{10}\left(\frac{11}{10} + \frac{6}{5}\right) = \frac{143}{100} \\ x_3 &= \frac{6}{5} + \frac{1}{10} = \frac{13}{10} \\ y_3 &= \frac{143}{100} + \frac{1}{10}\left(\frac{6}{5} + \frac{143}{100}\right) = \frac{1693}{1000}. \end{aligned}$$

5. Find $y(x)$ given that $e^{-x}y' = \sqrt{1 - y^2}$ with $y(0) = 0$. What is the exact value of $y(x)$ when $x = \ln\left(1 - \frac{\pi}{6}\right)$?

Solution: Separating the variables, we obtain

$$\frac{dy}{\sqrt{1-y^2}} = e^x\,dx.$$

Thus,

$$\begin{aligned}
\int_0^y \frac{dt}{\sqrt{1-t^2}} &= \int_0^x e^s\,ds; \\
\arcsin t\Big|_0^y &= e^s\Big|_0^x; \\
\arcsin(y) - \arcsin(0) &= e^x - e^0; \\
\arcsin(y) &= e^x - 1; \\
y = \sin(e^x - 1) &
\end{aligned}$$

is the solution to the given initial value problem. When $x = \ln(1-\pi/6)$, we have

$$\begin{aligned}
y &= \sin\left(e^{\ln(1-\pi/6)} - 1\right) \\
&= \sin\left[\left(1 - \frac{\pi}{6}\right) - 1\right] \\
&= -\sin\frac{\pi}{6} = -\frac{1}{2}.
\end{aligned}$$

6. Use the accompanying figures to determine how many subdivisions will be required to obtain a Midpoint Rule approximation, accurate to within 10^{-5} for

$$\int_0^2 f(x)\,dx,$$

where $f(x) = \cos x^3$. Be sure to explain all of your reasoning. (*Do not* try to find the Midpoint Rule approximation in question.)

Solution: The error in the Midpoint Rule approximation to $\int_a^b f(x)\,dx$ is bounded by $\dfrac{M(b-a)^3}{24n^2}$, where n is the number of subdivisions and M is any number for which $|f''(x)| \le M$ for all x such that $a \le x \le b$. From Figure 4, we see that we can take $M = 110$ for $f(x) = \cos x^3$. We are given $a = 0$ and $b = 2$, so $(b-a)^3 = 8$. Hence, we must choose n so that it satisfies

$$\frac{110 \cdot 8}{24n^2} \le 10^{-5};$$

$$\begin{aligned} \frac{1.1 \times 10^7}{3} &\leq n^2; \\ \sqrt{3.6666667 \times 10^6} &\leq n^2; \\ 1914.85 &\leq n. \end{aligned}$$

We will need at least 1915 subdivisions.

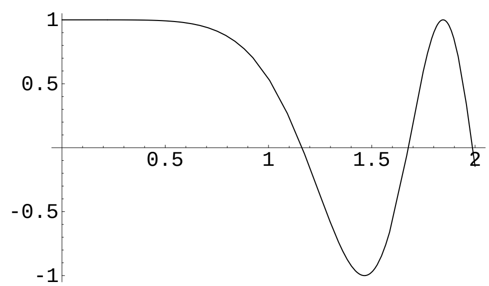

Figure 2: Problem 6, Graph of $f(x)$

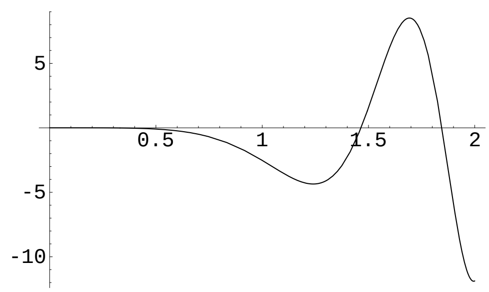

Figure 3: Problem 6, Graph of $f'(x)$

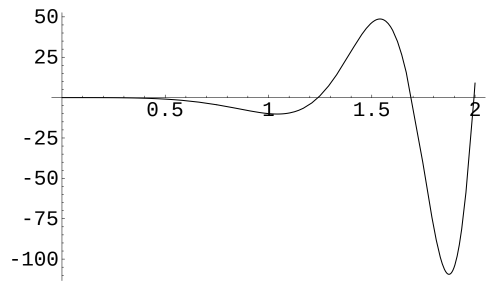

Figure 4: Problem 6, Graph of $f''(x)$

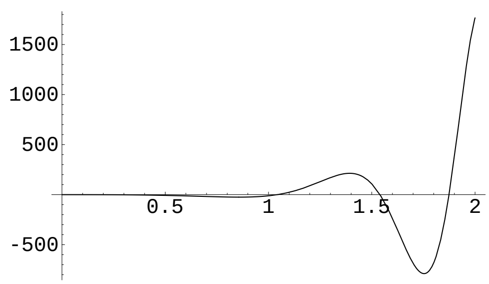

Figure 5: Problem 6, Graph of $f^{(3)}(x)$

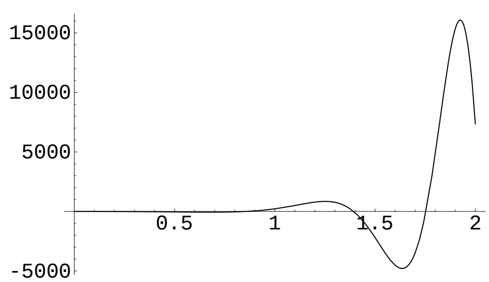

Figure 6: Problem 6, Graph of $f^{(4)}(x)$

MTH 2410, Calculus II—Optional Makeup Exam **Mar. 30, 2006**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Do not use decimal approximations unless the nature of a problem requires them. Your work is due at 2:50 pm. You may keep this copy of the exam.

1. Use Newton's Method to find an approximate solution for the equation
$$x^4 - 3x + 1 = 0.$$
Take $x_0 = 1.1$ and give both x_1 and x_2 to at least five digits to the right of the decimal.

2. A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?

3. Are the improper integrals convergent or divergent? If convergent, give the value.
   (a) $\displaystyle\int_0^{\pi/2} \sec x\, dx$
   (b) $\displaystyle\int_0^{\infty} \frac{dx}{1+x^2}$

4. Find the length of the polar curve $r = e^{2\theta}$, $0 \le \theta \le 2\pi$.

5. Find the volume generated when the region bounded by the curve $2x+y-6=0$ and the lines $x = 1$, $y = 2$ is rotated about the line $x = -1$.

6. Find the solution of the differential equation $\dfrac{dP}{dt} = \sqrt{Pt}$ that satisfies the initial condition $P(1) = 2$.

7. Æthelbert needs an approximate value for the integral
$$\int_0^2 \frac{x^3}{\sqrt{1+x^3}}\, dx.$$
He plans to use Simpson's Rule, and he wants to be sure that the error in his approximation is no more than 1×10^{-5}. Make approriate use of one or more of the accompanying figures, in which f is the integrand, to determine how many subdivisions he will need.

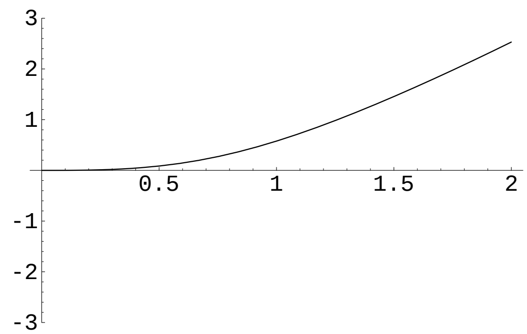

Figure 1: $y = f(x)$

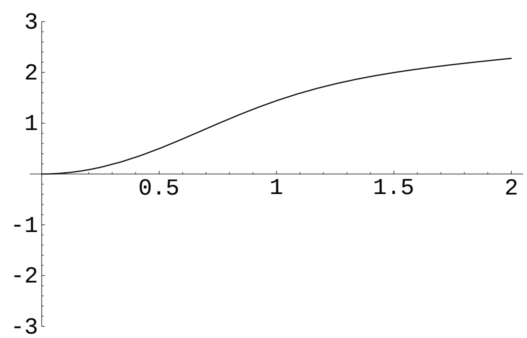

Figure 2: $y = f'(x)$

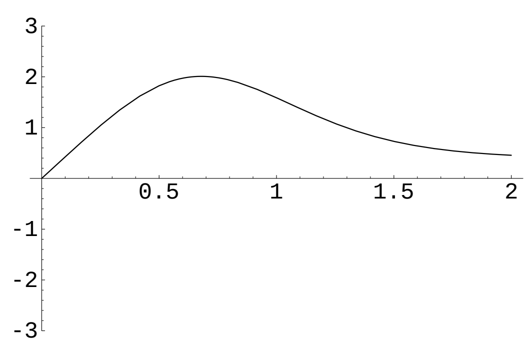

Figure 3: $y = f''(x)$

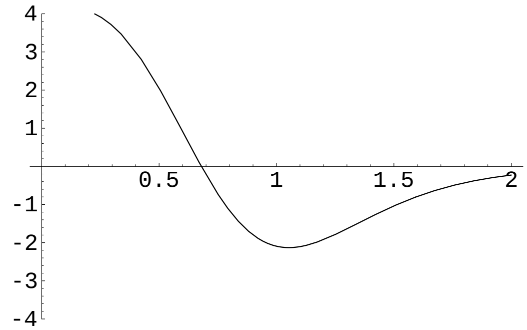

Figure 4: $y = f^{(3)}(x)$

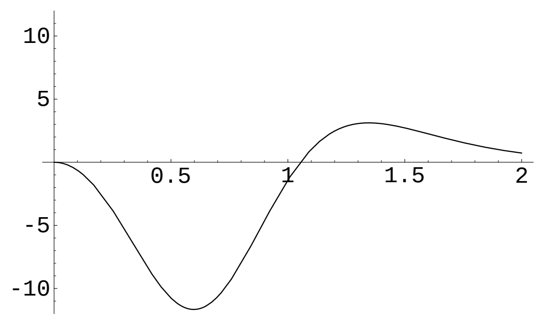

Figure 5: $y = f^{(4)}(x)$

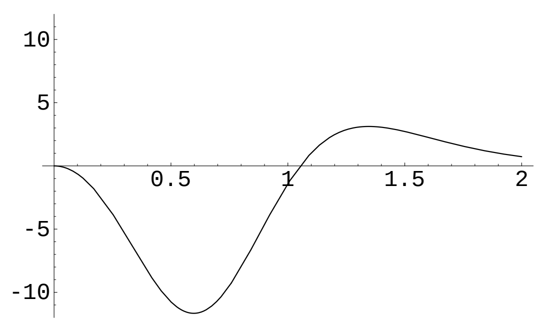

Figure 6: $y = f^{(5)}(x)$

MTH 2410, Calculus II—Optional Makeup Exam **Mar. 30, 2006**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Do not use decimal approximations unless the nature of a problem requires them. Your work is due at 2:50 pm. You may keep this copy of the exam.

1. Use Newton's Method to find an approximate solution for the equation

$$x^4 - 3x + 1 = 0.$$

Take $x_0 = 1.1$ and give both x_1 and x_2 to at least five digits to the right of the decimal.

Solution: Taking $f(x) = x^4 - 3x + 1$, Newton's Method for the equation $f(x) = 0$ is

$$\begin{aligned} x_{k+1} &= x_k - \frac{f(x_k)}{f'(x_k)} \\ &= x_k - \frac{(x_k)^4 - 3x_k + 1}{4(x_k)^3 - 3} \\ &= \frac{3(x_k)^4 - 1}{4(x_k)^3 - 3}. \end{aligned}$$

Hence, putting $x_0 = 1.1$, we find that

$$\begin{aligned} x_1 &= \frac{3(x_0)^4 - 1}{4(x_0)^3 - 3} = 1.4596816; \\ x_2 &= \frac{3(x_1)^4 - 1}{4(x_1)^3 - 3} = 1.3367301. \end{aligned}$$

2. A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?

Solution: The cable weighs 2 lb per foot, so the force exerted when x feet of cable still hang from the building is $2x$ pounds. Hence total work is

$$\begin{aligned} \int_0^{100} 2x\,dx &= x^2 \Big|_0^{100} \\ &= 10,000 \text{ ft-lb}. \end{aligned}$$

3. Are the improper integrals convergent or divergent? If convergent, give the value.

(a) $\displaystyle\int_0^{\pi/2} \sec x \, dx$

(b) $\displaystyle\int_0^{\infty} \frac{dx}{1+x^2}$

Solution:

(a)

$$\begin{aligned}
\int_0^{\pi/2} \sec x \, dx &= \lim_{T \to \pi/2^-} \int_0^T \sec x \, dx \\
&= \lim_{T \to \pi/2^-} \ln|\sec x + \tan x| \Big|_0^T \\
&= \lim_{T \to \pi/2^-} \ln|\sec T + \tan T| - \ln 1.
\end{aligned}$$

This limit is infinite, so the improper integral diverges.

(b)

$$\begin{aligned}
\int_0^{\infty} \frac{dx}{1+x^2} &= \lim_{T \to \infty} \int_0^T \frac{dx}{1+x^2} \\
&= \lim_{T \to \infty} \arctan x \Big|_0^T \\
&= \lim_{T \to \infty} \arctan T - \arctan 0 \\
&= \frac{\pi}{2}.
\end{aligned}$$

This improper integral converges to the value $\pi/2$.

4. Find the length of the polar curve $r = e^{2\theta}$, $0 \leq \theta \leq 2\pi$.

Solution:

$$
\begin{aligned}
s &= \int_0^{2\pi} \sqrt{[r'(\theta)]^2 + [r(\theta)]^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{4e^{4\theta} + e^{4\theta}}\, d\theta \\
&= \sqrt{5} \int_0^{2\pi} e^{2\theta}\, d\theta \\
&= \frac{\sqrt{5}}{2} e^{2\theta} \Big|_0^{2\pi} \;=\; \frac{\sqrt{5}}{2}\big(e^{4\pi} - 1\big).
\end{aligned}
$$

5. Find the volume generated when the region bounded by the curve $2x+y-6=0$ and the lines $x=1$, $y=2$ is rotated about the line $x=-1$.

   Solution: We set up the integral on the y-axis, using the method of washers. The equation $2x+y-6=0$ gives $x=3-y/2)$. We are revolving the region about the line $x=-1$, so the outer radius of a washer is $(3-y/2)-(-1)=4-y/2$, while the inner radius is $1-(-1)=2$. Thus

$$
\begin{aligned}
V &= \pi \int_2^4 \left[\left(4 - \frac{y}{2}\right)^2 - 2^2\right] dy \\
&= \pi \int_2^4 \left(\frac{y^2}{4} - 4y + 12\right) dy \\
&= \frac{14\pi}{3}.
\end{aligned}
$$

   Note: If we set this up along the x-axis, we arrive, through the method of cylindrical shells, at the integral

$$
2\pi \int_1^2 (x+1)(4-2x)\, dx,
$$

   which has the same value.

6. Find the solution of the differential equation $\dfrac{dP}{dt} = \sqrt{Pt}$ that satisfies the initial condition $P(1)=2$.

Solution: If $\frac{dP}{dt} = \sqrt{Pt}$, then

$$\begin{aligned}
\frac{dP}{\sqrt{P}} &= \sqrt{t}\,dt \\
\int \frac{dP}{\sqrt{P}} &= \int \sqrt{t}\,dt \\
2P^{1/2} &= \frac{2}{3}t^{3/2} + c \\
P &= \left(\frac{1}{3}t^{3/2} + c\right)^2
\end{aligned}$$

But $P = 2$ when $t = 1$, so

$$2 = \left(\frac{1}{3} + c\right)^2.$$

It follows that $c = \pm\sqrt{2} - 1/3$, and we must make a choice. If we choose the minus sign, then

$$P = \left(\frac{1}{3}t^{3/2} - \sqrt{2} - \frac{1}{3}\right)^2, \text{ so that} \tag{1}$$

$$\left.\frac{dP}{dt}\right|_{t=1} = \sqrt{1}\left(\frac{1}{3}\cdot 1^{3/2} - \sqrt{2} - \frac{1}{3}\right) = -\sqrt{2} < 0. \tag{2}$$

But from the original differential equation, we see that $\frac{dP}{dt}$ must be *positive*. It follows that that the solution we seek is given by

$$P = \left(\frac{1}{3}t^{3/2} + \sqrt{2} - \frac{1}{3}\right)^2.$$

7. Æthelbert needs an approximate value for the integral

$$\int_0^2 \frac{x^3}{\sqrt{1+x^3}}\,dx.$$

He plans to use Simpson's Rule, and he wants to be sure that the error in his approximation is no more than 1×10^{-5}. Make appropriate use of one or more of the accompanying figures, in which f is the integrand, to determine how many subdivisions he will need.

Solution: The error E in the Simpson's Rule approximation for $\int_a^b f(x)\,dx$ with n subdivisions always satisfies the inequality

$$|E| \;\leq\; \frac{M(b-a)^5}{180n^4},$$

where M may be any number for which $|f^{(4)}(x)| \leq M$ whenever $a \leq x \leq b$. In this case, we have $a = 0$, $b = 2$, and, from Figure 5 we see that we may take M to be just about anything larger than 12. To be on the safe side, let's take $M = 15$. We must then solve the inequality

$$\frac{15 \cdot 2^5}{180n^4} \;\leq\; 10^{-5},$$

which is equivalent to

$$\frac{800000}{3} \;\leq\; n^4, \text{ or}$$

$$n \;\geq\; \sqrt[4]{\frac{800000}{3}} \;\sim\; 22.73$$

Thus, we will want n to be an even number larger than 22.73. $n = 24$ will work. The smallest acceptable value for M is 12. Taking $M = 12$ would allow $n = 22$.

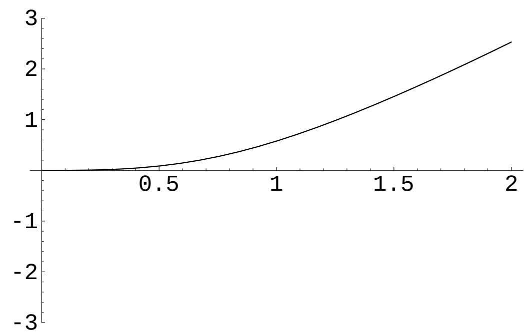

Figure 1: $y = f(x)$

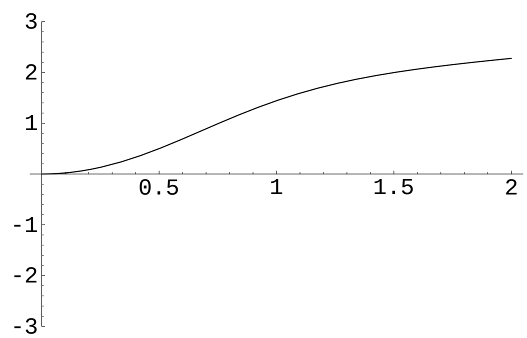

Figure 2: $y = f'(x)$

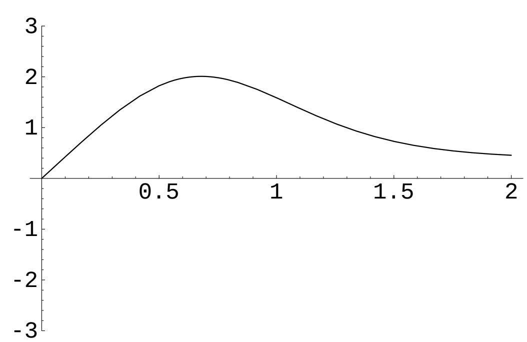

Figure 3: $y = f''(x)$

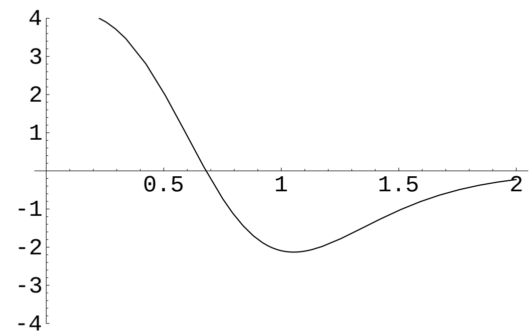

Figure 4: $y = f^{(3)}(x)$

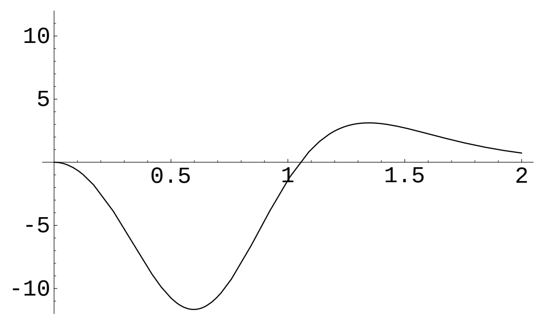

Figure 5: $y = f^{(4)}(x)$

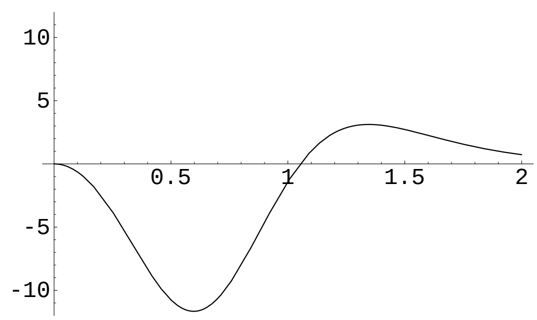

Figure 6: $y = f^{(5)}(x)$

MTH 2410, Calculus II—Exam IV **April 27, 2006**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Do not use decimal approximations unless the nature of a problem requires them. Your work is due at 2:50 pm. You may keep this copy of the exam.

1. Find the Taylor series in powers of $(x-3)$ for the function
$$f(x) = 1 + x + x^2 + x^3.$$

2. Find the radius of convergence and the interior of the interval of convergence for the series
$$\sum_{k=0}^{\infty} \frac{(k+1)^2}{2^k}(x+5)^k.$$

3. Find the radius of convergence and the interior of the interval of convergence for the series
$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}.$$

4. Find the radius of convergence and the interior of the interval of convergence for the series
$$\sum_{k=1}^{\infty} \frac{k!\, x^k}{k^k}.$$

5. Find the Taylor series in powers of $(x-2)$ for the function
$$f(x) = \ln(3-x).$$
Give both the radius of convergence and the interior of the interval of convergence, together with the reasoning that supports your assertions. [Hint: Use the geometric series trick and the fact that $3 - x = 1 - (x-2)$.]

6. Show how to find N so that whenever $n \geq N$ we have
$$\left| \frac{1}{1-x} - (1 + x + x^2 + \cdots + x^n) \right| < \frac{1}{75}$$
for every x that satisfies $-7/8 \leq x \leq 7/8$. Give the reasoning that supports your procedure.

7. Determine how many terms of the Maclaurin series for the cosine function are needed to guarantee that using the Maclaurin polynomial in place of the cosine function results in an error no bigger than $1/500$ throughout the interval $[-\pi/2, \pi/2]$. Give your reasoning.

MTH 2410, Calculus II—Exam IV **April 27, 2006**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Do not use decimal approximations unless the nature of a problem requires them. Your work is due at 2:50 pm. You may keep this copy of the exam.

1. Find the Taylor series in powers of $(x-3)$ for the function

$$f(x) = 1 + x + x^2 + x^3.$$

Solution:

$$\begin{aligned}
f(x) &= 1 + x + x^2 + x^3 \Rightarrow f(3) = 1 + 3 + 9 + 27 = 40;\\
f'(x) &= 1 + 2x + 3x^2 \Rightarrow f'(3) = 1 + 6 + 27 = 34;\\
f''(x) &= 2 + 6x \Rightarrow f''(3) = 2 + 18 = 20;\\
f^{(3)}(x) &= 6 \Rightarrow f^{(3)}(3) = 6;\\
f^{(k)}(x) &= 0 \text{ for all } k > 3 \Rightarrow f^{(k)}(3) = 0 \text{ for all } k > 3.
\end{aligned}$$

Hence, $f(x) = f(3) + f'(3)(x-3) + \frac{1}{2}f''(3)(x-3)^2 + \frac{1}{6}f^{(3)}(3)(x-3)^3$, or

$$f(x) = 40 + 34(x-3) + 10(x-3)^2 + (x-3)^3.$$

2. Find the radius of convergence and the interior of the interval of convergence for the series

$$\sum_{k=0}^{\infty} \frac{(k+1)^2}{2^k}(x+5)^k.$$

Solution: We apply the Ratio Test:

$$\begin{aligned}
\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} &= \lim_{k\to\infty} \frac{(k+2)^2}{2^{k+1}} \cdot \frac{2^k}{(k+1)^2}\\
&= \frac{1}{2} \lim_{k\to\infty} \left(\frac{1+2/k}{1+1/k}\right)^2\\
&= \frac{1}{2}.
\end{aligned}$$

The radius of convergence is therefore 2 and the interior of the interval of convergence is $(-7, -3)$.

3. Find the radius of convergence and the interior of the interval of convergence for the series

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}.$$

Solution: We apply the Ratio Test:

$$\begin{aligned}\lim_{k\to\infty}\frac{|a_{k+1}|}{|a_k|} &= \lim_{k\to\infty}\frac{1}{(2k+2)!}\cdot\frac{(2k)!}{1}\\ &= \lim_{k\to\infty}\frac{1}{(2k+1)(2k+2)}\\ &= 0.\end{aligned}$$

The radius of convergence is therefore ∞ and the interior of the interval of convergence is $(-\infty,\infty)$.

4. Find the radius of convergence and the interior of the interval of convergence for the series

$$\sum_{k=1}^{\infty}\frac{k!\,x^k}{k^k}.$$

Solution: We apply the Ratio Test:

$$\begin{aligned}\lim_{k\to\infty}\frac{|a_{k+1}|}{|a_k|} &= \lim_{k\to\infty}\frac{(k+1)!}{(k+1)^{k+1}}\cdot\frac{k^k}{k!}\\ &= \lim_{k\to\infty}\left(\frac{k}{k+1}\right)^k\\ &= \lim_{k\to\infty} e^{k\ln[k/(k+1)]}\\ &= e^{\lim_{k\to\infty}\{k\ln[k/(k+1)]\}}\end{aligned}$$

To evaluate the limit in the exponent of the latter expression, we rewrite it:

$$\lim_{k\to\infty} k\ln\frac{k}{k+1} = \lim_{k\to\infty}\frac{\left(\ln\dfrac{k}{k+1}\right)}{\left(\dfrac{1}{k}\right)},$$

and we observe that both numerator and denominator of the resulting compound fraction go to zero as k grows without bound. We may therefore attempt l'Hôpital's rule, which leads us to evaluate

$$\lim_{k\to\infty}\frac{\dfrac{k+1}{k}\cdot\dfrac{(k+1)-k}{(k+1)^2}}{-\dfrac{1}{k^2}} = -\lim_{k\to\infty}\frac{1}{\left(1+\dfrac{1}{k}\right)} = -1.$$

Consequently,

$$\lim_{k\to\infty}\frac{|a_{k+1}|}{|a_k|} = e^{-1},$$

so that the radius of convergence is e and the interior of the interval of convergence is $(-e,e)$.

5. Find the Taylor series in powers of $(x-2)$ for the function

$$f(x) = \ln(3-x).$$

Give both the radius of convergence and the interior of the interval of convergence, together with the reasoning that supports your assertions. [Hint: Use the geometric series trick and the fact that $3-x = 1-(x-2)$.]

Solution: If $f(x) = \ln(3-x)$, then

$$\begin{aligned} f'(x) &= -\frac{1}{3-x} \\ &= -\frac{1}{1-(x-2)} \\ &= -\sum_{k=0}^{\infty}(x-2)^k, \text{ when } |x-2|<1. \end{aligned}$$

Hence,

$$f(x) = c - \sum_{k=0}^{\infty} \frac{(x-2)^{k+1}}{k+1}, \text{ also when } |x-2|<1.$$

But $f(2) = \ln(3-2) = \ln 1 = 0$, and putting $x=2$ in the series then gives $c=0$. Hence,

$$f(x) = -\sum_{k=0}^{\infty} \frac{(x-2)^{k+1}}{k+1} \text{ when } |x-2|<1.$$

6. Show how to find N so that whenever $n \geq N$ we have

$$\left|\frac{1}{1-x} - (1+x+x^2+\cdots+x^n)\right| < \frac{1}{75}$$

for every x that satisfies $-7/8 \leq x \leq 7/8$. Give the reasoning that supports your procedure.

Solution: We have

$$\left|\frac{1}{1-x} - (1+x+x^2+\cdots+x^n)\right| = \left|\frac{x^{n+1}}{1-x}\right|,$$

, and we want to be sure that this latter fraction is smaller than $1/75$ when $-7/8 \leq x \leq 7/8$. Under the latter circumstances, $|1-x|$, which is the distance from 1 to x must be $1/8$ or larger. Hence

$$\left|\frac{x^{n+1}}{1-x}\right| \leq \frac{|x|^{n+1}}{(1/8)} = 8|x|^{n+1},$$

and it suffices to be sure that $|x|^{n+1} \leq 1/(8 \cdot 75) = 1/600$. But $|x| < 7/8$, so we must be sure that $(7/8)^{n+1} \leq 1/600$. This latter inequality is equivalent, in turn, to the inequalities:

$$(n+1)\ln\frac{7}{8} \leq -\ln 600;$$
$$n+1 \geq -\frac{\ln 600}{\ln(7/8)};$$
$$n \geq -\frac{\ln 600}{\ln(7/8)} - 1 \sim 46.9.$$

We will therefore need to take $N = 47$.

7. Determine how many terms of the Maclaurin series for the cosine function are needed to guarantee that using the Maclaurin polynomial in place of the cosine function results in an error no bigger than $1/500$ throughout the interval $[-\pi/2, \pi/2]$. Give your reasoning.

Solution: Taylor's theorem says that for $-\pi/2 \leq x \leq \pi/2$

$$\cos x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + R_{2n}(x),$$

where

$$|R_{2n}(x)| \leq \frac{M|x|^{2n+1}}{(2n+1)!},$$

provided that $|f^{(2n+1)}(x)| \leq M$ for every x in $[-\pi/2, \pi/2]$. But every derivative of f is either a sine function or a cosine function, possibly with a minus sign attached, and so $|f^{(2n+1)}(x)| \leq 1$ for every x in $[-\pi/2, \pi/2]$. Also, we are given $|x| < \pi/2$, so we need to find n for which

$$\frac{(\pi/2)^{2n+1}}{(2n+1)!} \leq \frac{1}{500}.$$

By direct calculation, we find that when $n = 3$, the quantity on the left side of this inequality is about 0.0047, which is too large. When $n = 4$ the quantity on the left is about 0.00016, and this satisfies the inequality. We therefore need all terms through $n = 4$. Because we started counting with $k = 0$, we will need five (non-zero) terms of the Maclaurin series for the cosine function to guarantee the required accuracy. We can be sure that

$$\left|\cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} - \frac{x^8}{40320}\right| \leq \frac{1}{500}$$

when $-\pi/2 \leq x \leq \pi/2$.

MTH 2410, Calculus II—Final Exam **May 9, 2006**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Do not use decimal approximations unless the nature of a problem requires them. Your work is due at 2:50 pm. You may keep this copy of the exam.

1. Find the limits:

   (a) $\lim_{x \to 0^+} x \ln x$

   (b) $\lim_{x \to 0} \dfrac{x - \sin x}{x^3}$

2. Solve the initial value problem:

$$\frac{x^2}{y^2 - 8}\frac{dy}{dx} = \frac{1}{2y};$$
$$y(1) = 3$$

3. Is the integral $\int_0^\infty xe^{-x^2}\,dx$ convergent or divergent? If it is convergent, find its value.

4. Find the volume generated when the region bounded by the curve $y = \sqrt{x}$, the x-axis, and the lines $x = 4$, $x = 9$ is rotated about the line $y = -1$.

5. The base of a solid is the region in the first quadrant bounded by the y-axis, the line $y = 4$, and the curve $y = x^2$. Every cross-section of the solid perpendicular to the y-axis is a semi-circle. (See Figure 1, which depicts the solid and two of the intersecting planes.) Find the volume of the solid.

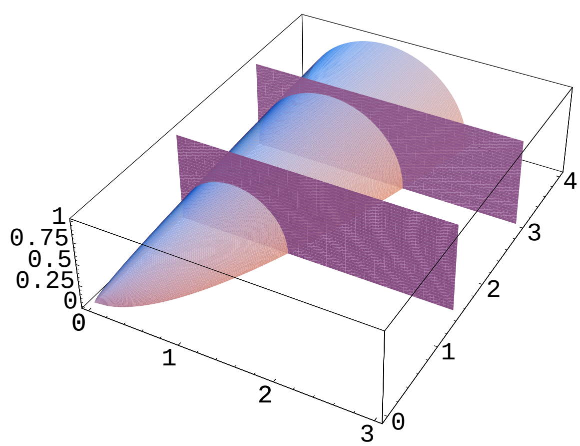

Figure 1: Problem 5

6. Make appropriate use of any of Figures 2–8 to determine how many subdivisions are needed to evaluate π to six decimal places by approximating the integral

$$\pi = \int_0^1 \frac{4\,dx}{1+x^2}$$

to within 1×10^{-6} using Simpson's Rule. Be sure to explain which figures you use—as well as the uses to which you put them.

7. Determine how many terms of the Maclaurin series for the cosine function are needed to guarantee that using the Maclaurin polynomial in place of the cosine function results in an error no bigger than $1/500$ throughout the interval $[-\pi/2, \pi/2]$. Give your reasoning.

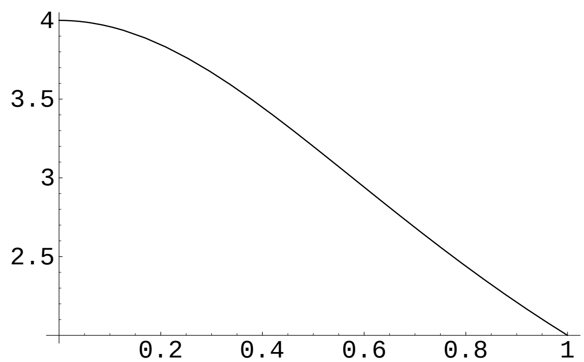

Figure 2: Problem 6: Graph of $f(x)$

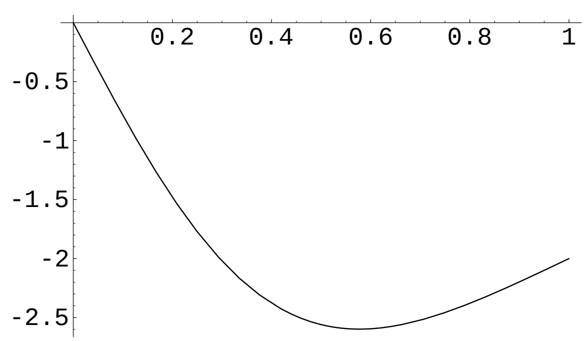

Figure 3: Problem 6: Graph of $f'(x)$

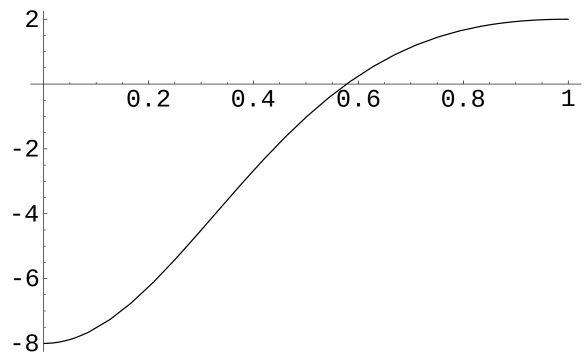

Figure 4: Problem 6: Graph of $f''(x)$

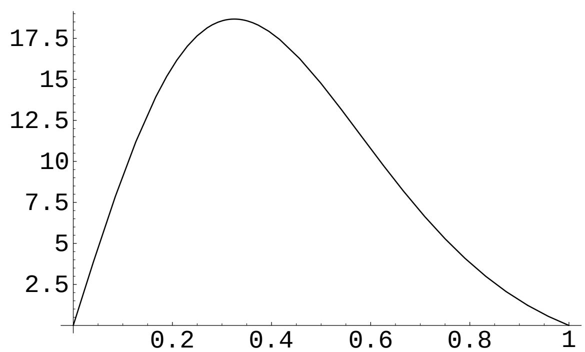

Figure 5: Problem 6: Graph of $f^{(3)}(x)$

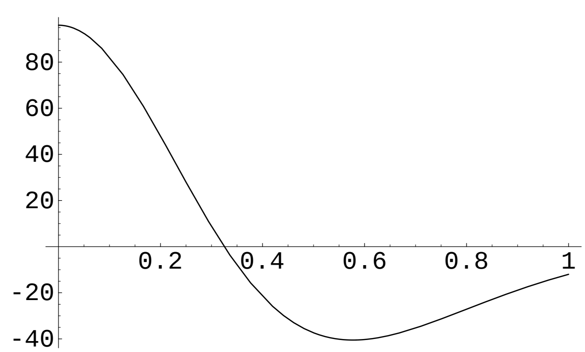

Figure 6: Problem 6: Graph of $f^{(4)}(x)$

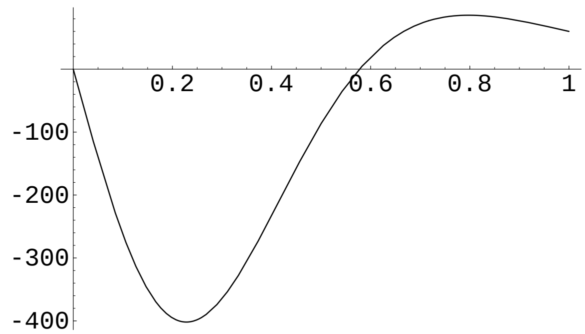

Figure 7: Problem 6: Graph of $f^{(5)}(x)$

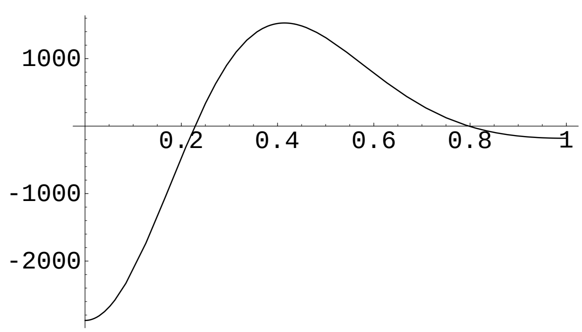

Figure 8: Problem 6: Graph of $f^{(6)}(x)$

MTH 2410, Calculus II—Final Exam **May 9, 2006**

Instructions: Solve the following problems. Present your solutions, *including your reasoning*, on your own paper. Do not use decimal approximations unless the nature of a problem requires them. Your work is due at 2:50 pm. You may keep this copy of the exam.

1. Find the limits:

   (a) $\lim_{x\to 0^+} x\ln x$

   (b) $\lim_{x\to 0} \dfrac{x-\sin x}{x^3}$

   Solution:

   (a) $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}(\ln x)/(x^{-1})$, where both numerator and denominator become infinite as $x \to 0^+$. We may therefore attempt L'Hôpital's Rule. We get:

$$\begin{aligned}\lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+} \frac{\ln x}{x^{-1}}\\ &= \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}}\\ &= \lim_{x\to 0^+} -x \;=\; 0.\end{aligned}$$

   (b) $\lim_{x\to 0}(x-\sin x) = 0$ and $\lim_{x\to 0} x^3 = 0$. We may therefore attempt L'Hôpital's Rule. This leads to:

$$\begin{aligned}\lim_{x\to 0}\frac{x-\sin x}{x^3} &= \lim_{x\to 0}\frac{1-\cos x}{3x^2}\\ &= \lim_{x\to 0}\frac{1-\cos^2 x}{3x^2(1+\cos x)}\\ &= \frac{1}{3}\lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)}\\ &= \frac{1}{3}\cdot\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\lim_{x\to 0}\left(\frac{1}{1+\cos x}\right)\\ &= \frac{1}{3}\cdot 1\cdot\frac{1}{2} \;=\; \frac{1}{6}.\end{aligned}$$

2. Solve the initial value problem:

$$\frac{x^2}{y^2-8}\frac{dy}{dx} = \frac{1}{2y};$$
$$y(1) = 3$$

   Solution: Separating variables, we have:

$$\frac{2y\,dy}{y^2-8} = \frac{dx}{x^2}.$$

Thus,

$$\int \frac{2y\,dy}{y^2-8} = \int \frac{dx}{x^2},$$

or

$$\ln|y^2-8| = -\frac{1}{x} + c.$$

This latter is equivalent to

$$y^2 - 8 = Ce^{-1/x},$$

or

$$y = \pm\sqrt{Ce^{-1/x}+8}.$$

But $y(1) = 3$, so

$$3 = \pm\sqrt{Ce^{-1}+8}.$$

We therefore take the "+" sign and put $C = e$. This gives the solution

$$y = \sqrt{e^{1-1/x}+8}.$$

3. Is the integral $\displaystyle\int_0^\infty xe^{-x^2}\,dx$ convergent or divergent? If it is convergent, find its value.

Solution: The integrand is continuous on $(0,\infty)$, so we must consider

$$\lim_{T\to\infty}\int_0^T xe^{-x^2}\,dx.$$

Let $u = x^2$. Then $du = 2x\,dx$, or $x\,dx = du/2$. Also, $u = 0$ when $x = 0$, and $u = T^2$ when $x = T$. Thus

$$\begin{aligned}
\lim_{T\to\infty}\int_0^T xe^{-x^2}\,dx &= \lim_{T\to\infty}\frac{1}{2}\int_0^{T^2} e^{-u}\,du \\
&= \frac{1}{2}\lim_{T\to\infty}\big(-e^{-u}\big)\Big|_0^{T^2} \\
&= \frac{1}{2}\left[\lim_{T\to 0}\left(-e^{-T^2}+1\right)\right] \\
&= \frac{1}{2}.
\end{aligned}$$

The improper integral converges to the value $1/2$.

4. Find the volume generated when the region bounded by the curve $y = \sqrt{x}$, the x-axis, and the lines $x = 4$, $x = 9$ is rotated about the line $y = -1$.

   Solution:

$$\begin{aligned} V &= \pi \int_4^9 \left([\sqrt{x} - (-1)]^2 - [0 - (-1)]^2 \right) dx \\ &= \pi \int_4^9 \left(x + 2\sqrt{x} \right) dx = \pi \left(\frac{x^2}{2} + \frac{4x^{3/2}}{3} \right) \Big|_4^9 = \frac{347\pi}{6} \end{aligned}$$

5. The base of a solid is the region in the first quadrant bounded by the y-axis, the line $y = 4$, and the curve $y = x^2$. Every cross-section of the solid perpendicular to the y-axis is a semi-circle. (See Figure 1, which depicts the solid and two of the intersecting planes.) Find the volume of the solid.

   Solution: We solve for x in terms of y, and we find that the diameter of the semi-circle at $y = t$ is $\sqrt{t}$, so the area of that semi-circle is $(1/2)\pi(\sqrt{t}/2)^2 = (\pi/8)t$. Hence

$$\begin{aligned} V &= \frac{\pi}{8} \int_0^4 t\, dt \\ &= \frac{\pi}{16} t^2 \Big|_0^4 = \pi. \end{aligned}$$

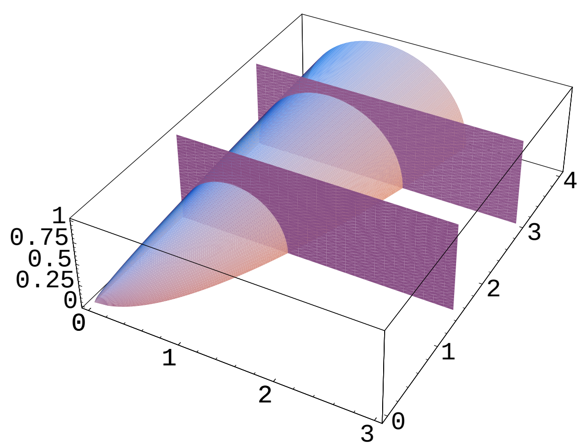

Figure 1: Problem 5

6. Make appropriate use of any of Figures 2–8 to determine how many subdivisions are needed to evaluate π to six decimal places by approximating the integral

$$\pi = \int_0^1 \frac{4\, dx}{1 + x^2}$$

   to within 1×10^{-6} using Simpson's Rule. Be sure to explain which figures you use—as well as the uses to which you put them.

Solution: The error in the Simpson's Rule approximation, with n subdivisions, for $\int_a^b f(x)\,dx$ does not exceed $M(b-a)^5/(180n^4)$, provided that M is chosen so that $\left|f^{(4)}(x)\right| \leq M$ for all x in $[a,b]$. For $f(x) = 4/(1+x^2)$, we see from Figure 6 that we may take $M = 100$. Hence we must find n so that

$$\frac{100(1-0)^5}{180n^4} \leq \frac{1}{10^6}$$

or

$$\frac{10^7}{18} \leq n^4.$$

Thus, we need

$$n \geq \left(\frac{5 \cdot 10^6}{9}\right)^{1/4},$$

or

$$n \geq 27.302.$$

n must be a positive even integer, so we take $n = 28$.

Note: Careful evaluation shows that $f^{(4)}(0) = 96$, so we may take $M = 96$, but no smaller. (See Figure 6.) This gives $n \geq 27.025$, which does not change our conclusion.

7. Determine how many terms of the Maclaurin series for the cosine function are needed to guarantee that using the Maclaurin polynomial in place of the cosine function results in an error no bigger than $1/500$ throughout the interval $[-\pi/2, \pi/2]$. Give your reasoning.

Solution: Taylor's theorem says that for $-\pi/2 \leq x \leq \pi/2$

$$\cos x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + R_{2n}(x),$$

where

$$|R_{2n}(x)| \leq \frac{M|x|^{2n+1}}{(2n+1)!},$$

provided that $|f^{(2n+1)}(x)| \leq M$ for every x in $[-\pi/2, \pi/2]$. But every derivative of f is either a sine function or a cosine function, possibly with a minus sign attached, and so $|f^{(2n+1)}(x)| \leq 1$ for every x in $[-\pi/2, \pi/2]$. Also, we are given $|x| \leq \pi/2$, so we need to find n for which

$$\frac{(\pi/2)^{2n+1}}{(2n+1)!} \leq \frac{1}{500}.$$

By direct calculation, we find that when $n = 3$, the quantity on the left side of this inequality is about 0.0047, which is too large. When $n = 4$ the quantity on the left is about 0.00016, and this satisfies the inequality. We therefore need all terms through $n = 4$. Because we started counting with $k = 0$, we will need five (non-zero) terms of the Maclaurin series for the cosine function to guarantee the required accuracy. We can be sure that

$$\left| \cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} - \frac{x^8}{40320} \right| \leq \frac{1}{500}$$

when $-\pi/2 \leq x \leq \pi/2$.

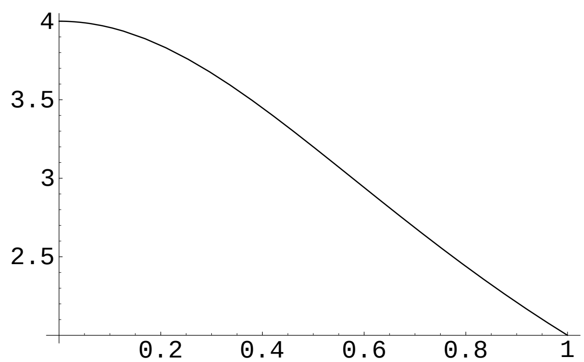

Figure 2: Problem 6: Graph of $f(x)$

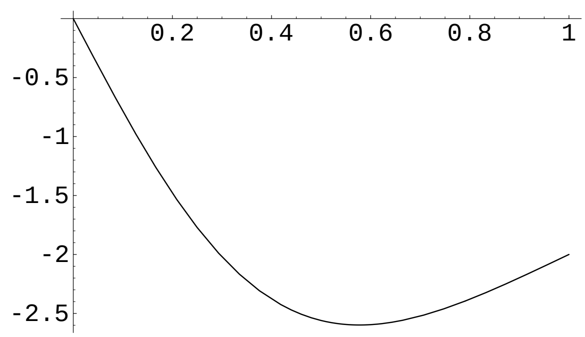

Figure 3: Problem 6: Graph of $f'(x)$

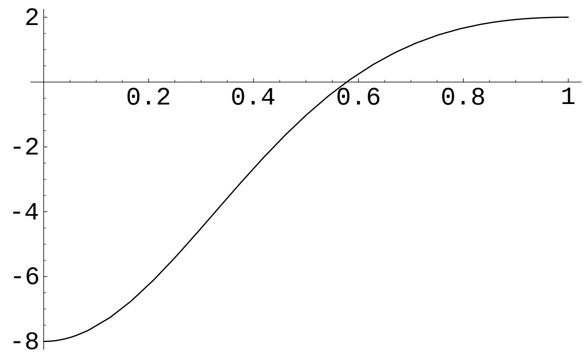

Figure 4: Problem 6: Graph of $f''(x)$

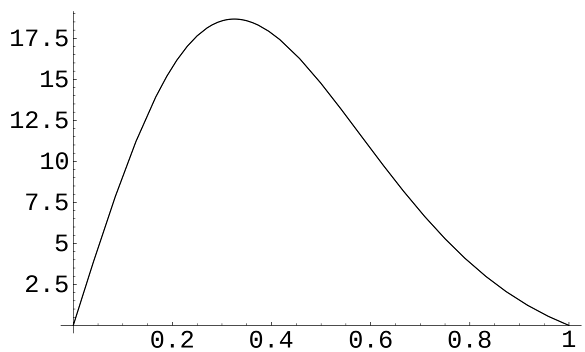

Figure 5: Problem 6: Graph of $f^{(3)}(x)$

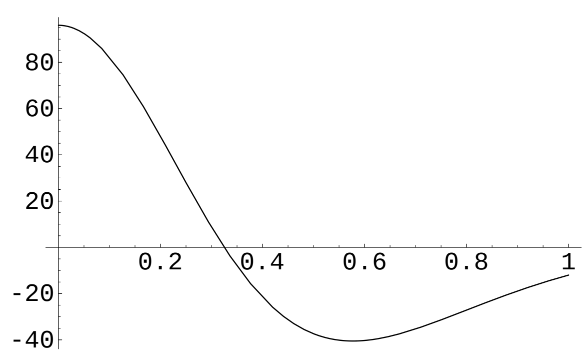

Figure 6: Problem 6: Graph of $f^{(4)}(x)$

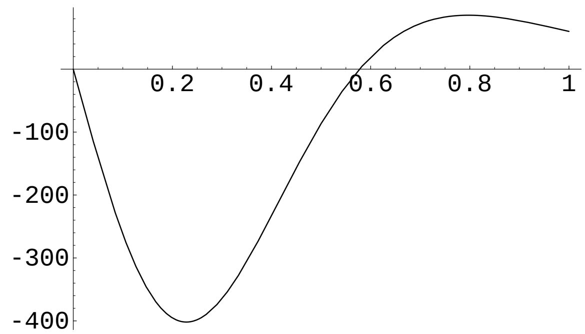

Figure 7: Problem 6: Graph of $f^{(5)}(x)$

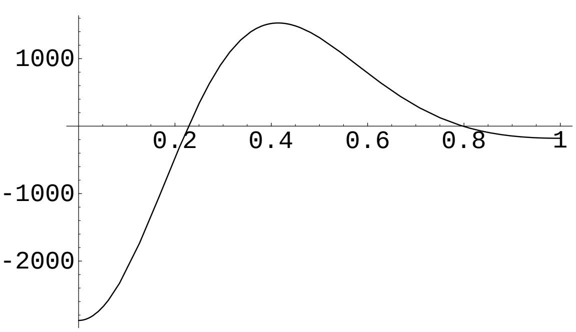

Figure 8: Problem 6: Graph of $f^{(6)}(x)$

MTH 2410, Calculus II—Exam I **June 5, 2008**

Instructions: Write out, *on your own paper*, complete solutions of following problems. Do not give decimal approximations unless a problem requires them. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your work is due at 5:15 pm. You may keep this copy of the exam.

1. Use an appropriate substitution to evaluate the definite integral $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$.

2. Use integration by parts to find $\displaystyle\int x\cos(3x)\,dx$.

3. Find $\displaystyle\int \frac{8x-1}{(2x-1)(x+1)}\,dx$.

4. Use the Mid-point Rule with four subdivisions to find an approximate value for the integral $\displaystyle\int_1^3 \frac{dx}{x}$. Show your calculations and give your answer correct to four digits to the right of the decimal.

5. Is the integral $\displaystyle\int_0^\infty xe^{-x^2}\,dx$ convergent or divergent? If it is convergent, find its value.

6. Find the area of the region bounded by the curves $x = y^2 - 4y$ and $x = 2y - y^2$.

7. Let R denote the plane region bounded by the curves $y = x^2$ and $x = y^2$. Find the volume generated by revolving R about the line $y = -2$.

8. Æthelbert wants to use Simpson's Rule to estimate the value of the integral $\displaystyle\int_{-1}^{1} \sqrt{4 - x^3}\,dx$.

   He needs to be sure his error is at most 0.001. Use the plot (Figure 1) of $\displaystyle\frac{d^4}{dx^4}\left(\sqrt{4-x^3}\right)$ and the fact that if M is a number for which $|f^{(4)}(x)| \le M$ for every $x \in [a,b]$, then the magnitude of the error in an n-subdivision Simpson's Rule approximation for the integral $\displaystyle\int_a^b f(x)\,dx$ is at most $\displaystyle\frac{M(b-a)^5}{180n^4}$ to determine how many subdivisions he needs to use.

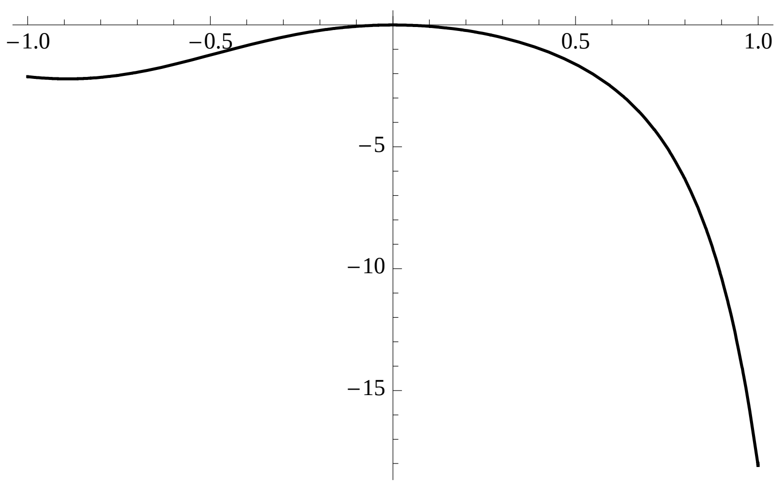

Figure 1: Graph of $\frac{d^4}{dx^4}\left(\sqrt{4-x^3}\right)$

MTH 2410, Calculus II—Exam I **June 5, 2008**

Instructions: Write out, *on your own paper*, complete solutions of following problems. Do not give decimal approximations unless a problem requires them. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your work is due at 5:15 pm. You may keep this copy of the exam.

1. Use an appropriate substitution to evaluate the definite integral $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$.

   Solution: Let $u = \ln x$. Then $du = dx/x$. When $x = e$, $u = 1$, and when $x = e^4$, $u = 4$. Therefore

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_1^4 \frac{du}{\sqrt{u}} = \int_1^4 u^{-1/2}\,du = 2u^{1/2}\Big|_1^4 = 4.$$

2. Use integration by parts to find $\displaystyle\int x\cos(3x)\,dx$.

   Solution: Let $u = x$; $dv = \cos 3x\,dx$. Then $du = dx$ and $v = \frac{1}{3}\sin 3x$. Hence

$$\int x\cos 3x\,dx = \frac{1}{3}x\sin 3x - \frac{1}{3}\int \sin 3x\,dx = \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + c.$$

3. Find $\displaystyle\int \frac{8x-1}{(2x-1)(x+1)}\,dx$.

   Solution: We must first find A and B so that, for all x,

$$\frac{8x-1}{(2x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+1} = \frac{A(x+1)+B(2x-1)}{(2x-1)(x+1)}$$
$$= \frac{(A+2B)x+(A-B)}{(2x-1)(x+1)}.$$

   For this to be so, we must have $8 = A + 2B$ and $-1 = A - B$. The second of these equations is equivalent to $A = B - 1$, and substituting this for A in the first of the pair gives

$$8 = (B-1) + 2B = 3B - 1,$$

   from which it follows that $B = 3$. But then $A = B - 1 = 2$. Thus

$$\int \frac{8x-1}{(2x-1)(x+1)}\,dx = \int \frac{2}{2x-1}\,dx + \int \frac{3}{x+1}\,dx$$
$$= \ln|2x-1| + 3\ln|x+1| + c = \ln\left|\frac{2x-1}{(x+1)^3}\right| + c$$

4. Use the Mid-point Rule with four subdivisions to find an approximate value for the integral $\int_1^3 \frac{dx}{x}$. Show your calculations and give your answer correct to four digits to the right of the decimal.

Solution:

$$\int_1^3 \frac{dx}{x} \sim \frac{1}{2}\left(\frac{4}{5}+\frac{4}{7}+\frac{4}{9}+\frac{4}{11}\right) = \frac{3776}{3465} \sim 1.0898.$$

5. Is the integral $\int_0^\infty xe^{-x^2}\,dx$ convergent or divergent? If it is convergent, find its value.

Solution: We must decide whether or not $\lim_{T\to\infty}\int_0^T xe^{-x^2}\,dx$ exists. To this end, let $u = -x^2$. Then $du = -2x\,dx$, or $x\,dx = -du/2$. When $x = 0$, $u = 0$, and when $x = T$, $u = -T^2$. Thus

$$\lim_{T\to\infty}\int_0^T xe^{-x^2}\,dx = -\frac{1}{2}\lim_{T\to\infty}\int_0^{-T^2} e^u\,du = -\frac{1}{2}\lim_{T\to\infty}\left(e^{-T^2}-1\right) = \frac{1}{2}.$$

The improper integral $\int_0^\infty xe^{-x^2}\,dx$ therefore converges to the value $1/2$.

6. Find the area of the region bounded by the curves $x = y^2 - 4y$ and $x = 2y - y^2$.

Solution: The curves are parabolae with horizontal axes of symmetry; the first opens rightward, the second leftward. We find the intersection points by observing that at such points we must have $y^2 - 4y = 2y - y^2$, which is equivalent to $y^2 - 3y = 0$, or to $y(y-3) = 0$. Thus, the intersection points are $(0, 0)$ and $(-3, 3)$. The area in question is thus

$$\int_0^3 [(2y-y^2)-(y^2-4y)]\,dy = \int_0^3 (6y-2y^2)\,dy = \left(3y^2 - \frac{2}{3}y^3\right)\Bigg|_0^3 = 9.$$

7. Let R denote the plane region bounded by the curves $y = x^2$ and $x = y^2$. Find the volume generated by revolving R about the line $y = -2$.

Solution: The curves are both parabolae. The first opens upward, while the second opens rightward. Substituting y^2 for x in the first equation, we find that $y = y^4$, or $y(y^3 - 1) = 0$. The intersection points are consequently $(0, 0)$ and $(1, 1)$. We rewrite the second equation as $y = \sqrt{x}$ (legitimate because we are interested only in that part of the curve which lies in the first quadrant and for which $0 \le x \le 1$), and we consider the washers generated by revolving vertical line segments with endpoints (x, x^2) and $(x, \sqrt{x})$ about the the line $y = -2$. The area of such a washer is $\pi[(\sqrt{x}+2)^2 - (x^2+2)^2]$, or $\pi(4\sqrt{x} - 3x - x^2)$. The desired volume is therefore

$$\pi\int_0^1 (4\sqrt{x} - 3x - x^2)\,dx = \pi\left(\frac{8}{3}x^{3/2} - \frac{3}{2}x^2 - \frac{1}{3}x^3\right)\Bigg|_0^1 = \frac{5\pi}{6}.$$

8. Æthelbert wants to use Simpson's Rule to estimate the value of the integral $\int_{-1}^{1} \sqrt{4-x^3}\,dx$.

He needs to be sure his error is at most 0.001. Use the plot (Figure 1) of $\frac{d^4}{dx^4}\left(\sqrt{4-x^3}\right)$ and the fact that if M is a number for which $|f^{(4)}(x)| \leq M$ for every $x \in [a,b]$, then the magnitude of the error in an n-subdivision Simpson's Rule approximation for the integral $\int_a^b f(x)\,dx$ is at most $\frac{M(b-a)^5}{180n^4}$ to determine how many subdivisions he needs to use.

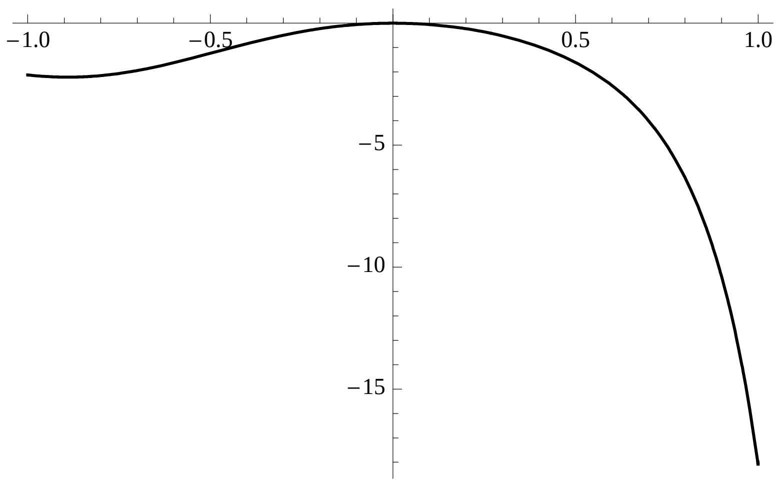

Figure 1: Graph of $\frac{d^4}{dx^4}\left(\sqrt{4-x^3}\right)$

Solution: It is evident from the figure that $|f^{(4)}(x)| \leq 20$ for all x in the interval $[-1,1]$. We must therefore solve the inequality

$$\frac{20[1-(-1)]^5}{180n^4} \leq \frac{1}{1000}, \text{ or } \frac{32}{9n^4} \leq \frac{1}{1000}.$$

Thus, we must take n so that

$$n^4 \geq \frac{32000}{9}, \text{ or } n \geq \sqrt[4]{\frac{32000}{9}} \sim 7.722.$$

But n must be an even integer, so Æthelbert needs to take $n = 8$. (Of course, larger even integers will also work.)

MTH 2410, Calculus II—Exam II **June 19, 2008**

Instructions: Write out, *on your own paper*, complete solutions of following problems. Do not give decimal approximations unless a problem requires them. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your work is due at 5:15 pm. You may keep this copy of the exam.

1. Find the x-coordinate of the centroid of the region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 9$.

2. Find the area of the region in the first quadrant that lies below the polar curve $r = \sec(2\theta)$ and inside the curve $r = 2$.

3. Find the length of the arc of the curve $y = \frac{1}{3}(2 + x^2)^{3/2}$ over the interval $0 \le x \le 2$.

4. (a) What are the constant solutions of the differential equation

$$\frac{dy}{dt} = (2y^4 + 5y^3 - 12y^2)\sin t?$$

   (b) Find all values of a for which $y = e^{ax}$ gives a solution of the differential equation

$$\frac{d^3y}{dx^3} - 6\frac{dy^2}{dx^2} + 5\frac{dy}{dx} = 0.$$

5. The last page of this exam contains a figure showing a a portion of the slope field for the differential equation $3y' = x^2 + y$. Remove the page from the exam, put your name on it, and sketch, on the displayed figure, an approximate solution curve for the initial value problem

$$3y' = x^2 + y;$$
$$y(2) = -1.$$

   Don't forget to submit your drawing as part of your exam.

6. Show how to use Euler's method with three steps to estimate $y(3/4)$ for the solution of the initial value problem

$$y' = 3y + x;$$
$$y(0) = -1.$$

   Show your supporting calculations and give your answer as a fraction of integers.

7. Bismuth-210 has a half-life of 5.0 days. At noon on June 1, Murgatroyd placed a sample containing of 800 mg of bismuth-210 in a suitable container in his laboratory and put the container into long-term storage.

   (a) Give a formula for the amount of the isotope remaining after t days.

   (b) How much remains at noon on July 1? (There are 30 days in June.)

   (c) To the nearest hour, when is the amount of bismuth-210 in the sample reduced to 1 mg?

8. (a) Find the general solution of the differential equation

$$xy' = y^2 - 5y + 6.$$

   (b) Using your work in the first part of this problem, find the particular solution of the differential equation for which $y(3) = 2$.

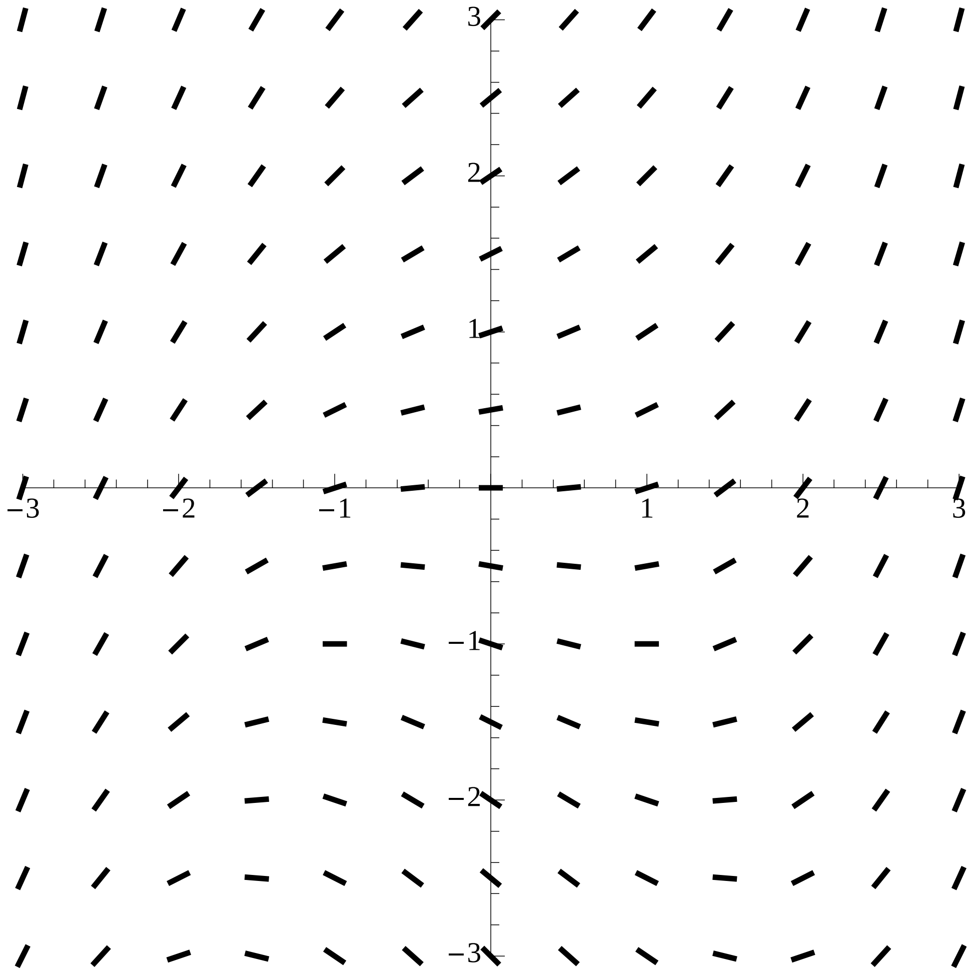

Figure 1: Slope field for $3y' = x^2 + y$

MTH 2410, Calculus II—Exam II **June 19, 2008**

Instructions: Write out, *on your own paper*, complete solutions of following problems. Do not give decimal approximations unless a problem requires them. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your work is due at 5:15 pm. You may keep this copy of the exam.

1. Find the x-coordinate of the centroid of the region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 9$.

   Solution:

$$\overline{x} = \frac{\int_0^9 x\sqrt{x}\,dx}{\int_0^9 \sqrt{x}\,dx} = \frac{27}{5}.$$

2. Find the area of the region in the first quadrant that lies below the polar curve $r = \sec(2\theta)$ and inside the curve $r = 2$.

   Solution: We must first determine the intersection point, which lies in the first quadrant where $\sec(2\theta) = 2$, or where $\cos(2\theta) = 1/2$. Thus $2\theta = \pi/3$, or $\theta = \pi/6$. Area is thus

$$\begin{aligned}\frac{1}{2}\int_0^{\pi/6} [2^2 - \sec^2(2\theta)]\,d\theta &= \frac{1}{2}\left(4\theta - \frac{1}{2}\tan(2\theta)\right)\Big|_0^{\pi/6} \\ &= \frac{4\pi - 3\sqrt{3}}{12}.\end{aligned}$$

3. Find the length of the arc of the curve $y = \frac{1}{3}(2 + x^2)^{3/2}$ over the interval $0 \le x \le 2$.

   Solution:

$$\begin{aligned} s &= \int_0^2 \sqrt{1 + (y')^2}\,dx \\ &= \int_0^2 \sqrt{1 + x^2(2 + x^2)}\,dx \\ &= \int_0^2 (1 + x^2)\,dx \\ &= \left(x + \frac{x^3}{3}\right)\Big|_0^2 = \frac{14}{3}. \end{aligned}$$

4. (a) What are the constant solutions of the differential equation

$$\frac{dy}{dt} = (2y^4 + 5y^3 - 12y^2)\sin t?$$

   (b) Find all values of a for which $y = e^{ax}$ gives a solution of the differential equation

$$\frac{d^3y}{dx^3} - 6\frac{dy^2}{dx^2} + 5\frac{dy}{dx} = 0.$$

   Solution:

(a)

$$\begin{aligned}\frac{dy}{dt} &= (2y^4 + 5y^3 - 12y^2)\sin t\\ &= y^2(2y-3)(y+4)\sin t,\end{aligned}$$

and it follows that the constant solutions of the differential equation are $y \equiv 0$, $y \equiv 3/2$, and $y \equiv -4$.

(b) Putting $y = e^{ax}$, we have $y^{(n)} = a^n e^{ax}$. Thus,

$$y''' - 6y'' + 5y' = a^3 e^{ax} - 6a^2 e^{ax} + 5ae^{ax}.$$

This latter quantity can be identically zero just when $0 = a^3 - 6a^2 + 5a = a(a-1)(a-5)$, and it follows that the desired values are $a = 0$, $a = 1$, and $a = 5$.

5. The last page of this exam contains a figure showing a a portion of the slope field for the differential equation $3y' = x^2 + y$. Remove the page from the exam, put your name on it, and sketch, on the displayed figure, an approximate solution curve for the initial value problem

$$\begin{aligned}3y' &= x^2 + y;\\ y(2) &= -1.\end{aligned}$$

Don't forget to submit your drawing as part of your exam.

Solution: See the figure.

6. Show how to use Euler's method with three steps to estimate $y(3/4)$ for the solution of the initial value problem

$$\begin{aligned}y' &= 3y + x;\\ y(0) &= -1.\end{aligned}$$

Show your supporting calculations and give your answer as a fraction of integers.

Solution: The equations for Euler's Method are

$$\begin{aligned}x_0 &= 0;\\ y_0 &= -1;\end{aligned}$$

and for $k > 0$,

$$\begin{aligned}x_k &= x_0 + kh;\\ y_k &= y_{k-1} + F(x_{k-1}, y_{k-1})h,\end{aligned}$$

where we take $F(x, y) = 3y + x$, from the differential equation. We are to take three steps from $x = 0$ to reach $x = 3/4$, so we will need $h = 1/4$. Thus

$$\begin{aligned}x_1 &= \frac{1}{4};\\ y_1 &= y_0 + F(x_0, y_0)h = -1 + [3\cdot(-1) + 0]\frac{1}{4} = -\frac{7}{4};\\ x_2 &= \frac{1}{2};\\ y_2 &= y_1 + F(x_1, y_1)h = -\frac{7}{4} + \left[3\cdot\left(-\frac{7}{4}\right) + \frac{1}{4}\right]\frac{1}{4} = -3;\\ x_3 &= \frac{3}{4};\\ y_3 &= y_2 + F(x_2, y_2)h = -3 + \left[3(-3) + \frac{1}{2}\right]\frac{1}{4} = -\frac{41}{8}.\end{aligned}$$

Our approximate value of $y(3/4)$ is $-41/8$.

7. Bismuth-210 has a half-life of 5.0 days. At noon on June 1, Murgatroyd placed a sample containing of 800 mg of bismuth-210 in a suitable container in his laboratory and put the container into long-term storage.

   (a) Give a formula for the amount of the isotope remaining after t days.
   (b) How much remains at noon on July 1? (There are 30 days in June.)
   (c) To the nearest hour, when is the amount of bismuth-210 in the sample reduced to 1 mg?

   Solution:

   (a) If N designates the amount, in mg, of the isotope present at time t, measured in days after June 1, then $N = 800 \cdot 2^{-t/5}$.
   (b) On July 1, $t = 30$. Therefore $N = 800 \cdot 2^{-6} = 25/2$ mg.
   (c) The remaining amount of the isotope will be 1 mg when

$$\begin{aligned} 1 &= 800 \cdot 2^{-t/5}; \\ 2^{t/5} &= 800; \\ \frac{t}{5}\ln 2 &= \ln 800; \\ t &= \frac{5\ln 800}{\ln 2} \sim 48.219281 \text{ days.} \end{aligned}$$

   There will be 1 mg remaining at about 5 pm on the 48th day after June 1, or at about 5 pm on July 19.

8. (a) Find the general solution of the differential equation

$$xy' = y^2 - 5y + 6.$$

   (b) Using your work in the first part of this problem, find the particular solution of the differential equation for which $y(3) = 2$.

   Solution:

   (a) We note first that, because $y^2 - 5y + 6 = (y-2)(y-3)$, the constant solutions of this differential equation are $y \equiv 2$ and $y \equiv 3$. Separating variables, we find that we must integrate:

$$\begin{aligned} \int \frac{dy}{(y-2)(y-3)} &= \int \frac{dx}{x}; \\ \int \frac{dy}{y-3} - \int \frac{dy}{y-2} &= \int \frac{dx}{x}; \\ \ln|y-3| - \ln|y-2| &= \ln|x| + c; \\ \ln\left|\frac{y-3}{y-2}\right| &= \ln|x| + c; \\ \left|\frac{y-3}{y-2}\right| &= C|x|, \text{ assuming that } C > 0; \\ y - 3 &= Cx(y-2), \text{where } C \text{ may now be arbitrary;} \\ y - Cxy &= 3 - 2Cx; \\ y &= \frac{3-2Cx}{1-Cx}. \end{aligned}$$

   The general solution of the differential equation consists then of the constant solutions $y \equiv 3$ and $y \equiv 2$, together with $y = (3-2Cx)/(1-Cx)$, C being an arbitrary constant.

   (b) The particular solution for which $y(3) = 2$ is the constant solution $y \equiv 2$.

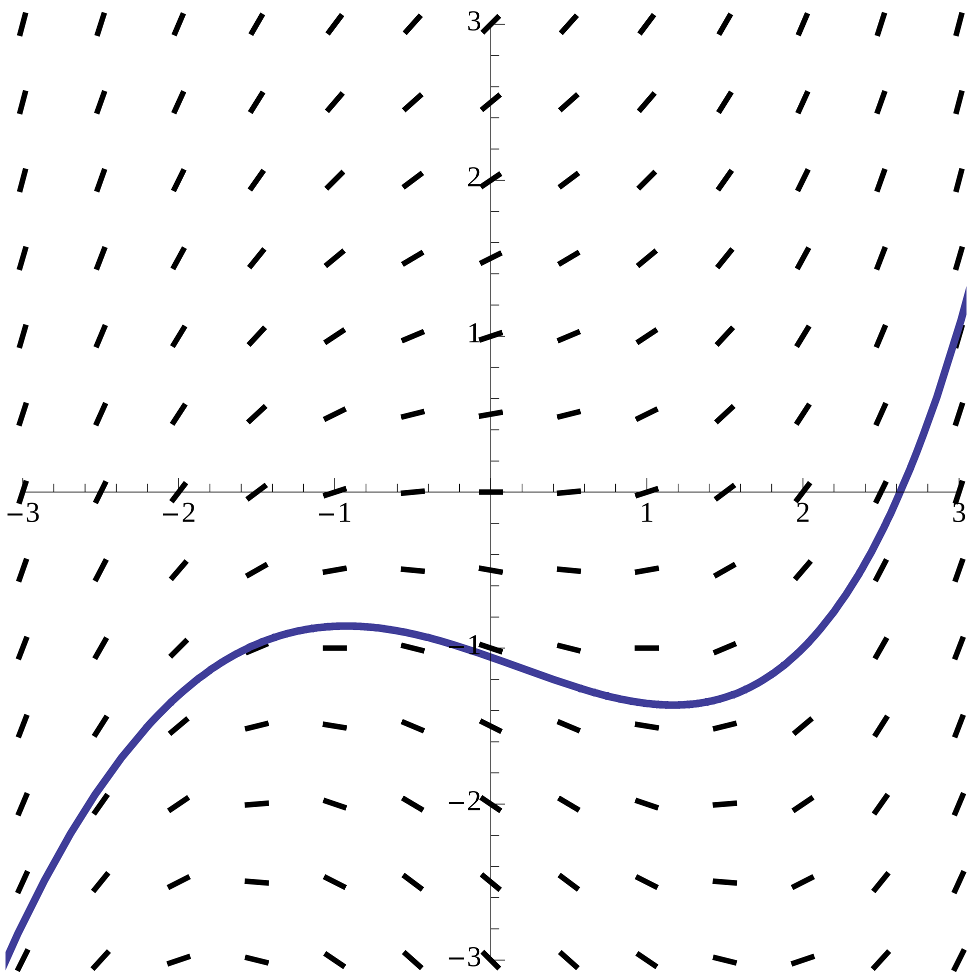

Figure 1: Slope field for $3y' = x^2 + y$

MTH 2410, Calculus II—Exam III **July 10, 2008**

Instructions: Write out, *on your own paper*, complete solutions of following problems. Do not give decimal approximations unless a problem requires them. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your work is due at 5:15 pm. You may keep this copy of the exam.

1. Use the Comparison Test to determine whether

$$\sum_{k=1}^{\infty} \frac{1}{k^2+k+1}$$

is convergent or divergent.

2. Convergent or divergent?

$$\sum_{k=1}^{\infty} \frac{k+5}{\sqrt[3]{k^7+k^2}}$$

3. Convergent or divergent?

$$\sum_{k=1}^{\infty} (-1)^k \frac{\ln k}{k}$$

4. Convergent or divergent?

$$\sum_{k=1}^{\infty} \frac{1-3k+k^2}{1+k+k^2}$$

5. Find the radius of convergence and the endpoints of the interval of convergence:

$$\sum_{k=1}^{\infty} \frac{(k+1)^2}{2^k}(x+5)^k$$

6. Determine for which values of $p > 0$ the series

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$$

converges. Use the Integral Test.

7. Find all values of s for which the series

$$\sum_{k=1}^{\infty} \frac{(s^2-5s+2)^k}{k2^{k+1}}$$

converges.

MTH 2410, Calculus II—Exam III **July 10, 2008**

Instructions: Write out, *on your own paper*, complete solutions of following problems. Do not give decimal approximations unless a problem requires them. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your work is due at 5:15 pm. You may keep this copy of the exam.

1. Use the Comparison Test to determine whether

$$\sum_{k=1}^{\infty} \frac{1}{k^2+k+1}$$

is convergent or divergent.

Solution: The series $\sum(1/k^2)$ is a convergent p-series with $p = 2 > 1$. For every $k \geq 1$, we have $k^2 + k + 1 > k^2 > 0$, so it follows that for every $k \geq 1$ we also have $1/(k^2 + k + 1) < 1/k^2$. By the Comparison Test, $\sum_{k=1}^{\infty} 1/(k^2 + k + 1)$ converges.

2. Convergent or divergent?

$$\sum_{k=1}^{\infty} \frac{k+5}{\sqrt[3]{k^7+k^2}}$$

Solution: The series $\sum 1/k^{4/3}$ is a convergent p-series with $p = 4/3 > 1$. Moreover,

$$\begin{aligned}\lim_{k\to\infty} \frac{(k+5)/\sqrt[3]{k^7+k^2}}{1/k^{4/3}} &= \lim_{k\to\infty} \frac{k^{7/3}+5k^{4/3}}{\sqrt[3]{k^7+k^2}} \\ &= \lim_{k\to\infty} \frac{1+5/k}{\sqrt[3]{1+1/k^5}} = 1.\end{aligned}$$

Consequently, by the Limit Comparison Test, $\sum(k + 5)/\sqrt[3]{k^7 + k^2}$ converges.

3. Convergent or divergent?

$$\sum_{k=1}^{\infty} (-1)^k \frac{\ln k}{k}$$

Solution: Let $f(x) = (\ln x)/x$. Then $f'(x) = (1-\ln x)/x^2$. When $x > e$, $\ln x > 1$, so $f'(x) < 0$ on the interval (e, ∞). It follows that f is a decreasing function on (e, ∞), so that the sequence $a_k = (\ln k)/k$ is a decreasing sequence when $k \geq 3$. Hence, by the Alternating Series Test,

$$\sum_{k=1}^{\infty} (-1)^k \frac{\ln k}{k}$$

converges iff $\lim_{k\to\infty}(\ln k)/k = 0$. Because $k \to \infty$ implies that both $\lim_{k\to\infty} \ln k = \infty$ and $\lim_{k\to\infty} k = \infty$ (Duh!), we have from l'Hôpital's rule

$$\begin{aligned}\lim_{k\to\infty} \frac{\ln k}{k} &= \lim_{k\to\infty} \frac{(1/k)}{1} \\ &= 0.\end{aligned}$$

We conclude that the series converges.

4. Convergent or divergent?

$$\sum_{k=1}^{\infty} \frac{1-3k+k^2}{1+k+k^2}$$

Solution: We have

$$\begin{aligned}\lim_{k\to\infty} \frac{1-3k+k^2}{1+k+k^2} &= \lim_{k\to\infty} \frac{(1/k^2)-(3/k)+1}{(1/k^2)+(1/k)+1} \\ &= 1 \neq 0.\end{aligned}$$

We conclude, by the Test for Divergence, that the series diverges.

5. Find the radius of convergence and the endpoints of the interval of convergence:

$$\sum_{k=1}^{\infty} \frac{(k+1)^2}{2^k}(x+5)^k$$

Solution: We apply the Ratio Test:

$$\begin{aligned}\lim_{k\to\infty} \left|(k+2)^2(x+5)^{k+1}/2^{k+1}\right| / \left|(k+1)^2(x+5)^k/2^k\right| &= \lim_{k\to\infty} \frac{(k+2)^2|x+5|}{2(k+1)^2} \\ &= \frac{|x+5|}{2} \lim_{k\to\infty} \left(\frac{k+2}{k+1}\right)^2 \\ &= \frac{|x+5|}{2} \lim_{k\to\infty} \left(\frac{1+2/k}{1+1/k}\right)^2 \\ &= \frac{|x+5|}{2}\end{aligned}$$

The interior of the interval of convergence thus consists of all numbers x for which $|x+5| < 2$, so the radius of convergence is 2 and the endpoints of the interval of convergence are -7 and -3.

6. Determine for which values of $p > 0$ the series

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$$

converges. Use the Integral Test.

Solution: Let f be the function given by

$$f(x) = \frac{1}{x(\ln x)^p}.$$

Then f is continuous when $x > 0$ and

$$f'(x) = -\frac{p+\ln x}{x^2(\ln x)^{1+p}}.$$

Because p is positive, this quantity is negative for all $x > 1$. Hence the function f is decreasing on (x, ∞). By the Integral Test, the series

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$$

converges iff the improper integral $\int_2^\infty f(t)\,dt$ converges. When $p \neq 1$, we have (making the substitution $u = \ln t$; $du = dt/t$)

$$\begin{aligned}
\int_2^\infty \frac{1}{t(\ln t)^p}\,dt &= \lim_{T\to\infty} \int_2^T \frac{1}{t(\ln t)^p}\,dt \\
&= \lim_{T\to\infty} \int_{\ln 2}^{\ln T} \frac{1}{u^p}\,du \\
&= \lim_{T\to\infty} \left.\frac{1}{(1-p)u^{p-1}}\right|_{\ln 2}^{\ln T} \\
&= \lim_{T\to\infty} \frac{1}{(1-p)(\ln T)^{p-1}} - \frac{1}{(1-p)(\ln 2)^{p-1}} \\
&= \begin{cases} \infty & \text{when } 0 < p < 1 \\ 1/[(p-1)(\ln 2)^{p-1}] & \text{when } 1 < p. \end{cases}
\end{aligned}$$

When $p = 1$,

$$\begin{aligned}
\int_2^\infty \frac{1}{t(\ln t)^p}\,dt &= \lim_{T\to\infty} \int_2^T \frac{1}{t(\ln t)}\,dt \\
&= \lim_{T\to\infty} \int_{\ln 2}^{\ln T} \frac{1}{u}\,du \\
&= \lim_{T\to\infty} \ln u \Big|_{\ln 2}^{\ln T} \\
&= \lim_{T\to\infty} \ln\ln T - \ln\ln 2 = \infty.
\end{aligned}$$

We conclude that the series converges when $p > 1$ and diverges when $0 < p \leq 1$.

7. Find all values of s for which the series

$$\sum_{k=1}^{\infty} \frac{(s^2 - 5s + 2)^k}{k2^{k+1}}$$

converges.

Solution: We begin by applying the Ratio Test:

$$\begin{aligned}
\lim_{k\to\infty} \frac{|(s^2-5s+2)^{k+1}/([k+1]2^{k+2})|}{|(s^2-5s+2)^k/(k2^{k+1})|} &= \frac{|s^2-5s+2|}{2} \lim_{k\to\infty} \frac{k}{2(k+1)} \\
&= \frac{|s^2-5s+2|}{2}.
\end{aligned}$$

The Ratio Test now allows us to conclude that our series converges whenever $-2 < s^2 - 5s + 2 < 2$. The first of these two inequalities is equivalent to $0 < s^2 - 5s + 4 = (s-1)(s-4)$, and this inequality is correct when $s < 1$ and when $s > 4$. The second inequality can be rewritten as $0 > s^2 - 5s = s(s-5)$, and this inequality it true when $0 < s < 5$. But *both* inequalities must be correct in order for the Ratio Test to give a limit which is less than one, so the Ratio Test gives convergence when s lies in $(0,1) \cup (4,5)$. When s takes on any of the values 0, 1, 4, or 5, the Ratio Test fails because the limit is one. Otherwise, the series diverges. It remains, then, to test the four endpoints.

If $s = 0$ or $s = 5$, then $s^2 - 5s + 2 = 2$, and the series becomes

$$\sum_{k=1}^{\infty} \frac{1}{2k},$$

which is a divergent harmonic series.

If $s = 1$ or $s = 4$, then $s^2 - 5s + 2 = -2$, and the series becomes

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{2k},$$

which is a convergent alternating harmonic series.

Thus, the series

$$\sum_{k=1}^{\infty} \frac{(s^2 - 5s + 2)^k}{k2^{k+1}}$$

converges when s lies in $(0, 1] \cup [4, 5)$ and diverges otherwise.

MTH 2410, Calculus II—Exam I **Sept. 10, 2009**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 4:50 pm.

1. Evaluate the following limits:

   (a) $\lim_{x\to 0} \dfrac{e^{-x} - 1 + x}{x^2}$

   (b) $\lim_{x\to\infty} xe^{-x^2}$

2. Use Newton's Method to find an approximate solution for the equation $x^3 + x^2 + x = 1$. Begin with $x_0 = 1$; give x_2 correct to at least four digits to the right of the decimal.

3. Find the following:

   (a) $\displaystyle\int \frac{x+1}{x^2+2x+5}\,dx$

   (b) $\displaystyle\int \sin x \cos^3 x\,dx$

4. Find the following:

   (a) $\displaystyle\int t^2 \ln 4t\,dt$

   (b) $\displaystyle\int \frac{dt}{(9+t^2)^{3/2}}$

5. (a) Write out the form of the partial fractions decomposition for
$$\frac{3x^2+12x-7}{(x-3)^3(x+1)(x^2+2x+4)^2}.$$

   (b) Evaluate the integral: $\displaystyle\int_3^5 \frac{dt}{t^2-3t+2}$.

6. Give the four-subdivision Midpoint Rule approximation to the integral $\displaystyle\int_1^9 \sqrt{1+u^3}\,du$. Give this approximation to at least four digits to the right of the decimal.

7. Brünhilde wants an approximate value for $\ln 2$ correct to at least 6 digits (that is, accurate to within ± 0.0000005). She has decided to do a Simpson's Rule approximation of the definite integral $\int_1^2 dx/x$, but she is unsure how many subdivisions she needs in order to get the desired accuracy. What is the smallest number of subdivisions that will guarantee her the accuracy she wants?

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 2410, Calculus II—Exam I **Sept. 10, 2009**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 4:50 pm.

1. Evaluate the following limits:

   (a) $\lim_{x\to 0} \dfrac{e^{-x} - 1 + x}{x^2}$

   (b) $\lim_{x\to\infty} xe^{-x^2}$

   Solution:

   (a) $\lim_{x\to 0}(e^{-x} - 1 + x) = 0$ and $\lim_{x\to 0} x^2 = 0$, so we may attempt l'Hôpital's Rule. The derivative in the numerator is $-e^{-x} + 1$ and that in the denominator is $2x$. Both $\to 0$ as $x \to 0$, so we may attempt l'Hôpital's Rule a second time. We find that

   $$\lim_{x\to 0} \frac{e^{-x} - 1 + x}{x^2} = \lim_{x\to 0} \frac{-e^{-x} + 1}{2x} = \lim_{x\to 0} \frac{e^{-x}}{2} = \frac{1}{2}.$$

   (b) We first note that $xe^{-x^2} = x/e^{x^2}$, where both numerator and denominator $\to \infty$ as $x \to \infty$. We may therefore attempt l'Hôpital's Rule, and we find that

   $$\lim_{x\to\infty} xe^{-x^2} = \lim_{x\to\infty} \frac{x}{e^{x^2}} = \lim_{x\to\infty} \frac{1}{2xe^{x^2}} = 0.$$

2. Use Newton's Method to find an approximate solution for the equation $x^3 + x^2 + x = 1$. Begin with $x_0 = 1$; give x_2 correct to at least four digits to the right of the decimal.

   Solution: According to Newton's Method, we put $x_0 = 1$, $f(x) = x^3 + x^2 + x - 1$, and apply the recurrence relation

   $$\begin{aligned} x_k &= x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})} \\ &= x_{k-1} - \frac{x_{k-1}^3 + x_{k-1}^2 + x_{k-1} - 1}{3x_{k-1}^2 + 2x_{k-1} + 1}. \end{aligned}$$

   Thus,

   $$\begin{aligned} x_1 &= 1 - \frac{2}{6} = \frac{2}{3}; \\ x_2 &= \frac{2}{3} - \frac{(2/3)^3 + (2/3)^2 + (2/3) - 1}{3(2/3)^2 + 2(2/3) + 1} = \frac{5}{9}. \end{aligned}$$

3. Find the following:

   (a) $\displaystyle\int \frac{x+1}{x^2 + 2x + 5}\, dx$

(b) $\int \sin x \cos^3 x \, dx$

Solution:

(a) Let $u = x^2 + 2x + 5$. Then $du = (2x+2)\,dx$, or $(x+1)\,dx = \frac{1}{2}\,du$. Thus

$$\int \frac{x+1}{x^2+2x=5}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln(x^2+2x+5) + c.$$

(We may remove the absolute value symbols because $x^2 + 2x + 5 > 0$ for all x.)

(b) Let $u = \cos x$. Then $du = -\sin x\,dx$, so

$$\int \sin x \cos^3 x\,dx = -\int u^3\,du = -\frac{u^4}{4} + c = -\frac{1}{4}\cos^4 x + c.$$

4. Find the following:

(a) $\int t^2 \ln 4t\,dt$

(b) $\int \frac{dt}{(9+t^2)^{3/2}}$

Solution:

(a) We integrate by parts, taking $u = \ln 4t$ and $dv = t^2\,dt$. Then $du = dt/t$, $v = t^3/3$, and

$$\begin{aligned}\int t^2 \ln 4t\,dt &= uv - \int v\,du = \frac{t^3}{3}\ln 4t - \int \frac{t^3}{3}\cdot\frac{dt}{t} \\ &= \frac{t^3}{3}\ln 4t - \frac{1}{3}\int t^2\,dt = \frac{t^3}{3}\left(\ln 4t - \frac{1}{3}\right) + c.\end{aligned}$$

(b) We let $t = 3\tan\theta$, so that $dt = 3\sec^2\theta\,d\theta$. Then

$$\begin{aligned}\int \frac{dt}{(9+t^2)^{3/2}} &= \int \frac{3\sec^2\theta\,d\theta}{(9+9\tan^2\theta)^{3/2}} = \int \frac{3\sec^2\theta\,d\theta}{27\sec^3\theta} \\ &= \frac{1}{9}\int \cos\theta\,d\theta = \frac{1}{9}\sin\theta + c = \frac{1}{9}\sin\arctan\frac{t}{3} + c \\ &= \frac{t}{9\sqrt{9+t^2}} + c.\end{aligned}$$

5. (a) Write out the form of the partial fractions decomposition for

$$\frac{3x^2+12x-7}{(x-3)^3(x+1)(x^2+2x+4)^2}.$$

(b) Evaluate the integral: $\int_3^5 \frac{dt}{t^2-3t+2}$.

Solution:

(a) The partial fractions decomposition of the fraction $\dfrac{3x^2+12x-7}{(x-3)^3(x+1)(x^2+2x+4)^2}$ has the form

$$\frac{A}{(x-3)^3}+\frac{B}{(x-3)^2}+\frac{C}{x-3}+\frac{D}{x+1}+\frac{Ex+F}{(x^2+2x+4)^2}+\frac{Gx+H}{x^2+2x+4}.$$

(b)

$$\begin{aligned}\int_3^5 \frac{dt}{t^2-3t+2} &= \int_3^5 \frac{dt}{(t-1)(t-2)} = \int_3^5 \left[\frac{1}{t-2}-\frac{1}{t-1}\right] dt \\ &= \big(\ln|t-2|-\ln|t-1|\big)\Big|_3^5 \\ &= \big(\ln 3-\ln 4\big)-\big(\ln 1-\ln 2\big)=\ln\frac{3}{2}.\end{aligned}$$

6. Give the four-subdivision Midpoint Rule approximation to the integral $\displaystyle\int_1^9 \sqrt{1+u^3}\,du$. Give this approximation to at least four digits to the right of the decimal.

Solution: Subdividing $[1,9]$ into four equal sub-intervals gives $u_k = 1+2k$, $k=0$, 1, 2, 3, 4. The midpoints of these sub-intervals are 2, 4, 6, and 8, and the width of each sub-interval is 2. Thus, the Midpoint Rule approximation for $\displaystyle\int_1^9 \sqrt{1+u^3}\,du$ is

$$2\cdot\big(\sqrt{1+2^3}+\sqrt{1+4^3}+\sqrt{1+6^3}+\sqrt{1+8^3}\big)=2(3+\sqrt{65}+\sqrt{217}+3\sqrt{57}\big).$$

This is about 96.8853618. [Note: The integral is 97.43785354676, approximately.]

7. Brünhilde wants an approximate value for $\ln 2$ correct to at least 6 digits (that is, accurate to within ± 0.0000005). She has decided to do a Simpson's Rule approximation of the definite integral $\int_1^2 dx/x$, but she is unsure how many subdivisions she needs in order to get the desired accuracy. What is the smallest number of subdivisions that will guarantee her the accuracy she wants?

Solution: The magnitude of the error in a Simpson's Rule approximation to the integral $\int_a^b f(t)\,dt$ is at most $M(b-a)^5/(180n^4)$, where n is the number of subdivisions and M is any number for which $|f^{(4)}(t)|\le M$ when $a\le t\le b$. In the integral given, we have $f(t)=1/t$, so that $f^{(4)}(t)=24/t^5$. Thus, $f^{(4)}$ is positive and decreasing on the interval $[1,2]$, and so takes on its maximum value in $[1,2]$ at the left-hand endpoint, where $t=1$. Consequently, $|f^{(4)}(t)|\le f^{(4)}(1)=24$ for all t in $[1,2]$. We must therefore find a positive even integer n which satisfies the inequality

$$\frac{24(2-1)^5}{180n^4}\le\frac{1}{2000000},$$

or

$$\frac{800000}{3}\le n^4.$$

Thus, we require that $n\ge\sqrt[4]{800000/3}\sim 22.72$. Because n must be even, the smallest choice that Brünhilde can make is $n=24$.

MTH 2410, Calculus II—Exam II **Oct. 8, 2009**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 4:50 pm.

1. Find the volume of the solid generated by revolving the region bounded by the curves $x = y^2$ and $x = 2y$ about the line $y = 3$.

2. A certain solid has the unit circle as its base. The intersection of any plane perpendicular to the x-axis with the solid is an equilateral triangle. Find the volume of the solid.

3. When a particle is located at a distance x meters from the origin, a force of $\cos(\pi x/3)$ newtons acts on it. How much work is done by this force in moving the particle from $x = 1$ m to $x = 2$ m?

4. Find the length of the arc, from $x = 4$ to $x = 7$, of the curve $y = \dfrac{2\ln x - 4x^2}{8}$.

5. Evaluate the definite integral: $\displaystyle\int_0^1 \frac{dx}{(1+x^2)^{3/2}}$.

6. For which values of p does the integral $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^p}$ converge? Diverge? Give your reasoning.

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 2410, Calculus II—Exam II **Oct. 8, 2009**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 4:50 pm.

1. Find the volume of the solid generated by revolving the region bounded by the curves $x = y^2$ and $x = 2y$ about the line $y = 3$.

   Solution: The curves intersect at $(0,0)$ and $(4,2)$. We will use the method of cylindrical shells. A typical shell has radius $(3-y)$ and height $2y - y^2$. Thus

$$\begin{aligned} V &= 2\pi \int_0^2 [\text{radius}(y) \times \text{height}(y)]\, dy \\ &= 2\pi \int_0^2 (3-y)(2y-y^2)\, dy = 2\pi \int_0^2 (6y - 5y^2 + y^3)\, dy \\ &= 2\pi \left(3y^2 - \frac{5}{3}y^3 + \frac{1}{4}y^4\right)\Bigg|_0^2 = \frac{16\pi}{3}. \end{aligned}$$

   Alternate Solution: Use the method of washers. A typical washer has outer radius equal to $(3 - x/2)$ and inner radius $(3 - \sqrt{x})$. Thus

$$\begin{aligned} V &= \pi \int_0^4 \left([\text{outer radius}(x)]^2 - [\text{inner radius}(x)]^2\right)\, dx \\ &= \pi \int_0^4 \left[\left(3 - \frac{x}{2}\right)^2 - \left(3 - \sqrt{x}\right)^2\right] dx = \pi \int_0^4 \left(6\sqrt{x} - 4x + \frac{1}{4}x^2\right) dx \\ &= \left(4x^{3/2} - 2x^2 + \frac{1}{12}x^3\right)\Bigg|_0^4 = \frac{16\pi}{3}. \end{aligned}$$

2. A certain solid has the unit circle as its base. The intersection of any plane perpendicular to the x-axis with the solid is an equilateral triangle. Find the volume of the solid.

   Solution: This is Example 7, from Section 2 of Chapter 7 (page 369) of the textbook.

3. When a particle is located at a distance x meters from the origin, a force of $\cos(\pi x/3)$ newtons acts on it. How much work is done by this force in moving the particle from $x = 1$ m to $x = 2$ m?

   Solution: Work is the integral of force over the distance through which it acts. Hence

$$\begin{aligned} W &= \int_1^2 \cos\left(\frac{\pi}{3}x\right)\, dx = \frac{3}{\pi}\sin\frac{\pi}{3}x\Bigg|_1^2 \\ &= \frac{3}{\pi}\left(\sin\frac{2}{3}\pi - \sin\frac{1}{3}\pi\right) = \frac{3}{\pi}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = 0. \end{aligned}$$

4. Find the length of the arc, from $x = 4$ to $x = 7$, of the curve $y = \dfrac{2\ln x - 4x^2}{8}$.

Solution: Arc-length s is given by

$$
\begin{aligned}
s &= \int_4^7 \sqrt{1 + [f'(x)]^2}\, dx = \int_4^7 \sqrt{1 + \left(\frac{1}{4x} - x\right)^2}\, dx \\
&= \int_4^7 \sqrt{\frac{1}{16x^2} + \frac{1}{2} + x^2}\, dx = \int_4^7 \sqrt{\left(\frac{1}{4x} + x\right)^2}\, dx \\
&= \int_4^7 \left(\frac{1}{4x} + x\right) dx = \left(\frac{1}{4}\ln x + \frac{1}{2}x^2\right)\Bigg|_4^7 = \frac{33}{2} + \frac{1}{4}\ln\frac{7}{4}.
\end{aligned}
$$

5. Evaluate the definite integral: $\displaystyle\int_0^1 \frac{dx}{(1+x^2)^{3/2}}$.

Solution: We make the substitution $x = \tan\theta$. Then $dx = \sec^2\theta\, d\theta$. Moreover, when $x = 0$ we have $\theta = 0$ and when $x = 1$ we have $\theta = \pi/4$. Thus

$$
\begin{aligned}
\int_0^1 \frac{dx}{(1+x^2)^{3/2}} &= \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^{3/2}} = \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^{3/2}} \\
&= \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{\sec^3\theta} \\
&= \int_0^{\pi/4} \cos\theta\, d\theta = \sin\theta\Big|_0^{\pi/4} = \sin\frac{\pi}{4} - \sin 0 = \frac{1}{\sqrt{2}}.
\end{aligned}
$$

6. For which values of p does the integral $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^p}$ converge? Diverge? Give your reasoning.

Solution: We must evaluate $\displaystyle\lim_{T\to\infty}\int_2^T \frac{dx}{x(\ln x)^p}$. To this end, we substitute $u = \ln x$, which gives $du = dx/x$. Moreover, $u = \ln x$ means that $u = \ln 2$ when $x = 2$ and $u = \ln T$ when $x = T$. Thus,

$$
\begin{aligned}
\int_2^T \frac{dx}{x(\ln x)^p} &= \int_{\ln 2}^{\ln T} \frac{du}{u^p} \\
&= \begin{cases} \dfrac{1}{1-p}[(\ln T)^{1-p} - (\ln 2)^{1-p}], & \text{when } p \neq 1. \\[2ex] \ln\ln T - \ln\ln 2, & \text{when } p = 1. \end{cases}
\end{aligned}
$$

Thus, when $p = 1$ we have

$$
\int_2^\infty \frac{dx}{x(\ln x)^p} = \lim_{T\to\infty}[\ln\ln T - \ln\ln 2],
$$

and this limit does not exist. So the integral diverges when $p = 1$.

When $p \neq 1$, we have

$$\int_2^\infty \frac{dx}{x(\ln x)^p} = \lim_{T\to\infty} \frac{1}{1-p}[(\ln T)^{1-p} - (\ln 2)^{1-p}],$$
$$= \begin{cases} \dfrac{(\ln 2)^{1-p}}{p-1}, & \text{when } p > 1, \\ \\ \infty, & \text{when } p < 1, \end{cases}$$

because $(\ln T)^{1-p} \to 0$ when $T \to \infty$ if $1 - p < 0$, while $(\ln T)^{1-p} \to \infty$ when $T \to \infty$ if $1 - p > 0$.

We conclude that $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^p}$ converges when $p > 1$ and diverges when $p \leq 1$.

MTH 2410, Calculus II—Exam III **Nov. 12, 2009**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 4:50 pm.

1. Find $\lim_{x\to 0} \dfrac{6\sin x - 6x + x^3}{x^5}$.

2. Solve the initial value problem:

$$\begin{aligned} \frac{dS}{dt} &= 6 - \frac{3}{100}S; \\ S(0) &= 50. \end{aligned}$$

3. Find the volume generated when the triangle whose vertices are $(1,0)$, $(2,0)$, and $(1,1)$ is rotated about the y-axis.

4. A curve C is given by the parametric equations

$$\begin{aligned} x &= \cos^3 t, \\ y &= \sin^3 t, \end{aligned}$$

with $0 \le t \le 2\pi$. Locate all first-quadrant points on C where the slope of the tangent line is -1. What is $\dfrac{d^2y}{dx^2}$ at each of these points?

5. Find the area of the region that lies outside the polar curve $r = 1/2$ but inside the polar curve $r = \cos\theta$.

6. A curve γ is given by the parametric equations

$$\begin{aligned} x(t) &= 3t^2 + 48t + 3, \\ y(t) &= 4t^2 + 64t - 1 \end{aligned}$$

Find the length of the portion of γ that corresponds to $0 \le t \le 5$.

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 2410, Calculus II—Exam III **Nov. 12, 2009**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 4:50 pm.

1. Find $\lim_{x\to 0} \dfrac{6\sin x - 6x + x^3}{x^5}$.

   Solution: Noting at every step that numerator and denominator both approach 0 as $x \to 0$, we apply l'Hôpital's Rule repeatedly to obtain:

$$\begin{aligned}\lim_{x\to 0} \frac{6\sin x - 6x + x^3}{x^5} &= \lim_{x\to 0} \frac{6\cos x - 6 + 3x^2}{5x^4} \\ &= \lim_{x\to 0} \frac{-6\sin x + 6x}{20x^3} \\ &= \lim_{x\to 0} \frac{-6\cos x + 6}{60x^2} \\ &= \lim_{x\to 0} \frac{6\sin x}{120x} = \frac{1}{20},\end{aligned}$$

   where the final equation follows from the fact that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.

2. Solve the initial value problem:

$$\begin{aligned}\frac{dS}{dt} &= 6 - \frac{3}{100}S; \\ S(0) &= 50.\end{aligned}$$

   Solution: Separating variables and integrating, we have

$$\begin{aligned}\frac{100\,dS}{600 - 3S} &= dt \\ \int \frac{100\,dS}{600 - 3S} &= \int dt \\ -\frac{100}{3}\ln|600 - 3S| &= t + c,\end{aligned}$$

   for a certain constant c. Thus,

$$\begin{aligned}|600 - 3S| &= Ce^{-3t/100},\ C > 0,\ \text{or} \\ 600 - 3S &= Ce^{-3t/100},\ \text{with } C \text{ arbitrary.}\end{aligned}$$

   From this it follows that $S = 200 - Ce^{-3t/100}$ for a certain constant C. But then

$$\begin{aligned}50 = S(0) &= 200 - Ce^{-3\cdot 0/100} \\ &= 200 - C,\end{aligned}$$

   whence $C = 150$. Consequently

$$S(t) = 200 - 150e^{-3t/100}.$$

3. Find the volume generated when the triangle whose vertices are $(1,0)$, $(2,0)$, and $(1,1)$ is rotated about the y-axis.

Solution: The left side of the triangle is the vertical line $x = 1$, while the right side is the line $x = 2 - y$. Using the method of washers, we find that the volume V obtained by revolving the triangle about the y-axis is

$$\begin{aligned} V &= \pi \int_0^1 [(2-y)^2 - 1^2]\, dy \\ &= \pi \int_0^1 (3 - 4y + y^2)\, dy \\ &= \pi \left(3y - 2y^2 + \frac{y^3}{3}\right)\Big|_0^1 \\ &= \pi \left(3 - 2 + \frac{1}{3}\right) = \frac{4\pi}{3}. \end{aligned}$$

Note: Using the method of cylindrical shells leads to the integral $2\pi \int_1^2 x(2-x)\, dx$.

4. A curve C is given by the parametric equations

$$\begin{aligned} x &= \cos^3 t, \\ y &= \sin^3 t, \end{aligned}$$

with $0 \le t \le 2\pi$. Locate all first-quadrant points on C where the slope of the tangent line is -1. What is $\frac{d^2y}{dx^2}$ at each of these points?

Solution: We have

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3\sin^2 t \cos t}{-3\cos^2 t \sin t} = -\tan t.$$

The point $(x(t), y(t))$ lies in the first quadrant if, and only if, both $x(t)$ and $y(t)$ are non-negative. The only values of t which lie in the given interval $[0, 2\pi]$ and meet this requirement are those which lie in the interval $[0, \pi/2]$, and the only value of t in this interval for which $-\tan t = -1$ is $t = \pi/4$. Consequently, the only point in the first quadrant at which the slope of the tangent line to the curve C is the point $(\cos^3 \pi/4, \sin^3 \pi/4) = (\sqrt{2}/4, \sqrt{2}/4)$.

We must compute $\frac{d^2y}{dx^2}$ at this point. We note that

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &= \frac{d}{dt}\left(-\tan t\right) / \frac{d}{dt} x(t) \\ &= \frac{\sec^2 t}{3\cos^2 t \sin t} = \frac{1}{3} \csc t \sec^4 t. \end{aligned}$$

We conclude that

$$\left.\frac{d^2y}{dx^2}\right|_{t=\pi/4} = \frac{4\sqrt{2}}{3}.$$

5. Find the area of the region that lies outside the polar curve $r = 1/2$ but inside the polar curve $r = \cos\theta$.

Solution: We begin with a figure: The area A that we are to find is that of the rightmost

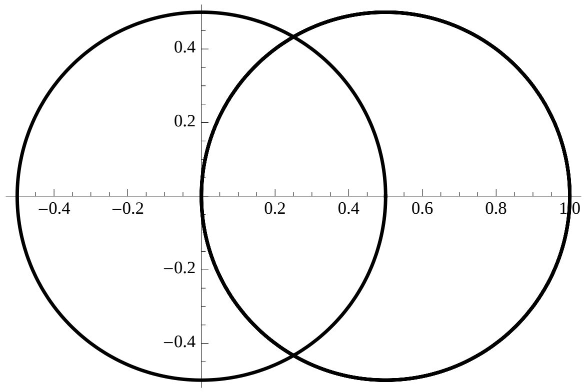

Figure 1: The curves of Problem 5

of the three bounded regions. Solving the equations of the curves simultaneously, we find that the polar coordinates of the two intersection points are $(1/2, \pi/3)$ for the upper point and $(1/2, -\pi/3)$ for the lower point. We therefore compute A using

$$\begin{aligned} A &= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left[\cos^2\theta - \left(\frac{1}{2}\right)^2\right] d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(\cos^2\theta - \frac{1}{4}\right) d\theta \\ &= \frac{1}{4}\int_{-\pi/3}^{\pi/3} (1+\cos 2\theta)\, d\theta - \frac{1}{8}\int_{-\pi/3}^{\pi/3} d\theta \\ &= \frac{1}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Bigg|_{-\pi/3}^{\pi/3} - \frac{1}{8}\theta\Bigg|_{-\pi/3}^{\pi/3} = \frac{\sqrt{3}}{8} + \frac{\pi}{12}. \end{aligned}$$

6. A curve γ is given by the parametric equations

$$\begin{aligned} x(t) &= 3t^2 + 48t + 3, \\ y(t) &= 4t^2 + 64t - 1 \end{aligned}$$

Find the length of the portion of γ that corresponds to $0 \le t \le 5$.

Solution: Arclength s is given by

$$\begin{aligned} s &= \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \int_0^5 \sqrt{(6t+48)^2 + (8t+64)^2}\, dt \\ &= \int_0^5 \sqrt{36(t+8)^2 + 64(t+8)^2}\, dt = \int_0^5 \sqrt{100(t+8)^2}\, dt \\ &= 10 \int_0^5 (t+8)\, dt = 10 \left(\frac{t^2}{2} + 8t \right) \Bigg|_0^5 = 525. \end{aligned}$$

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 2410, Calculus II—Final Exam **Dec. 8, 2009**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 4:50 pm.

1. Use Newton's Method to find an approximate solution to the equation $x^3+x=1$. Begin with $x_0 = 0.75$, and give the resulting value of x_2 correct to at least five digits to the right of the decimal.

2. The region bounded by the curve $4x+y^2-16=0$ and the line $x=0$ is revolved about the line $x=-1$. Find the volume of the solid so generated.

3. Solve the initial value problem:

$$\frac{dS}{dt} = 8 - \frac{7}{100}S;$$
$$S(0) = 75.$$

   Give the solution S as a function of t.

4. Æthelbert wants to use Simpson's Rule to estimate the value of the integral $\int_{-1}^{1} \sqrt{8-x^3}\,dx$. He needs to be sure his error is at most 0.00001. Use the plot (Figure 1) of $\frac{d^4}{dx^4}\left(\sqrt{8-x^3}\right)$ and the fact that if M is a number for which $|f^{(4)}(x)| \le M$ for every $x \in [a,b]$, then the magnitude of the error in an n-subdivision Simpson's Rule approximation for the integral $\int_a^b f(x)\,dx$ is at most $\frac{M(b-a)^5}{180n^4}$ to determine how many subdivisions he needs to use.

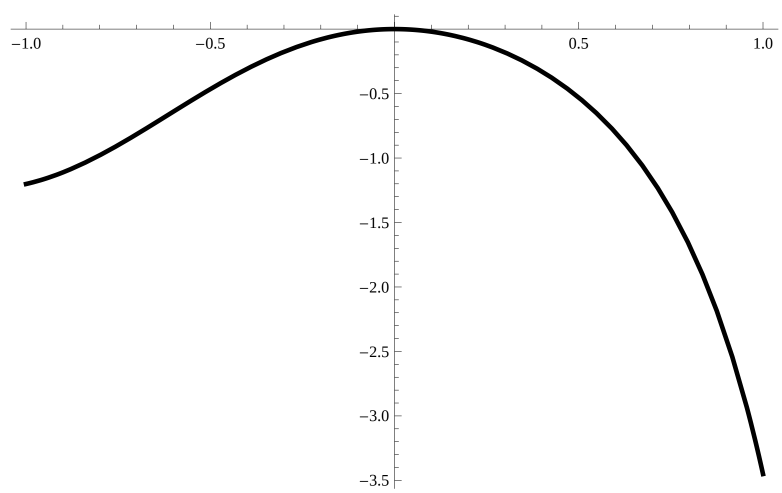

Figure 1: Graph of $\frac{d^4}{dx^4}\left(\sqrt{8-x^3}\right)$

5. A curve C is given by the parametric equations

$$x = \cos^2 t,$$
$$y = \sin^3 t,$$

with $0 \le t \le \pi/2$. Locate all points on C where the slope of the tangent line is $-3/4$. What is $\dfrac{d^2y}{dx^2}$ at each of these points?

6. Find the Maclaurin series for the arctangent function. Give the first three non-zero terms, the general term, and the interval of convergence. [Hint: $\dfrac{d}{dx}\arctan x = \dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)}$.]

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 2410, Calculus II—Final Exam **Dec. 8, 2009**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 4:50 pm.

1. Use Newton's Method to find an approximate solution to the equation $x^3+x=1$. Begin with $x_0=0.75$, and give the resulting value of x_2 correct to at least five digits to the right of the decimal.

   Solution: The Newton's Method iteration scheme for this problem is

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} = x_{n-1} - \frac{x_{n-1}^3 + x_{n-1} - 1}{3x_{n-1}^2+1} = \frac{2x_{n-1}^3+1}{3x_{n-1}^2+1}.$$

   Thus, with $x_0 = 0.75$, we have

$$x_1 = \frac{2\cdot(0.75)^3+1}{3\cdot(0.75)^2+1} \sim 0.6860465;$$
$$x_2 \sim \frac{2\cdot(0.6860465)^3+1}{3\cdot(0.6860465)^2+1} \sim 0.6823396.$$

   To five digits' accuracy, $x_0 \sim 0.68234$.

2. The region bounded by the curve $4x+y^2-16=0$ and the line $x=0$ is revolved about the line $x=-1$. Find the volume of the solid so generated.

   Solution: We use the method of washers. Solving for x in terms of y, we obtain $x = 4 - y^2/4$. Thus, the volume V is given by

$$V = \pi\int_{-4}^{4}\left[\left(4-\frac{y^2}{4}+1\right)^2 - 1^2\right]dy = \pi\int_{-4}^{4}\left(24-\frac{5}{2}y^2+\frac{1}{16}y^4\right)dy$$
$$= \pi\left(24y-\frac{5}{2}y^2+\frac{1}{16}y^4\right)\Big|_{-4}^{4} = \frac{1664}{15}\pi.$$

3. Solve the initial value problem:

$$\frac{dS}{dt} = 8 - \frac{7}{100}S;$$
$$S(0) = 75.$$

   Give the solution S as a function of t.

   Solution: Separating variables and integrating leads to

$$\frac{100dS}{800-7S} = dt;$$
$$\int\frac{100dS}{800-7S} = \int dt;$$
$$-\frac{100}{7}\ln|800-7S| = t+c;$$
$$\ln|800-7S| = -\frac{7}{100}t+c;$$

   for a certain constant c. But $S=75$ when $t=0$, so

$$\ln 275 = c.$$

Thus

$$800 - 7S = 275e^{-7t/100}, \text{ and}$$
$$S = \frac{800}{7} - \frac{275}{7}e^{-7t/100}.$$

4. Æthelbert wants to use Simpson's Rule to estimate the value of the integral $\int_{-1}^{1} \sqrt{8 - x^3}\, dx$. He needs to be sure his error is at most 0.00001. Use the plot (Figure 1) of $\frac{d^4}{dx^4}\left(\sqrt{8 - x^3}\right)$ and the fact that if M is a number for which $|f^{(4)}(x)| \leq M$ for every $x \in [a, b]$, then the magnitude of the error in an n-subdivision Simpson's Rule approximation for the integral $\int_a^b f(x)\, dx$ is at most $\frac{M(b-a)^5}{180n^4}$ to determine how many subdivisions he needs to use.

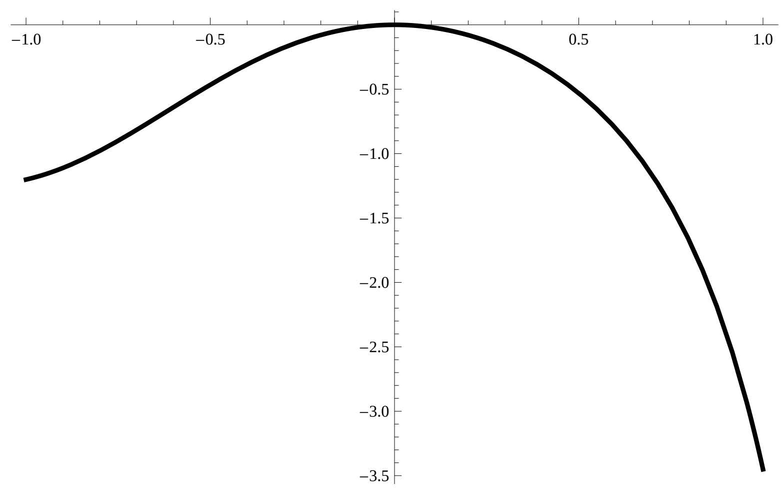

Figure 1: Graph of $\frac{d^4}{dx^4}\left(\sqrt{8 - x^3}\right)$

Solution: Taking $f(x) = \sqrt{8 - x^3}$, we see that it is clear from the graph that $|f^{(4)}(x)| \leq 7/2$ for all $x \in [-1, 1]$. Because $b = 1$ and $a = -1$, $(b - a)^5 = 2^5 = 32$, and we need to solve the inequality

$$\frac{(7/2) \cdot 32}{180n^4} \leq \frac{1}{100000},$$

or, equivalently,

$$n^4 \geq \frac{560000}{9}.$$

We therefore require that $n \geq \sqrt[4]{560000/9} \sim 15.7938$. Because n must be an even integer for Simpson's Rule, we will have to take $n = 16$.

5. A curve C is given by the parametric equations

$$x = \cos^2 t,$$
$$y = \sin^3 t,$$

with $0 \le t \le \pi/2$. Locate all points on C where the slope of the tangent line is $-3/4$. What is $\frac{d^2y}{dx^2}$ at each of these points?

Solution: If $x(t) = \cos^2 t$ and $y(t) = \sin^3 t$, then

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3\sin^2 t\cos t}{-2\cos t\sin t} = -\frac{3}{2}\sin t.$$

This means that the slope of this curve is $-3/4$ when $\sin t = 1/2$, or when $t = \pi/6$. The corresponding point on the curve is the point with coordinates $\left(\frac{3}{4}, \frac{1}{8}\right)$.

Now we have

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\frac{d}{dt}\left(-\frac{3}{2}\sin t\right)}{\frac{d}{dt}\left(\cos^2 t\right)} = \frac{3}{4}\csc t.$$

Consequently, $\left.\frac{d^2y}{dx^2}\right|_{t=\pi/6} = \frac{3}{2}$.

6. Find the Maclaurin series for the arctangent function. Give the first three non-zero terms, the general term, and the interval of convergence. [Hint: $\frac{d}{dx}\arctan x = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$.]

   Solution: We have

$$\begin{aligned}\frac{d}{dx}\arctan x &= \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} \\ &= 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots,\end{aligned}$$

   whenever $|x| < 1$, by our analysis of the geometric series. Thus, term-by-term integration gives

$$\begin{aligned}\arctan x &= \int \frac{dx}{1+x^2} \\ &= c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n\frac{x^{2n+1}}{2n+1} + \cdots,\end{aligned}$$

   for a certain constant c, also when $|x| < 1$. But $\arctan 0 = 0$, and every term containing a power of x on the right side of the last displayed equation above vanishes when $x = 0$. Consequently, $c = 0$, and

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n\frac{x^{2n+1}}{2n+1} + \cdots.$$

   The interval of convergence for the integrated series is the same as the interval of convergence for the integrand, so this representation is correct throughout the interval $(-1, 1)$, which is the interval of convergence. [Note: This is Example 8.6.7, page 456, of the textbook.]

MTH 2410, Calculus II—Exam I **Feb. 20, 2014**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 1:50 pm.

1. A particle traveling along a straight line at 144 ft/sec experiences a deceleration, beginning at time $t = 0$, of $8t$ ft/sec^2. Show how to determine how many feet it travels before coming to a complete stop.
2. Consider the region bounded by the curve $y = 1 + \sqrt{x}$, the line $y = 1$, and the lines $x = 4$ and $x = 9$. Show how to find the volume generated when this region is rotated about the x-axis.
3. Show how to find $\displaystyle\int \frac{x+1}{x^2+2x+5}\,dx$.
4. Show how to find $\displaystyle\int \sin x \cos^3 x\,dx$.
5. Show how to find $\displaystyle\int t^2 \ln 4t\,dt$.
6. Show how to find $\displaystyle\int \frac{dt}{(9+t^2)^{3/2}}$.
7. Write out the form of the partial fractions decomposition for

$$\frac{3x^2+12x-7}{(x-3)^3(x+1)(x^2+2x+4)^2}.$$

   Do not solve for the coefficients or carry out any integrations.

8. Show how to evaluate the integral: $\displaystyle\int_3^5 \frac{dt}{t^2-3t+2}$.

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 2410, Calculus II—Exam I **Feb. 20, 2014**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 1:50 pm.

1. A particle traveling along a straight line at 144 ft/sec experiences a deceleration, beginning at time $t = 0$, of $8t$ ft/sec^2. How many feet does it travel before coming to a complete stop?

   Solution: Acceleration, $a(t)$ is given by $a(t) = -8t$. Thus,

   $$\begin{aligned} v(t) &= v(0) + \int_0^t a(\tau)\,d\tau = v(0) - \int_0^t 8\tau\,d\tau \\ &= 144 - 4t^2. \end{aligned}$$

   The particle comes to a complete stop when $v(t) = 0$, or when $t = 6$. (We ignore the solution $t = -6$ because it has no significance for us.)

   For position $s(t)$, we have

   $$\begin{aligned} s(t) &= s(0) + \int_0^t v(\tau)\,d\tau = s(0) + \int_0^t [144 - 4\tau^2]\,d\tau \\ &= 144t - \frac{4}{3}t^3. \end{aligned}$$

   Thus, setting $s(0) = 0$, we find that the distance traveled before stopping is

   $$s(6) = 144 \cdot 6 - \frac{4}{3} \cdot 6^3 = 576 \text{ ft}$$

2. Consider the region bounded by the curve $y = 1 + \sqrt{x}$, the line $y = 1$, and the lines $x = 4$ and $x = 9$. Show how to find the volume generated when this region is rotated about the x-axis.

   Solution: If τ is a number between 4 and 9, the plane $x = \tau$ intersects this volume in a washer whose center is at the point $x = \tau$ on the x-axis, whose inner radius is 1 and whose outer radius is $1 + \sqrt{\tau}$. The area of such a washer is $\pi\left[(1 + \sqrt{\tau})^2 - 1^2\right] = \pi\left[2\sqrt{\tau} + \tau\right]$, so the volume in question is

   $$\pi \int_4^9 \left[2\tau^{1/2} + \tau\right]\,d\tau = \pi\left(\frac{4}{3}\tau^{3/2} + \frac{1}{2}\tau^2\right)\Bigg|_4^9 = \pi\left(\frac{153}{2} - \frac{56}{3}\right) = \frac{347}{6}\pi.$$

3. Show how to find $\displaystyle\int \frac{x+1}{x^2 + 2x + 5}\,dx$.

   Solution: To find $\displaystyle\int \frac{x+1}{x^2+2x+5}\,dx$, we let $u = x^2 + 2x + 5$. Then $du = 2(x+1)\,dx$, or $(x+1)\,dx = \dfrac{1}{2}\,du$. Thus

   $$\int \frac{x+1}{x^2+2x+5}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+2x+5) + C.$$

4. Show how to find $\int \sin x \cos^3 x\,dx$.

Solution: To find $\int \sin x \cos^3 x\,dx$, we let $u = \cos x$. Then $du = -\sin x\,dx$, and we have

$$\int \sin x \cos^3 x\,dx = -\int u^3\,du = -\frac{1}{4}u^4 + C = -\frac{1}{4}\cos^4 x + C.$$

5. Show how to find $\int t^2 \ln 4t\,dt$.

Solution To find $\int t^2 \ln 4t\,dt$, we integrate by parts, taking $u = \ln 4t$ and $dv = t^2\,dt$. We have $du = \frac{1}{t}\,dt$, and we may take $v = \frac{1}{3}t^3$. Then

$$\begin{aligned}\int t^2 \ln 4t\,dt &= \int u\,dv = uv - \int v\,du \\ &= \frac{1}{3}t^3 \ln 4t - \frac{1}{3}\int t^3 \cdot \frac{1}{t}\,dt = \frac{1}{3}t^3 \ln 4t - \frac{1}{3}\int t^2\,dt \\ &= \frac{1}{3}t^3 \ln 4t - \frac{1}{9}t^3 + C.\end{aligned}$$

6. Show how to find $\int \frac{dt}{(9+t^2)^{3/2}}$

Solution: To find $\int \frac{dt}{(9+t^2)^{3/2}}$, we let $t = 3\tan\theta$. Then $dt = 3\sec^2\theta\,d\theta$, and

$$\begin{aligned}\int \frac{dt}{(9+t^2)^{3/2}} &= \int \frac{3\sec^2\theta\,d\theta}{(9+9\tan^2\theta)^{3/2}} = \frac{1}{9}\int \frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^{3/2}} \\ &= \frac{1}{9}\int \cos\theta\,d\theta = \frac{1}{9}\sin\theta + C.\end{aligned}$$

But $\tan\theta = t/3$, and it is easily seen that this means $\sin\theta = t/\sqrt{t^2+9}$. It follows that

$$\int \frac{dt}{(9+t^2)^{3/2}} = \frac{t}{9\sqrt{t^2+9}} + C.$$

7. Write out the form of the partial fractions decomposition (but *do not* solve or integrate) for

$$\frac{3x^2+12x-7}{(x-3)^3(x+1)(x^2+2x+4)^2}.$$

Solution:

$$\frac{3x^2+12x-7}{(x-3)^3(x+1)(x^2+2x+4)^2} = \frac{A}{(x-3)^3} + \frac{B}{(x-3)^2} + \frac{C}{x-3} + \frac{D}{x+1} + \frac{Ex+F}{(x^2+2x+4)^2} + \frac{Gx+H}{x^2+2x+4}.$$

8. Show how to evaluate the integral: $\displaystyle\int_3^5 \frac{dt}{t^2 - 3t + 2}$.

Solution: We note first that

$$\int_3^5 \frac{dt}{t^2 - 3t + 2} = \int_3^5 \frac{dt}{(t-2)(t-1)}.$$

Consequently, we must find the partial fractions decomposition:

$$\begin{aligned}\frac{1}{(t-2)(t-1)} &= \frac{A}{t-2} + \frac{B}{t-1} \\ &= \frac{(A+B)t - (A+2B)}{(t-2)(t-1)}.\end{aligned}$$

If the numerator of the latter fraction is to equal the numerator of the first fraction above for all values of t, then we must have $A + B = 0$ and $-(A + 2B) = 1$. Solving these two equations simultaneously gives $A = 1$ and $B = -1$. Thus,

$$\begin{aligned}\int_3^5 \frac{dt}{t^2 - 3t + 2} &= \int_3^5 \frac{dt}{t-2} - \int_3^5 \frac{dt}{t-1} \\ &= \ln(t-2)\Big|_3^5 - \ln(t-1)\Big|_3^5 \\ &= \ln 3 - \ln 1 - \ln 4 + \ln 2 \\ &= \ln \frac{3}{2}.\end{aligned}$$

MTH 2410, Calculus II—Exam II **Mar. 20, 2014**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 1:50 pm.

1. A particle traveling along a straight line at 144 ft/sec experiences a deceleration, beginning at time $t = 0$, of $8t$ ft/sec^2. Show how to determine how many feet it travels before coming to a complete stop.

2. Use the relations $\tanh u = \dfrac{\sinh u}{\cosh u}$ and $\cosh^2 u - \sinh^2 u = 1$ to derive an algebraic expression for $\tanh(\sinh^{-1} x)$.

3. Show how to find the arc-length of the curve $y = \dfrac{1}{4}x^2 - \dfrac{1}{2}\ln x$ over the interval $[1, 2e]$.

4. Show how to evaluate $\displaystyle\int x^2 e^x\, dx$.

5. Show how to find the volume of the solid obtained by rotating the region bounded by the curves $y = 1/x^4$, $y = 0$, $x = 1$, and $x = 5$ about the y-axis.

6. Show how to determine whether the improper integral $\displaystyle\int_0^\infty \frac{x\,dx}{x^2+1}$ converges or diverges. If it diverges, explain why. If it converges, find its value.

7. (a) Give the trapezoidal approximation, with 3 subdivisions, for the definite integral $\displaystyle\int_4^7 \frac{dx}{x}$. Either give your answer as a fraction of integers, or give it accurate to at least 6 decimals.

   (b) The error in the trapezoidal approximation with n subdivisions for the integral $\displaystyle\int_a^b f(x)\,dx$ does not exceed the quantity $\dfrac{M(b-a)^3}{12n^2}$ for a certain number M.

   i. Explain what M is and how to find it.

   ii. According to this statement, what is the maximum possible error in the approximation you computed in the first part of this problem?

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 2410, Calculus II—Exam II **Mar. 20, 2014**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 1:50 pm.

1. A particle traveling along a straight line at 144 ft/sec experiences a deceleration, beginning at time $t = 0$, of $8t$ ft/sec^2. Show how to determine how many feet it travels before coming to a complete stop.

   Solution: Acceleration, $a(t)$ is given by $a(t) = -8t$. Thus,

$$\begin{aligned} v(t) &= v(0) + \int_0^t a(\tau)\,d\tau = v(0) - \int_0^t 8\tau\,d\tau \\ &= 144 - 4t^2. \end{aligned}$$

   The particle comes to a complete stop when $v(t) = 0$, or when $t = 6$. (We ignore the solution $t = -6$ because it has no significance for us.)

   For position $s(t)$, we have

$$\begin{aligned} s(t) &= s(0) + \int_0^t v(\tau)\,d\tau = s(0) + \int_0^t [144 - 4\tau^2]\,d\tau \\ &= 144t - \frac{4}{3}t^3, \end{aligned}$$

   where we hade set $s(0) = 0$. We find that the distance traveled before stopping is

$$s(6) = 144 \cdot 6 - \frac{4}{3} \cdot 6^3 = 576 \text{ ft}$$

2. Use the relations $\tanh u = \dfrac{\sinh u}{\cosh u}$ and $\cosh^2 u - \sinh^2 u = 1$ to derive an algebraic expression for $\tanh(\sinh^{-1} x)$.

   Solution: We have $\cosh^2 u - \sinh^2 u = 1$, so $\cosh^2 u = 1 + \sinh^2 u$. But $\cosh u > 0$ for all real u, so $\cosh u = +\sqrt{1 + \sinh^2 u}$. Therefore,

$$\begin{aligned} \tanh\left(\sinh^{-1} x\right) &= \frac{\sinh\left(\sinh^{-1} x\right)}{\cosh\left(\sinh^{-1} x\right)} = \frac{x}{\cosh\left(\sinh^{-1} x\right)} \\ &= \frac{x}{\sqrt{1 + \sinh^2\left(\sinh^{-1} x\right)}} = \frac{x}{\sqrt{1 + x^2}}. \end{aligned}$$

Thus,

$$\tanh\left(\sinh^{-1} x\right) = \frac{x}{\sqrt{1+x^2}}$$

3. Show how to find the arc-length of the curve $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$ over the interval $[1, 2e]$.

Solution: If $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, then $y' = \frac{1}{2}\left(x - \frac{1}{x}\right)$. Thus

$$\begin{aligned}\int_1^{2e} \sqrt{1+(y')^2}\,dx &= \int_1^{2e} \sqrt{1 + \frac{1}{4}\left(x^2 - 2 - \frac{1}{x^2}\right)}\,dx = \int_1^{2e} \sqrt{\frac{1}{4}\left(x^2 + 2 + \frac{1}{x^2}\right)}\,dx \\ &= \frac{1}{2}\int_1^{2e} \left(x + \frac{1}{x}\right)\,dx = \frac{1}{2}\left(\frac{x^2}{2} + \ln x\right)\Big|_1^{2e} \\ &= e^2 + \frac{1}{4} + \frac{1}{2}\ln 2.\end{aligned}$$

4. Show how to evaluate $\int x^2 e^x\,dx$.

Solution: We integrate by parts, taking $u = x^2$, and $dv = e^x\,dx$. Then $du = 2x\,dx$, and $v = e^x$, so

$$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx.$$

To integrate $\int x e^x\,dx$, we again integrate by parts, this time taking $U = x$, $dV = e^x\,dx$. Then we must take $dU = dx$, and $V = e^x$. Thus,

$$\int x e^x\,dx = x e^x - \int e^x\,dx.$$

Hence,

$$\begin{aligned}\int x^2 e^x\,dx &= x^2 e^x - 2\left[x e^x - \int e^x\,dx\right] \\ &= x^2 e^x - 2x e^x + 2e^x + C.\end{aligned}$$

5. Show how to find the volume of the solid obtained by rotating the region bounded by the curves $y = 1/x^4$, $y = 0$, $x = 1$, and $x = 5$ about the y-axis.

Solution: Using the method of shells, we find that

$$V = 2\pi \int_1^5 x \cdot \frac{dx}{x^4} = 2\pi \int_1^5 x^{-3}\, dx = -\pi x^{-2} \Big|_1^5 = \frac{24}{25}\pi.$$

6. Show how to determine whether the improper integral $\displaystyle\int_0^\infty \frac{x\, dx}{x^2+1}\, dx$ converges or diverges. If it diverges, explain why. If it converges, find its value.

Solution: Putting $u = x^2 + 1$ we have $du = 2x\, dx$, so that

$$\int \frac{x\, dx}{x^2+1}\, dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln(x^2+1).$$

But

$$\int_0^\infty \frac{x\, dx}{x^2+1} = \lim_{T\to\infty} \int_0^T \frac{x\, dx}{x^2+1},$$

provided the limit exists. We therefore have

$$\lim_{T\to\infty} \int_0^T \frac{x\, dx}{x^2+1} = \lim_{T\to\infty} \left[\frac{1}{2}\ln(x^2+1)\right]\Big|_0^T = \frac{1}{2}\lim_{T\to\infty} \ln(T^2+1),$$

which does not exist. It follows that the improper integral diverges.

7. (a) Give the trapezoidal approximation, with 3 subdivisions, for the definite integral $\displaystyle\int_4^7 \frac{dx}{x}$. Either giive your answer as a fraction of integers, or give it accurate to at least 6 decimals.

   (b) The error in the trapezoidal approximation with n subdivisions for the integral $\displaystyle\int_a^b f(x)\, dx$ does not exceed the quantity $\dfrac{M(b-a)^3}{12n^2}$ for a certain number M.

   i. Explain what M is and how to find it.

   ii. According to this statement, what is the maximum possible error in the approximation you computed in the first part of this problem?

Solution:

(a) If there are to be three subdivision of $[4, 7]$, then $\Delta x = \frac{7-4}{3} = 1$. We then have $x_k = 4 + k\Delta x = 4 + k$, with $k = 0, 1, 2, 3$. The desired trapezoidal approximation, T, is then

$$\begin{aligned} T &= \frac{1}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)]\Delta x \\ &= \frac{1}{2}\left[\frac{1}{4} + \frac{2}{5} + \frac{2}{6} + \frac{1}{7}\right] = \frac{473}{840} \sim 0.563095. \end{aligned}$$

(b) i. The number M in the error estimate for the trapezoidal rule may be any number for which $|f''(x)| \leq M$ for every x in the interval $[a, b]$. We can find it by finding the absolute maximum of $|f(x)|$ on the interval $[a, b]$.

ii. When $f(x) = \dfrac{1}{x}$, we have $f''(x) = \dfrac{2}{x^3}$. But x^3 increases as x increases, so $f''(x)$ is a positive, decreasing function on $[4, 7]$. Thus, $f''(x)$ assumes its largest value in $[4, 7]$ at $x = 4$. We may therefore take $M = f''(4) = \dfrac{1}{32}$. Thus, error, E is at most

$$E = \frac{M(b-a)^3}{12n^2} = \frac{1}{32} \cdot \frac{(7-4)^3}{12 \cdot 3^2} = \frac{1}{128} = 0.0078125.$$

MTH 2410, Calculus II—Exam III **Apr. 24, 2014**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 1:50 pm.

1. Let $y = f(x)$ be the solution to the initial value problem

$$\frac{dy}{dx} = 2x + y;$$
$$f(0) = 1.$$

Show how to use Euler's Method with step size 1/10 to find an approximate value for $f(1/5)$. **Show the calculations that support your answer.**

2. Show how to determine whether each of the following series is divergent or convergent. **Explain your reasoning fully.**

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n + 4^n}$

3. Show how to determine whether the following series is divergent or convergent. **Explain your reasoning fully.**

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

4. The region bounded by the y-axis, the curve $y = x^2$, and the line $y = 4$ is revolved about the y-axis. Find the volume of the solid generated in this fashion.

5. Find the solution, $y = f(x)$, to the differential equation $y' = y(1 - x)^{-1}$, given that $y = 1$ when $x = 2$. Show your reasoning.

6. Brünhilde is working a scientific problem, and she needs a decent approximate value for the integral

$$\int_{-2}^{2} \frac{2x - 1}{\sqrt{1 + x^4}}\, dx.$$

She has found that the Simpson's Rule approximation for the integral, using 10 subdivisions and carrying six significant digits, is -2.71538. But she doesn't know how many or those digits she can trust. However, she does know that the magnitude of the error in the Simpson's Rule approximation, with subdivisions of equal width h, to the integral $\int_a^b f(x)\, dx$ is at most

$$\frac{Mh^4}{180}(b - a),$$

where M is any number for which $|f^{(4)}(x)| \le M$ for every x in $[a, b]$. Make appropriate use of the figures below to estimate the potential error in Brünhilde's approximation. Explain your reasoning.

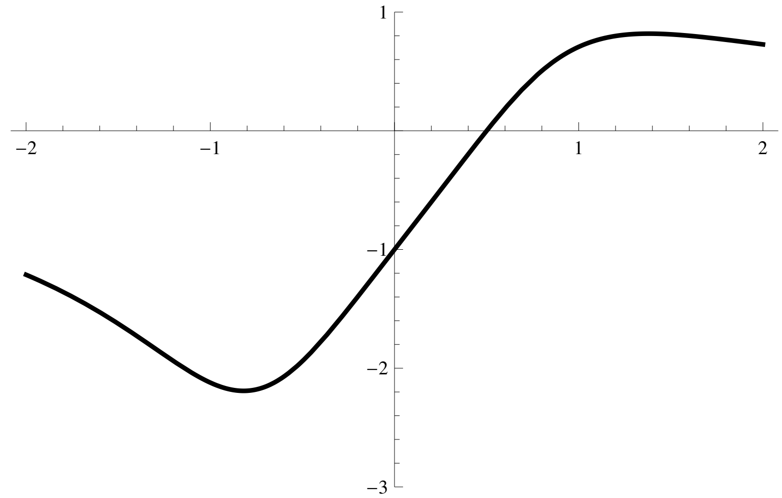

Figure 1: Graph of $f(x)$

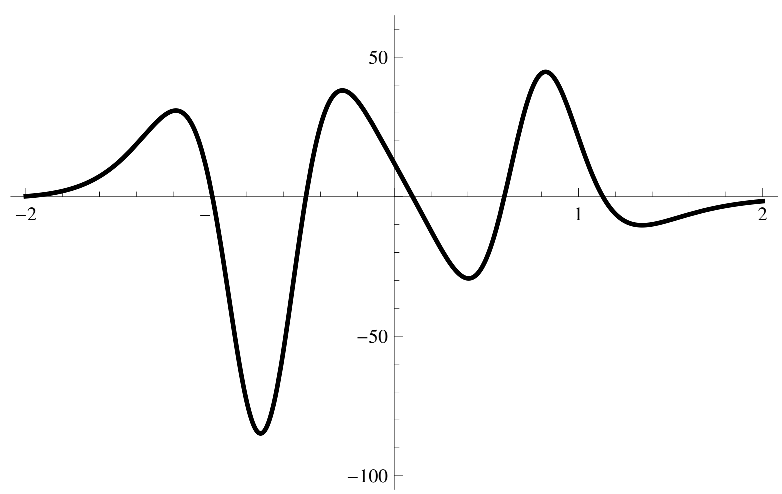

Figure 2: Graph of $f^{(4)}(x)$

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 2410, Calculus II—Exam III **Apr. 24, 2014**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 1:50 pm.

1. Let $y = f(x)$ be the solution to the initial value problem

$$\frac{dy}{dx} = 2x + y;$$
$$f(0) = 1.$$

Show how to use Euler's Method with step size $1/10$ to find an approximate value for $f(1/5)$. **Show the calculations that support your answer.**

Solution: The equations for the Euler's Method approximation, with step-size h, to the solution of $y' = F(x, y)$ with $y = y_0$ when $x = x_0$ are

$$x_k = x_0 + kh;$$
$$y_k = y_{k-1} + F(x_{k-1}, y_{k-1})h,$$

when $k = 1, 2, \ldots$. We begin with $x_0 = 0$, $y_0 = 1$, $h = 1/10$. Then we have

$$x_1 = x_0 + h = \frac{1}{10};$$
$$y_1 = y_0 + F(x_0, y_0)h = 1 + (2x_0 + y_0)\frac{1}{10} = 1 + 1 \cdot \frac{1}{10} = \frac{11}{10};$$
$$x_2 = x_0 + 2h = \frac{2}{10} = \frac{1}{5};$$
$$y_2 = y_1 + F(x_1, y_1)h = \frac{11}{10} + (2x_1 + y_1)h = \frac{11}{10} + \left(\frac{2}{10} + \frac{11}{10}\right) \cdot \frac{1}{10} = \frac{123}{100}.$$

The value of $f(1/5)$ is therefore approximately $123/100 = 1.23$.

Note: The actual solution of this initial value problem is $f(x) = 3e^x - 2 - 2x$, so that $f(0.2) \sim 1.2642$.

2. Show how to determine whether each of the following series is divergent or convergent. **Explain your reasoning fully.**

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n + 4^n}$

Solution:

(a) We apply the Ratio Test with $a_n = 3^n/n!$:

$$\begin{aligned} \lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{3^{n+1}/(n+1)!}{3^n/n!} \\ &= \lim_{n\to\infty} \left[\frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} \right] \\ &= \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1. \end{aligned}$$

By the Ratio Test, therefore, this series converges.

(b) We note that

$$\lim_{n\to\infty} \frac{5^n}{3^n + 4^n} = \lim_{n\to\infty} \frac{\left(\frac{5}{4}\right)^n}{\left(\frac{3}{4}\right)^n + 1}.$$

The numerator grows without bound when $n \to \infty$, while the denominator goes to one. Thus, the limit is infinite. We conclude that for this series, $\lim_{n\to\infty} a_n \neq 0$, so that, by the Divergence Test, this series diverges.

3. Show how to determine whether the following series is divergent or convergent. **Explain your reasoning fully.**

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Solution: We have

$$\begin{aligned} \lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \left(\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \right) \\ &= \lim_{n\to\infty} \left(\frac{\cancel{(n+1)!}}{(n+1)^n} \cdot \frac{n^n}{\cancel{(n+1)!}} \right) \\ &= \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n \\ &= \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^{-n} \\ &= \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^{-n} \end{aligned}$$

You should know that this limit is e^{-1}. But if you don't, you can calculate it as follows:

$$\begin{aligned} \lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^{-n} &= \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n \\ &= \lim_{n\to\infty} e^{n \ln\left(\frac{n}{n+1}\right)} \end{aligned}$$

But

$$\lim_{n\to\infty} n \ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} \frac{\ln\left(\dfrac{n}{n+1}\right)}{\left(\dfrac{1}{n}\right)},$$

to which we may apply l'Hôpital's Rule, the limits in numerator and denominator both being zero. Thus,

$$\begin{aligned}\lim_{n\to\infty} n \ln\left(\frac{n}{n+1}\right) &= \lim_{n\to\infty} \frac{\left[\dfrac{1}{(n+1)^2}\right]}{-\left[\dfrac{1}{n^2}\right]} \\ &= -\lim_{n\to\infty} \frac{n^2}{n^2+1} = -1\end{aligned}$$

Consequently, $\lim_{n\to\infty}(a_{n+1}/a_n) = e^{-1} < 1$. We conclude, by the Ratio Test, that this series converges.

4. The region bounded by the y-axis, the curve $y = x^2$, and the line $y = 4$ is revolved about the y-axis. Find the volume of the solid generated in this fashion.

Solution: If $y = x^2$, then $x = \sqrt{y}$. By the method of disks, the required volume is

$$\pi \int_0^4 (\sqrt{y})^2\, dy = \pi \int_0^4 y\, dy = \pi \frac{y^2}{2}\bigg|_0^4 = 8\pi.$$

Alternately, the method of cylindrical shells gives the volume as

$$2\pi \int_0^2 x(4-x^2)\, dx = 2\pi \left(2x^2 - \frac{x^4}{4}\right)\bigg|_0^2 = 2\pi\left(8 - \frac{16}{4}\right) = 8\pi.$$

5. Find the solution, $y = f(x)$, to the differential equation $y' = y(1-x)^{-1}$, given that $y = 1$ when $x = 2$. Show your reasoning.

Solution: We have

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{1-x}, \text{ or} \\ \frac{dy}{y} &= \frac{dx}{1-x}.\end{aligned}$$

Hence,

$$\begin{aligned}\int_1^y \frac{dv}{v} &= \int_1^x \frac{du}{1-u}, \text{ or} \\ \ln|v|\bigg|_1^y &= -\ln|1-u|\bigg|_2^x, \\ \ln|y| - \ln 1 &= -\ln|1-x| + \ln 1, \\ \ln|y| &= -\ln|1-x|.\end{aligned}$$

But $y = 1 > 0$ when $1 - x = -1 < 0$, so $|1 - x| = x - 1$ and $|y| = y$ for our solution. The solution we seek is therefore given by

$$\ln y = -\ln(x-1), \text{ or}$$
$$y = \frac{1}{x-1}.$$

6. Brünhilde is working a scientific problem, and she needs a decent approximate value for the integral

$$\int_{-2}^{2} \frac{2x-1}{\sqrt{1+x^4}}\, dx.$$

She has found that the Simpson's Rule approximation for the integral, using 10 subdivisions and carrying six significant digits, is -2.71538. But she doesn't know how many or those digits she can trust. However, she does know that the magnitude of the error in the Simpson's Rule approximation, with subdivisions of equal width h, to the integral $\int_a^b f(x)\, dx$ is at most

$$\frac{Mh^4}{180}(b-a),$$

where M is any number for which $|f^{(4)}(x)| \le M$ for every x in $[a, b]$. Make appropriate use of the figures below to estimate the potential error in Brünhilde's approximation. Explain your reasoning.

Solution: From Figure 2, we see that $-90 \le f^{(4)}(x) \le 90$, or $|f^{(4)}(x)| \le 90$, on the interval $[-2, 2]$, so we may take $M = 90$. Consequently, the magnitude of the error in Brünhilde's approximation to her integral is at most

$$\frac{Mh^4}{180}(b-a) = \frac{90(4/10)^4[2-(-2)]}{180} = \frac{32}{625} < 0.052$$

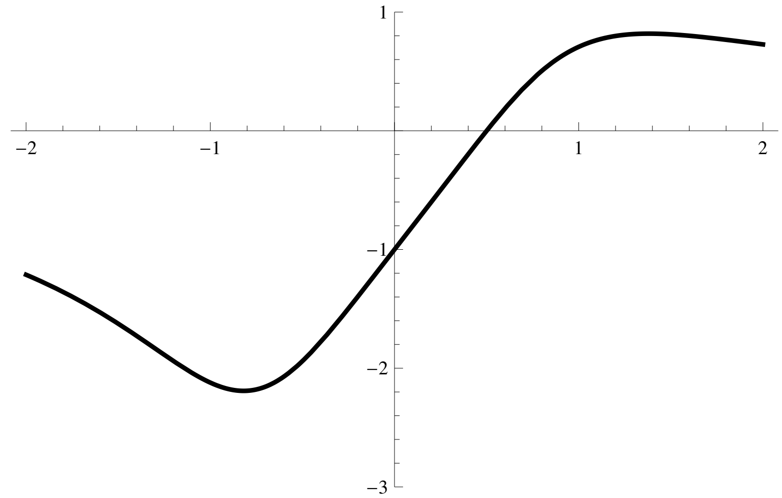

Figure 1: Graph of $f(x)$

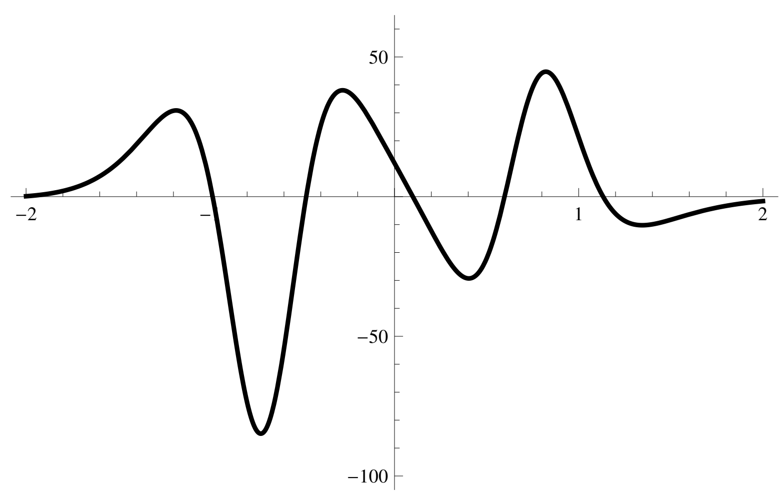

Figure 2: Graph of $f^{(4)}(x)$

MTH 2410, Calculus II—Final Exam **May 13, 2014**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your work is due at 1:00 pm.

1. A particle traveling along a straight line at 576 ft/sec experiences a deceleration, beginning at time $t = 0$, of $8t^2$ ft/sec^2. Show how to determine how many feet it travels before coming to a complete stop.

2. Consider the region bounded by the curve $y = 1 + 2\sqrt{x}$, the line $y = 1$, and the lines $x = 4$ and $x = 9$. Show how to find the volume generated when this region is rotated about x-axis.

3. Show how to find the arc-length of the curve $y = x^2 - \frac{1}{8}\ln x$ over the interval $[1, e]$.

4. Show how to find the solution of the differential equation:

$$\frac{dy}{dx} + 2y = 8,$$

   that satisfies the initial condition $y(0) = 0$.

5. Show how to find the Maclaurin series (equivalently, the Taylor series about $x = 0$) for $\tanh^{-1} x$, the inverse hyperbolic tangent function. Give the first three non-zero terms, the general term, and the radius of convergence. [Hint: It may be helpful to recall that $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1 - x^2}$.]

6. (a) Give the trapezoidal approximation, with 3 subdivisions, for the definite integral $\int_{-14}^{-8} \frac{dx}{x}$. Either give your answer as a fraction of integers, or give it accurate to at least 6 decimals.

   (b) If M is a number for which $|f''(x)| \le M$ for all x in the interval $[a, b]$, the error in the trapezoidal approximation with n subdivisions for the integral $\int_a^b f(x)\,dx$ does not exceed the quantity $\frac{M(b-a)^3}{12n^2}$.

      i. Explain how to choose a suitable M for this approximation.

      ii. According to this statement, how many subdivisions would we need in the trapezoidal approximation of this integral to guarantee that the resulting approximation is within 0.002 of the integral's true value?

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 2410, Calculus II—Final Exam **May 13, 2014**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your work is due at 1:00 pm.

1. A particle traveling along a straight line at 576 ft/sec experiences a deceleration, beginning at time $t = 0$, of $8t^2$ ft/sec^2. Show how to determine how many feet it travels before coming to a complete stop.

   Solution: Acceleration, $a(t)$ is given by $a(t) = -8t^2$. Thus,

$$\begin{aligned} v(t) &= v(0) + \int_0^t a(\tau)\, d\tau = v(0) - \int_0^t 8\tau^2\, d\tau \\ &= 576 - \frac{8}{3}t^3. \end{aligned}$$

   The particle comes to a complete stop when $v(t) = 0$, or when $t^3 = \dfrac{1728}{8} = 216$. This gives $t = 6$ seconds

   For position $s(t)$, we have

$$\begin{aligned} s(t) &= s(0) + \int_0^t v(\tau)\, d\tau = s(0) + \int_0^t \left[576 - \frac{8}{3}\tau^3\right] d\tau \\ &= s(0) + 576t - \frac{2}{3}t^4. \end{aligned}$$

   We find that the distance traveled before stopping is

$$s(6) = 576 \cdot 6 - \frac{2}{3} \cdot 6^4 = 2592 \text{ ft}$$

2. Consider the region bounded by the curve $y = 1 + 2\sqrt{x}$, the line $y = 1$, and the lines $x = 4$ and $x = 9$. Show how to find the volume generated when this region is rotated about x-axis.

   Solution: If τ is a number between 4 and 9, the plane $x = \tau$ intersects this volume in a washer whose center is at the point $x = \tau$ on the x-axis, whose inner radius is 1 and whose outer radius is $1 + 2\sqrt{\tau}$. The area of such a washer is $\pi\left[(1 + 2\sqrt{\tau})^2 - 1^2\right] = 4\pi\left[\sqrt{\tau} + \tau\right]$, so the volume in question is

$$4\pi \int_4^9 \left[\tau^{1/2} + \tau\right] d\tau = 4\pi \left(\frac{2}{3}\tau^{3/2} + \frac{1}{2}\tau^2\right)\Bigg|_4^9 = 4\pi\left(\frac{117}{2} - \frac{40}{3}\right) = \frac{542}{3}\pi.$$

3. Show how to find the arc-length of the curve $y = x^2 - \dfrac{1}{8}\ln x$ over the interval $[1, e]$.

Solution: If $y = x^2 - \frac{1}{8}\ln x$, then $y' = 2x - \frac{1}{8x}$. Thus

$$\int_1^{2e} \sqrt{1+(y')^2}\,dx = \int_1^e \sqrt{1+\left(4x^2 - \frac{1}{2} + \frac{1}{64x^2}\right)}\,dx = \int_1^e \sqrt{\left(4x^2 + \frac{1}{2} + \frac{1}{64x^2}\right)}\,dx$$
$$= \int_1^e \left(2x + \frac{1}{8x}\right)\,dx = \left(x^2 + \frac{1}{8}\ln x\right)\Big|_1^e = e^2 - \frac{7}{8}.$$

4. Show how to find the solution of the differential equation:

$$\frac{dy}{dx} + 2y = 8,$$

that satisfies the initial condition $y(0) = 0$.

Solution: We have

$$\begin{aligned}
\frac{dy}{dx} + 2y &= 8;\\
\frac{dy}{dx} &= 8 - 2y;\\
\frac{dy}{8-2y} &= dx;\\
\int \frac{(-2)dy}{8-2y} &= -2\int dx;\\
\ln|8-2y| &= -2x + c;\\
8 - 2y &= Ce^{-2x};\\
y &= 4 - Ce^{-2x}.
\end{aligned}$$

But it is given that $y(0) = 0$, so $4 - Ce^0 = 0$, whence $C = 4$. The solution we seek is therefore $y = 4 - 4e^{-2x}$.

5. Show how to find the Maclaurin series (equivalently, the Taylor series about $x = 0$) for $\tanh^{-1} x$, the inverse hyperbolic tangent function. Give the first three non-zero terms, the general term, and the interval of convergence. [Hint: It may be helpful to recall that $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$.]

Solution: Let $f(x) = \tanh^{-1} x$. Then, using what we know about the geometric series, we find that

$$f'(x) = \frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + x^8 + \cdots,$$

and the radius of convergence for this series is one. Integrating, we obtain

$$f(x) = c + x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \cdots.$$

But $f(0) = \tanh^{-1} 0 = 0$, which means that $c = 0$, and

$$\begin{aligned} \tanh^{-1} x &= x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2k+1}}{2k+1} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}. \end{aligned}$$

The radius of convergence for the integrated series is the same as that for the series we integrated. Thus, the radius of convergence for the series immediately above is one.

6. (a) Give the trapezoidal approximation, with 3 subdivisions, for the definite integral $\int_{-14}^{-8} \frac{dx}{x}$. Either give your answer as a fraction of integers, or give it accurate to at least 6 decimals.

   (b) If M is a number for which $|f''(x)| \le M$ for all x in the interval $[a, b]$, the error in the trapezoidal approximation with n subdivisions for the integral $\int_a^b f(x)\,dx$ does not exceed the quantity $\frac{M(b-a)^3}{12n^2}$.

   i. Explain how to choose a suitable M for this approximation.

   ii. According to this statement, how many subdivisions would we need in the trapezoidal approximation of this integral to guarantee that the resulting approximation is within 0.002 of the integral's true value?

Solution:

(a) The required approximation is

$$\begin{aligned} \int_{-14}^{-8} \frac{dx}{x} &\sim -\frac{1}{2}\left(\frac{1}{14} + 2 \cdot \frac{1}{12} + 2 \cdot \frac{1}{10} + \frac{1}{8}\right) \cdot \frac{(-8) - (-14)}{3} \\ &\sim -\frac{473}{840} \end{aligned}$$

(b) i. The second derivative of the function $f(x) = 1/x$ is $f''(x) = 2/x^3$. The third derivative is $f'''(x) = -6/x^4$, and this is negative throughout the interval $[-14, -8]$, so f'' is a decreasing function whose values are negative on that interval. Consequently, the maximum value of $|f''(x)|$ on the interval $[-14, -8]$ is to be found at the right end-point of the interval. We may therefore take $M = 1/256$.

ii. If error is not to exceed 0.002, then we must have

$$\begin{aligned} \frac{M(b-a)^3}{12n^2} &\le \frac{1}{500}; \\ \frac{1}{256} \frac{6^3}{12n^2} &\le \frac{1}{500}; \\ n^2 &\ge \frac{2^3 \cdot 3^3 \cdot 2^2 \cdot 5^3}{2^8 \cdot 2^2 \cdot 3} = \frac{3^2 \cdot 5^3}{2^5}, \end{aligned}$$

whence

$$n \geq \frac{3 \cdot 5}{4} \cdot \sqrt{\frac{5}{2}} \sim 5.93.$$

Being sure of accuracy to within 0.002 thus requires a minimum of 6 subdivisions.

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 2140, Calculus II, Exam 1 **February 18, 2015**

Instructions: Write out, *on your own paper*, complete solutions of the following problems. You must show enough of your calculations to support your answers. Do not give decimal approximations unless the nature of a problem requires them. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your paper is due at 5:50 pm.

1. Show how to evaluate $\displaystyle\int_0^{\pi/2} \sin^2\theta\cos\theta\,d\theta$; find the value of the integral.

2. Show how to evaluate $\displaystyle\int_1^2 \frac{(\ln x)^2}{x}\,dx$ and find the value of the integral.

3. Show how to use a trigonometric substitution to transform the definite integral
$$\int_0^a \frac{dx}{(a^2+x^2)^{3/2}}$$
into another definite integral. Then give the value of the definite integral.

4. Show how to evaluate $\displaystyle\int x^2\,e^{2x}\,dx$ and give the value.

5. Show how the substitution $3x-4=u^3$ reduces the integral
$$\int \frac{x\,dx}{\sqrt[3]{3x-4}}$$
to an integral of a polynomial. Then use what you have shown to evaluate the original integral.

6. The region below the curve $y=-x^2+\dfrac{17}{2}x-11$ and above the curve $x-2y+2=0$ is revolved about the vertical line $x=-1$. Show how to find the volume of the solid thus generated. What is that volume?

MTH 2140, Calculus II, Exam 1 Solutions **February 18, 2015**

Instructions: Write out, *on your own paper*, complete solutions of the following problems. You must show enough of your calculations to support your answers. Do not give decimal approximations unless the nature of a problem requires them. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your paper is due at 5:50 pm.

1. Show how to evaluate $\int_0^{\pi/2} \sin^2\theta\cos\theta\,d\theta$ and give the value of the integral.

   Solution: Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$. Also, $u = 0$ when $\theta = 0$ and $u = 1$ when $\theta = \pi/2$. Hence,

$$\int_0^{\pi/2} \sin^2\theta\cos\theta\,d\theta = \int_0^1 u^2\,du = \left.\frac{u^3}{3}\right|_0^1 = \frac{1}{3}.$$

2. Show how to evaluate $\int_1^2 \frac{(\ln x)^2}{x}\,dx$ and find the value of the integral.

   Solution: Let $u = \ln x$. Then $du = \frac{dx}{x}$. Also, $u = \ln 1 = 0$ when $x = 1$, and $u = \ln 2$ when $x = 2$. Hence

$$\int_1^2 \frac{(\ln x)^2}{x}\,dx = \int_1^2 (\ln x)^2\frac{dx}{x} = \int_0^{\ln 2} u^2\,du = \left.\frac{u^3}{3}\right|_0^{\ln 2} = \frac{(\ln 2)^3}{3}.$$

3. Show how to use a trigonometric substitution to transform the definite integral

$$\int_0^a \frac{dx}{(a^2+x^2)^{3/2}}$$

   into another definite integral. Then give the value of the definite integral.

   Solution: Let $x = a\tan\theta$. Then $dx = a\sec^2\theta\,d\theta$. If $x = 0$, then $\theta = 0$, while if $x = a$, then $\theta = \pi/4$. Hence,

$$\begin{aligned}\int_0^a \frac{dx}{(a^2+x^2)^{3/2}} &= \int_0^{\pi/4} \frac{a\sec^2\theta\,d\theta}{(a^2+a^2\tan^2\theta)^{3/2}} = \int_0^{\pi/4} \frac{a\sec^2\theta\,d\theta}{a^3(1+\tan^2\theta)^{3/2}}\\ &= \frac{1}{a^2}\int_0^{\pi/4} \frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^{3/2}} = \frac{1}{a^2}\int_0^{\pi/4} \frac{\sec^2\theta\,d\theta}{\sec^3\theta}\\ &= \frac{1}{a^2}\int_0^{\pi/4} \cos\theta\,d\theta = \frac{1}{a^2}\sin\theta\Big|_0^{\pi/4} = \frac{\sqrt{2}}{2a^2}.\end{aligned}$$

4. Show how to evaluate $\int x^2 e^{2x}\,dx$ and give the value.

Solution: We begin by integrating by parts, taking $u = x^2$ and $dv = e^{2x}\,dx$. Then $du = 2x\,dx$ and $v = e^{2x}/2$, so the parts formula $\int u\,dv = uv - \int v\,du$ gives

$$\begin{aligned}\int x^2 e^{2x}\,dx &= x^2 \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} \cdot 2x\,dx \\ &= \frac{1}{2}x^2 e^{2x} - \int x e^{2x}\,dx.\end{aligned}$$

In order to carry out the latter integration, we again integrate by parts, this time taking $U = x$ and $dV = e^{2x}\,dx$. Then $dU = dx$ and $V = e^{2x}/2$, so

$$\int x e^{2x}\,dx = x \cdot \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} \cdot dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4}.$$

Substituting into our earlier expression, we find that

$$\int x^2 e^{2x}\,dx = \frac{1}{2}x^2 e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} = \frac{1}{4}\left(2x^2 - 2x + 1\right)e^{2x} + c.$$

5. Show how the substitution $3x - 4 = u^3$ reduces the integral

$$\int \frac{x\,dx}{\sqrt[3]{3x-4}}$$

to an integral of a polynomial. Then use what you have shown to evaluate the original integral.

Solution: If $3x - 4 = u^3$, then $\sqrt[3]{3x-4} = u$. Moreover, $x = \dfrac{u^3+4}{3}$ and $dx = u^2\,du$. Thus,

$$\begin{aligned}\int \frac{x\,dx}{\sqrt[3]{3x-4}} &= \frac{1}{3}\int \frac{(u^3+4)u^2\,du}{u} = \frac{1}{3}\int (u^4 + 4u)\,du \\ &= \frac{1}{3}\left(\frac{1}{5}u^5 + 2u^2\right) + c \\ &= \frac{1}{15}(3x-4)^{5/3} + \frac{2}{3}(3x-4)^{2/3} + c.\end{aligned}$$

6. The region below the curve $y = -x^2 + \dfrac{17}{2}x - 11$ and above the curve $x - 2y + 2 = 0$ is revolved about the vertical line $x = -1$. Show how to find the volume of the solid thus generated. What is that volume?

Solution: See Figure 1 for the picture. The intersection points of the curves are the simultaneous solutions of the equations $y = -x^2 + \frac{17}{2}x - 11$ and $x - 2y + 2 = 0$, which we find by solving the quadratic equation $-x^2 + \frac{17}{2}x - 11 = \frac{x+2}{2}$. This leads to intersection points at $(2, 2)$ and at $(6, 4)$. A vertical line segment x units to the right of the y-axis

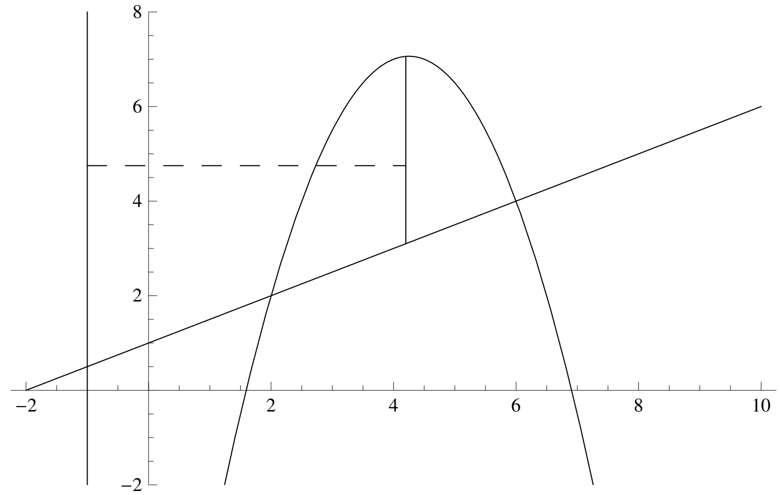

Figure 1: The curves of Problem 6

connecting the two curves as shown has length $\left(-x^2 + \frac{17}{2}x - 11\right) - \left(\frac{x+2}{2}\right)$, and is $x + 1$ units to the right of the line $x = -1$. When that line segment is revolved about the line $x = -1$, it therefore generates a cylinder whose area is

$$2\pi(x+1)\left[\left(-x^2 + \frac{17}{2}x - 11\right) - \left(\frac{x+2}{2}\right)\right] = 2\pi(x+1)(-x^2 + 8x - 12)$$
$$= 2\pi(-x^3 + 7x^2 - 4x - 12).$$

The desired volume is therefore

$$\begin{aligned} 2\pi \int_2^6 \left(-x^3 + 7x^2 - 4x - 12\right) dx &= 2\pi \left(-\frac{x^4}{4} + \frac{7}{3}x^3 - 2x^2 - 12x\right)\Big|_2^6 \\ &= 2\pi \left[-\frac{6^4}{4} + \frac{7 \cdot 6^3}{3} - 2 \cdot 6^2 - 12 \cdot 6\right] \\ &\quad - 2\pi \left[-\frac{2^4}{4} + \frac{7 \cdot 2^3}{3} - 2 \cdot 2^2 - 12 \cdot 2\right] \\ &= \frac{320\pi}{3}. \end{aligned}$$

MTH 2140, Calculus II, Exam 2 **March 18, 2015**

Instructions: Write out, *on your own paper*, complete solutions of the following problems. You must show enough of your calculations to support your answers. Do not give decimal approximations unless the nature of a problem requires them. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your paper is due at 5:50 pm.

1. (a) Show how to evaluate $\int \sin^3\theta \cos^{10}\theta \, d\theta$. What is the antiderivative?

   (b) Show how to evaluate $\int_0^1 \frac{dx}{(1+x^2)^{3/2}}$ and find the value of the integral.

2. Show how to find the indefinite integral: $\int \frac{x\,dx}{(x+1)(x+2)}$.

3. Find the length of the curve $y = \frac{x^2}{4} - \frac{\ln x}{2}$ over the interval $[1, 2e]$. Show your calculations.

4. Does the integral $\int_0^\infty xe^{-(x^2)}\,dx$ converge or diverge? Give all of your reasoning, including the value of the integral if it should happen to converge.

5. Show how to derive an algebraic expression for $\tanh(\sinh^{-1} x)$.

6. (a) Consider the integral $\int_4^7 \frac{dx}{x}$. Give the trapezoidal approximation with 3 subdivisions for this integral. Give your answer as a fraction of integers, or give it accurate to at least 6 decimals.

   (b) The error in the trapezoidal approximation with n subdivisions for the integral $\int_a^b f(x)\,dx$ is at most $\frac{M(b-a)^3}{12n^2}$, where M is any number for which $|f''(x)| \le M$ for all x in $[a,b]$. What, according to this statement, is the maximum possible error in the approximation you computed in the first part of this problem? In particular, explain the reasoning that underlies your choice of M.

MTH 2140, Calculus II, Exam 2 **March 18, 2015**

Instructions: Write out, *on your own paper*, complete solutions of the following problems. You must show enough of your calculations to support your answers. Do not give decimal approximations unless the nature of a problem requires them. Your grade will depend not only on the correctness of your conclusions but also on your presentation of correct reasoning that supports those conclusions. Your paper is due at 5:50 pm.

1. (a) Show how to evaluate $\int \sin^3\theta\cos^{10}\theta\,d\theta$. What is the antiderivative?

   (b) Show how to evaluate $\int_0^1 \frac{dx}{(1+x^2)^{3/2}}$ and find the value of the integral.

   Solution:

   (a)

$$\begin{aligned}\int \sin^3\theta\cos^{10}\theta\,d\theta &= \int \left(1-\cos^2\theta\right)\cos^{10}\theta\sin\theta\,d\theta\\ &= \int \cos^{10}\theta\sin\theta\,d\theta - \int \cos^{12}\theta\sin\theta\,d\theta\\ &= \frac{1}{13}\cos^{13}\theta - \frac{1}{11}\cos^{11}\theta + C.\end{aligned}$$

   (b) We let $x = \tan\theta$, so that $dx = \sec^2\theta\,d\theta$. Then $\theta = 0$ when $x = 0$, and $\theta = \pi/4$ when $x = 1$. Thus

$$\begin{aligned}\int_0^1 \frac{dx}{(1+x^2)^{3/2}} &= \int_0^{\pi/4}\frac{\sec^2\theta\,d\theta}{(1+\tan^2\theta)^{3/2}} = \int_0^{\pi/4}\frac{\cancel{\sec^2\theta}\,d\theta}{\sec^{\cancel{3}}\theta}\\ &= \int_0^{\pi/4}\cos\theta\,d\theta = \sin\theta\Big|_0^{\pi/4} = \frac{1}{\sqrt{2}}.\end{aligned}$$

2. Show how to find the indefinite integral: $\int \frac{x\,dx}{(x+1)(x+2)}$.

   Solution: We note first that if

$$\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2},$$

   then

$$(A+B)x + (2A+B) = x$$

so that

$$\begin{aligned} A + B &= 1; \\ 2A + B &= 0. \end{aligned}$$

Hence,

$$\begin{aligned} \int \frac{x\,dx}{(x+1)(x+2)} &= -\int \frac{dx}{x+1} + 2\int \frac{dx}{x+2} \\ &= 2\ln|x+2| - \ln|x+1| + C. \end{aligned}$$

3. Find the length of the curve $y = \dfrac{x^2}{4} - \dfrac{\ln x}{2}$ over the interval $[1, 2e]$. Show your calculations.

Solution: If $y = \dfrac{x^2}{4} - \dfrac{\ln x}{2}$, then $y' = \dfrac{x}{2} - \dfrac{1}{2x}$. Consequently, the arc-length over the interval $[1, 2e]$ is

$$\begin{aligned} \int_1^{2e} \sqrt{1 + (y')^2}\,dx &= \int_1^{2e} \sqrt{1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2}\,dx = \int_1^{2e} \sqrt{1 + \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}}\,dx \\ &= \int_1^{2e} \sqrt{\frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}}\,dx = \int_1^{2e} \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2}\,dx \\ &= \int_1^{2e} \left(\frac{x}{2} + \frac{1}{2x}\right)\,dx = \left[\frac{x^2}{4} + \frac{1}{2}\ln x\right]\Bigg|_1^{2e} = e^2 + \frac{1}{4} + \frac{1}{2}\ln 2. \end{aligned}$$

4. Does the integral $\displaystyle\int_0^\infty xe^{-(x^2)}\,dx$ converge or diverge? Give all of your reasoning, including the value of the integral if it should happen to converge.

Solution:

$$\begin{aligned} \int_0^\infty xe^{-(x^2)}\,dx &= \lim_{T\to\infty} \int_0^T xe^{-(x^2)}\,dx = -\lim_{T\to\infty} \frac{e^{(-x^2)}}{2}\Bigg|_0^T \\ &= -\frac{1}{2}\lim_{t\to\infty}\left(e^{-(T^2)} - 1\right) = \frac{1}{2}. \end{aligned}$$

The improper integral converges to the value $\dfrac{1}{2}$.

5. Show how to derive an algebraic expression for $\tanh(\sinh^{-1} x)$.

Solution: We have

$$\tanh(\sinh^{-1} x) = \frac{\sinh(\sinh^{-1} x)}{\cosh(\sinh^{-1} x)} = \frac{x}{\cosh(\sinh^{-1} x)},$$

because $\sinh(\sinh^{-1} x) = x$. But $\cosh^2 u - \sinh^2 u = 1$ is an identity, so

$$\cosh^2(\sinh^{-1} x) = 1 + \sinh^2(\sinh^{-1} x) = 1 + x^2$$

so that, $\cosh u$ always being positive (Why is this important?),

$$\cosh(\sinh^{-1} x) = \sqrt{1 + x^2}.$$

Thus,

$$\tanh(\sinh^{-1} x) = \frac{x}{\sqrt{1 + x^2}}.$$

6. (a) Consider the integral $\int_4^7 \frac{dx}{x}$. Give the trapezoidal approximation with 3 subdivisions for this integral. Give your answer as a fraction of integers, or give it accurate to at least 6 decimals.

   (b) The error in the trapezoidal approximation with n subdivisions for the integral $\int_a^b f(x)\,dx$ is at most $\frac{M(b-a)^3}{12n^2}$, where M is any number for which $|f''(x)| \leq M$ for all x in $[a, b]$. What, according to this statement, is the maximum possible error in the approximation you computed in the first part of this problem? In particular, explain the reasoning that underlies your choice of M.

Solution:

(a) Taking $f(x) = 1/x$, $a = 4$, $b = 7$, $n = 3$, we have $h = 1$, $x_0 = 4$, $x_1 = 5$, $x_2 = 6$, $x_3 = 7$. Thus, the required trapezoidal approximation is

$$\begin{aligned} T_3 &= \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right] \\ &= \frac{1}{2}\left[\frac{1}{4} + \frac{2}{5} + \frac{1}{3} + \frac{1}{7}\right] = \frac{473}{840} \sim 0.5630952381 \end{aligned}$$

(b) If $f(x) = \frac{1}{x}$, then $f'(x) = -\frac{1}{x^2}$, and $f''(x) = \frac{2}{x^3}$. On the interval $[4, 7]$, the function $|f''(x)| = \frac{2}{x^3}$ is a positive-valued decreasing function (because the function $x \mapsto x^3$ is increasing there), so when $4 \leq x \leq 7$, we know that

$$|f''(x)| = f(x) \leq f''(4) = \frac{2}{4^3} = \frac{1}{32}.$$

Consequently, the error E satisfies

$$|E| \leq \frac{M(b-a)^3}{12n^2} = \frac{1}{32} \cdot \frac{(7-4)^3}{12 \cdot 3^2} = \frac{1}{128} = 0.0078125.$$

MTH 2410, Calculus II—Exam III **Apr. 22, 2015**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 5:50 pm.

1. Let $y = f(x)$ be the solution to the initial value problem

$$\frac{dy}{dx} = 2x + y;$$
$$f(0) = 1.$$

Demonstrate the use of Euler's Method with step size 1/10 to find an approximate value for $f(1/5)$. **Show the calculations that support your answer.**

2. Demonstrate how to determine whether each of the following series is divergent or convergent. **Explain your reasoning fully.**

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n + 4^n}$

3. Demonstrate finding the arc length of the part of the curve

$$y = \frac{x^2}{4} - \frac{\ln x}{2}$$

that corresponds to $1 \le x \le 10e$.

4. The region bounded by the y-axis, the curve $y = x^2$, and the line $y = 4$ is revolved about the y-axis. Find the volume of the solid generated in this fashion.

5. Find the solution, $y = f(x)$, to the differential equation $y' = y(1-x)^{-1}$, given that $y = 1$ when $x = 2$. Show your reasoning.

6. Brünhilde is working a scientific problem, and she needs a decent approximate value for the integral

$$\int_{-2}^{2} \frac{2x-1}{\sqrt{1+x^4}}\,dx.$$

She has found that the Simpson's Rule approximation for the integral, using 10 subdivisions and carrying six significant digits, is -2.71538. But she doesn't know how many or those digits she can trust. However, she does know that the magnitude of the error in the Simpson's Rule approximation, with subdivisions of equal width h, to the integral $\int_a^b f(x)\,dx$ is at most

$$\frac{Mh^4}{180}(b-a),$$

where M is any number for which $|f^{(4)}(x)| \le M$ for every x in $[a, b]$. Make appropriate use of the figures below to estimate the potential error in Brünhilde's approximation. Explain your reasoning.

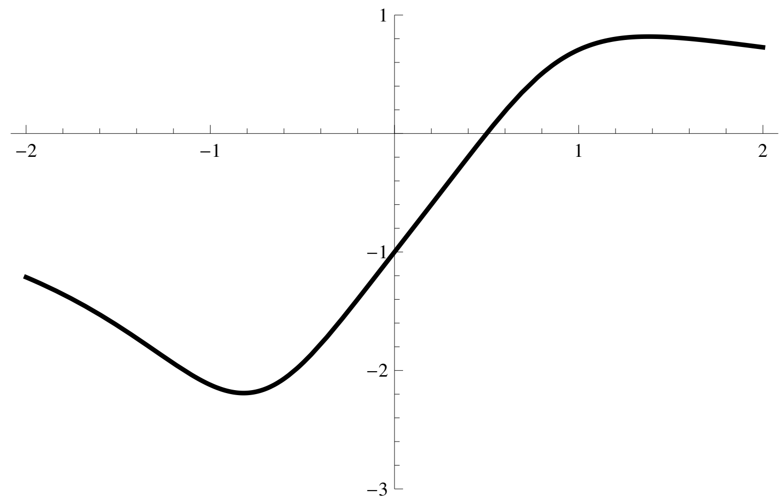

Figure 1: Graph of $f(x)$

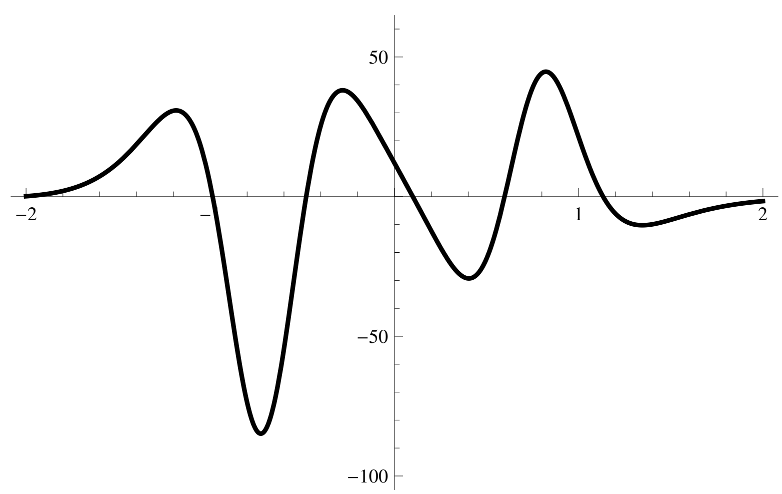

Figure 2: Graph of $f^{(4)}(x)$

MTH 2410, Calculus II—Exam III **Apr. 22, 2015**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your paper is due at 5:50 pm.

1. Let $y = f(x)$ be the solution to the initial value problem

$$\frac{dy}{dx} = 2x + y;$$
$$f(0) = 1.$$

Show how to use Euler's Method with step size $1/10$ to find an approximate value for $f(1/5)$. **Show the calculations that support your answer.**

Solution: The equations for the Euler's Method approximation, with step-size h, to the solution of $y' = F(x, y)$ with $y = y_0$ when $x = x_0$ are

$$x_k = x_0 + kh;$$
$$y_k = y_{k-1} + F(x_{k-1}, y_{k-1})h,$$

when $k = 1, 2, \ldots$. We begin with $x_0 = 0$, $y_0 = 1$, $h = 1/10$. Then we have

$$x_1 = x_0 + h = \frac{1}{10};$$
$$y_1 = y_0 + F(x_0, y_0)h = 1 + (2x_0 + y_0)\frac{1}{10} = 1 + 1 \cdot \frac{1}{10} = \frac{11}{10};$$
$$x_2 = x_0 + 2h = \frac{2}{10} = \frac{1}{5};$$
$$y_2 = y_1 + F(x_1, y_1)h = \frac{11}{10} + (2x_1 + y_1)h = \frac{11}{10} + \left(\frac{2}{10} + \frac{11}{10}\right) \cdot \frac{1}{10} = \frac{123}{100}.$$

The value of $f(1/5)$ is therefore approximately $123/100 = 1.23$.

Note: The actual solution of this initial value problem is $f(x) = 3e^x - 2 - 2x$, so that $f(0.2) \sim 1.2642$.

2. Show how to determine whether each of the following series is divergent or convergent. **Explain your reasoning fully.**

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n + 4^n}$

Solution:

(a) We apply the Ratio Test with $a_n = 3^n/n!$:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}/(n+1)!}{3^n/n!}$$
$$= \lim_{n\to\infty} \left[\frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n}\right]$$
$$= \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1.$$

By the Ratio Test, therefore, this series converges.

(b) We note that

$$\lim_{n\to\infty} \frac{5^n}{3^n + 4^n} = \lim_{n\to\infty} \frac{\left(\frac{5}{4}\right)^n}{\left(\frac{3}{4}\right)^n + 1}.$$

The denominator of this last fraction approaches 1 as $n \to \infty$, while the numerator increases without bound. We conclude that for this series, $\lim_{n\to\infty} a_n \neq 0$, so that this series diverges.

3. Demonstrate finding the arc length of the part of the curve

$$y = \frac{x^2}{4} - \frac{\ln x}{2}$$

that corresponds to $1 \leq x \leq 10e$.

Solution: Taking y as given, we obtain

$$y' = \frac{x}{2} - \frac{1}{2x},$$

so that

$$1 + (y')^2 = 1 + \left(\frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}\right)$$
$$= \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}$$
$$= \left(\frac{x}{2} + \frac{1}{2x}\right)^2.$$

Thus, the required arc length is

$$\int_1^{10e} \sqrt{1 + (y')^2}\, dx = \int_1^{10e} \left(\frac{x}{2} + \frac{1}{2x}\right) dx = \left(\frac{x^2}{4} + \frac{1}{2}\ln x\right)\Bigg|_1^{10e}$$
$$= \frac{100e^2 + 1 + \ln 100}{4}.$$

4. The region bounded by the y-axis, the curve $y = x^2$, and the line $y = 4$ is revolved about the y-axis. Find the volume of the solid generated in this fashion.

Solution: If $y = x^2$, then $x = \sqrt{y}$. By the method of disks, the required volume is

$$\pi \int_0^4 (\sqrt{y})^2 \, dy = \pi \int_0^4 y \, dy = \pi \frac{y^2}{2}\bigg|_0^4 = 8\pi.$$

Alternately, the method of cylindrical shells gives the volume as

$$2\pi \int_0^2 x(4 - x^2) \, dx = 2\pi \left(2x^2 - \frac{x^4}{4}\right)\bigg|_0^2 = 2\pi \left(8 - \frac{16}{4}\right) = 8\pi.$$

5. Find the solution, $y = f(x)$, to the differential equation $y' = y(1 - x)^{-1}$, given that $y = 1$ when $x = 2$. Show your reasoning.

Solution: We have

$$\begin{aligned} \frac{dy}{dx} &= \frac{y}{1-x}, \text{ or} \\ \frac{dy}{y} &= \frac{dx}{1-x}. \end{aligned}$$

Hence,

$$\begin{aligned} \int_1^y \frac{dv}{v} &= \int_1^x \frac{du}{1-u}, \text{ or} \\ \ln|v|\bigg|_1^y &= -\ln|1-u|\bigg|_2^x, \\ \ln|y| - \ln 1 &= -\ln|1-x| + \ln 1, \\ \ln|y| &= -\ln|1-x|. \end{aligned}$$

But $y = 1 > 0$ when $1 - x = -1 < 0$, so $|1 - x| = x - 1$ and $|y| = y$ for our solution. The solution we seek is therefore given by

$$\begin{aligned} \ln y &= -\ln(x-1), \text{ or} \\ y &= \frac{1}{x-1}. \end{aligned}$$

6. Brünhilde is working a scientific problem, and she needs a decent approximate value for the integral

$$\int_{-2}^{2} \frac{2x - 1}{\sqrt{1 + x^4}} \, dx.$$

She has found that the Simpson's Rule approximation for the integral, using 10 subdivisions and carrying six significant digits, is -2.71538. But she doesn't know how many or those digits she can trust. However, she does know that the magnitude of the error in the Simpson's Rule approximation, with subdivisions of equal width h, to the integral $\int_a^b f(x)\,dx$ is at most

$$\frac{Mh^4}{180}(b-a),$$

where M is any number for which $|f^{(4)}(x)| \le M$ for every x in $[a,b]$. Make appropriate use of the figures below to estimate the potential error in Brünhilde's approximation. Explain your reasoning.

Solution: From Figure 2, we see that $|f^{(4)}(x)| \le 90$, or

$$-90 \le f^{(4)}(x) \le 90$$

on the interval $[-2,2]$, so we may take $M = 90$. Consequently, the magnitude of the error in Brünhilde's approximation to her integral is at most

$$\frac{Mh^4}{180}(b-a) = \frac{90(4/10)^4[2-(-2)]}{180} = \frac{32}{625} < 0.052$$

Note: It is not at all unreasonable to take $M = 100$ in this solution. Doing so gives an error bound ot $\frac{64}{1125}$, which slightly less than 0.057.

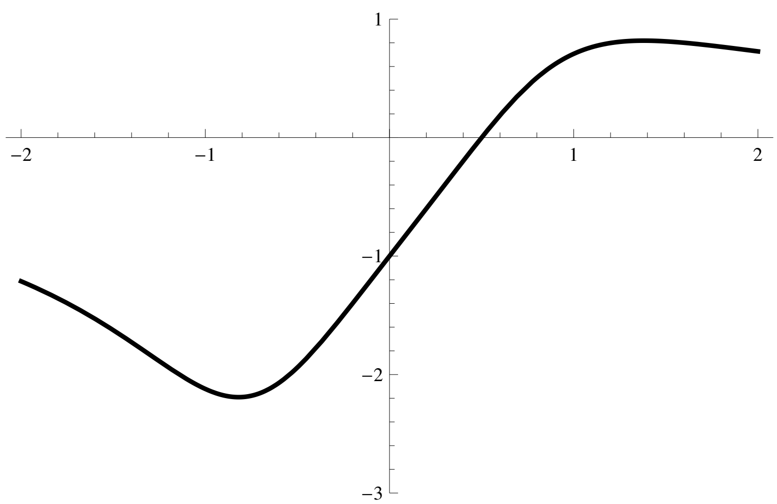

Figure 1: Graph of $f(x)$

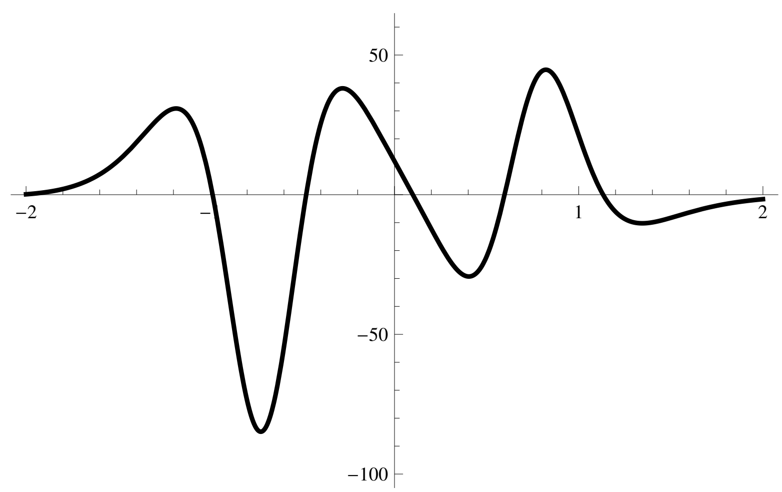

Figure 2: Graph of $f^{(4)}(x)$

MTH 2410, Calculus II—Final Exam **May 13, 2015**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your work is due at 5:30 pm.

1. A particle traveling along a straight line at 576 ft/sec experiences a deceleration, beginning at time $t = 0$ and ending when the particle comes to rest, of $8t^2$ ft/sec^2. Show how to determine how many feet it travels before coming to a complete stop.

2. Consider the region bounded by the curve $y = 1 + 2\sqrt{x}$, the line $y = 1$, and the lines $x = 4$ and $x = 9$. Show how to find the volume generated when this region is rotated about x-axis.

3. Show how to find the arc-length of the curve $y = x^2 - \frac{1}{8}\ln x$ over the interval $[1, e]$.

4. Show how to find the solution of the differential equation:

$$\frac{dy}{dx} + 2y = 8,$$

   that satisfies the initial condition $y(0) = 0$.

5. Show how to find the Maclaurin series (equivalently, the Taylor series about $x = 0$) for $\tanh^{-1} x$, the inverse hyperbolic tangent function. Give the first three non-zero terms, the general term, and the radius of convergence. [Hint: It may be helpful to recall that $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$.]

6. (a) Give the trapezoidal approximation, with 3 subdivisions, for the definite integral $\int_{-14}^{-8} \frac{dx}{x}$. Either give your answer as a fraction of integers, or give it accurate to at least 6 decimals.

   (b) If M is a number for which $|f''(x)| \le M$ for all x in the interval $[a, b]$, the error in the trapezoidal approximation with n subdivisions for the integral $\int_a^b f(x)\,dx$ does not exceed the quantity $\frac{M(b-a)^3}{12n^2}$.

      i. Explain how to choose a suitable M is for this approximation.

      ii. According to this statement, how many subdivisions would we need in the trapezoidal approximation of this integral to guarantee that the resulting approximation is within 0.002 of the integral's true value?

Complete solutions to the exam problems will be available from the course web-site later this evening.

MTH 2410, Calculus II—Final Exam **May 13, 2015**

Instructions: Work the following problems; give your reasoning as appropriate; show your supporting calculations. Do not give decimal approximations unless the nature of a problem requires them. Write your solutions on your own paper; your work is due at 5:30 pm.

1. A particle traveling along a straight line at 576 ft/sec experiences a deceleration, beginning at time $t = 0$ and ending when the particle comes to rest, of $8t^2$ ft/sec^2. Show how to determine how many feet it travels before coming to a complete stop.

   Solution: Acceleration, $a(t)$ is given by $a(t) = -8t^2$. Thus,

$$\begin{aligned} v(t) &= v(0) + \int_0^t a(\tau)\, d\tau = v(0) - \int_0^t 8\tau^2\, d\tau \\ &= 576 - \frac{8}{3}t^3. \end{aligned}$$

   The particle comes to a complete stop when $v(t) = 0$, or when $t^3 = \dfrac{1728}{8} = 216$. This gives $t = 6$ seconds

   For position $s(t)$, we have

$$\begin{aligned} s(t) &= s(0) + \int_0^t v(\tau)\, d\tau = s(0) + \int_0^t \left[576 - \frac{8}{3}\tau^3\right] d\tau \\ &= s(0) + 576t - \frac{2}{3}t^4. \end{aligned}$$

   We find that the distance traveled before stopping is

$$s(6) = 576 \cdot 6 - \frac{2}{3} \cdot 6^4 = 2592 \text{ ft}$$

2. Consider the region bounded by the curve $y = 1 + 2\sqrt{x}$, the line $y = 1$, and the lines $x = 4$ and $x = 9$. Show how to find the volume generated when this region is rotated about x-axis.

   Solution: If τ is a number between 4 and 9, the plane $x = \tau$ intersects this volume in a washer whose center is at the point $x = \tau$ on the x-axis, whose inner radius is 1 and whose outer radius is $1 + 2\sqrt{\tau}$. The area of such a washer is $\pi\left[(1 + 2\sqrt{\tau})^2 - 1^2\right] = 4\pi\left[\sqrt{\tau} + \tau\right]$, so the volume in question is

$$4\pi \int_4^9 \left[\tau^{1/2} + \tau\right] d\tau = 4\pi\left(\frac{2}{3}\tau^{3/2} + \frac{1}{2}\tau^2\right)\Bigg|_4^9 = 4\pi\left(\frac{117}{2} - \frac{40}{3}\right) = \frac{542}{3}\pi.$$

3. Show how to find the arc-length of the curve $y = x^2 - \dfrac{1}{8}\ln x$ over the interval $[1, e]$.

Solution: If $y = x^2 - \frac{1}{8}\ln x$, then $y' = 2x - \frac{1}{8x}$. Thus

$$\int_1^{2e} \sqrt{1+(y')^2}\,dx = \int_1^e \sqrt{1+\left(4x^2 - \frac{1}{2} + \frac{1}{64x^2}\right)}\,dx = \int_1^e \sqrt{\left(4x^2 + \frac{1}{2} + \frac{1}{64x^2}\right)}\,dx$$
$$= \int_1^e \left(2x + \frac{1}{8x}\right)\,dx = \left(x^2 + \frac{1}{8}\ln x\right)\Big|_1^e = e^2 - \frac{7}{8}.$$

4. Show how to find the solution of the differential equation:

$$\frac{dy}{dx} + 2y = 8,$$

that satisfies the initial condition $y(0) = 0$.

Solution: We have

$$\begin{aligned} \frac{dy}{dx} + 2y &= 8; \\ \frac{dy}{dx} &= 8 - 2y; \\ \frac{dy}{8-2y} &= dx; \\ \int \frac{(-2)dy}{8-2y} &= -2\int dx; \\ \ln|8-2y| &= -2x + c; \\ |8-2y| &= e^{-2x+c} = e^c \cdot e^{-2x} \end{aligned}$$

or, allowing $C \leq 0$,

$$\begin{aligned} 8 - 2y &= Ce^{-2x}; \\ y &= 4 - Ce^{-2x}. \end{aligned}$$

But it is given that $y(0) = 0$, so $4 - Ce^0 = 0$, whence $C = 4$. The solution we seek is therefore $y = 4 - 4e^{-2x}$.

5. Show how to find the Maclaurin series (equivalently, the Taylor series about $x = 0$) for $\tanh^{-1} x$, the inverse hyperbolic tangent function. Give the first three non-zero terms, the general term, and the interval of convergence. [Hint: It may be helpful to recall that $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$.]

Solution: Let $f(x) = \tanh^{-1} x$. Then, using what we know about the geometric series, we find that

$$f'(x) = \frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + x^8 + \cdots,$$

and the radius of convergence for this series is one. Integrating, we obtain

$$f(x) = c + x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \cdots .$$

But $f(0) = \tanh^{-1} 0 = 0$, which means that $c = 0$, and

$$\begin{aligned} \tanh^{-1} x &= x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2k+1}}{2k+1} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}. \end{aligned}$$

The radius of convergence for the integrated series is the same as that for the series we integrated. Thus, the radius of convergence for the series immediately above is one.

6. (a) Give the trapezoidal approximation, with 3 subdivisions, for the definite integral $\displaystyle\int_{-14}^{-8} \frac{dx}{x}$. Either give your answer as a fraction of integers, or give it accurate to at least 6 decimals.

   (b) If M is a number for which $|f''(x)| \le M$ for all x in the interval $[a, b]$, the error in the trapezoidal approximation with n subdivisions for the integral $\displaystyle\int_a^b f(x)\,dx$ does not exceed the quantity $\dfrac{M(b-a)^3}{12n^2}$.

      i. Explain how to choose a suitable M is for this approximation.

      ii. According to this statement, how many subdivisions would we need in the trapezoidal approximation of this integral to guarantee that the resulting approximation is within 0.002 of the integral's true value?

Solution:

(a) The required approximation is

$$\begin{aligned} \int_{-14}^{-8} \frac{dx}{x} &\sim -\frac{1}{2}\left(\frac{1}{14} + 2 \cdot \frac{1}{12} + 2 \cdot \frac{1}{10} + \frac{1}{8}\right) \cdot \frac{(-8) - (-14)}{3} \\ &\sim -\frac{473}{840} \sim -0.56309524 \end{aligned}$$

(b) i. The second derivative of the function $f(x) = 1/x$ is $f''(x) = 2/x^3$. The third derivative is $f'''(x) = -6/x^4$, and this is negative throughout the interval $[-14, -8]$, so f'' is a decreasing function whose values are negative on that interval. Consequently, the maximum value of $|f''(x)|$ on the interval $[-14, -8]$ is to be found at the right end-point of the interval. We may therefore take $M = |f''(-8)| = 1/256$.

   ii. If error is not to exceed 0.002, then we must have

$$\begin{aligned} \frac{M(b-a)^3}{12n^2} &\le \frac{1}{500}; \\ \frac{1}{256}\frac{6^3}{12n^2} &\le \frac{1}{500}; \\ n^2 &\ge \frac{2^3 \cdot 3^3 \cdot 2^2 \cdot 5^3}{2^8 \cdot 2^2 \cdot 3} = \frac{3^2 \cdot 5^3}{2^5}, \end{aligned}$$

whence

$$n \geq \frac{3 \cdot 5}{4} \cdot \sqrt{\frac{5}{2}} \sim 5.93.$$

Accuracy to within 0.002 thus requires a minimum of 6 subdivisions.

Complete solutions to the exam problems will be available from the course web-site later this evening.