AP Calculus 2019 BC FRQ Solutions

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1 Problem 1

1.1 Part a

Because fish enter the lake at the rate $E(t) = 20 + 15 \sin \frac{\pi t}{6}$, with t in hours after midnight, the number, F, of fish that enter the lake over the 5-hour period from $t = 0$ to $t = 5$, is given by

$$F = \int_0^5 E(t)\,dt = \int_0^5 \left[20 + 15 \sin \frac{\pi \tau}{6}\right]\,d\tau \tag{1}$$

$$= \left[20\tau - 90 \cos \frac{\pi \tau}{6}\right]\Bigg|_0^5 = 100 + \frac{45(2+\sqrt{3})}{\pi} \sim 153, \tag{2}$$

to the nearest whole number.

1.2 Part b

In like fashion, the average number $\overline{F}$ of fish that leave the lake per hour over the same 5-hour period is given by

$$\overline{F} = \frac{1}{5}\int_0^5 \left(4 + 2^{0.1\tau^2}\right)\,d\tau \sim 6.059. \tag{3}$$

The problem statement givess no instruction to give this answer to the nearest whole number, so we have given it through the first three decimal digits.

1.3 Part c

Let F_0 be the number of fish in the lake at time $t = 0$. The number $f(t)$ of fish in the lake at time t, for $0 \leq t \leq 8$, is then given by

$$f(t) = F_0 + \int_0^t \left[16 + 15 \sin \frac{\pi \tau}{6} - 2^{0.1\tau^2}\right] d\tau \tag{4}$$

By the Fundamental Theorem of Calculus,

$$f'(t) = 16 + 15 \sin \frac{\pi t}{6} - 2^{0.1t^2}. \tag{5}$$

Setting $f'(t) = 0$ and solving numerically, we find just one critical point in $[0, 8]$ at $t_0 \sim 6.204$. The maximum value of f on $[0, 8]$ occurs at an endpoint of the interval or at a critical point, so it must therefore be found at one of three points: at $t = 0$, $t = t_0$, or $t = 8$. Evaluating, gives

$$f(0) = F_0; \tag{6}$$

$$f(t_0) \sim F_0 + 135.015; \tag{7}$$

$$f(8) = F_0 + 80.920. \tag{8}$$

We conclude that the number of fish in the lake is greatest when $t = t_0 \sim 6.204$.

1.4 Part d

As we have seen in Part c of this problem, the rate of change of the number of fish in the lake during the relevant interval is

$$f'(t) = 16 + 15 \sin \frac{\pi t}{6} - 2^{0.1t^2}. \tag{9}$$

Thus,

$$f''(t) = \frac{5\pi}{2} \cos \frac{\pi t}{6} - \frac{2^{0.1t^2}}{5} t \ln 2. \tag{10}$$

So $f''(5) \sim -10.723$, and the rate of change of the number of fish in the lake is decreasing at about the rate of 10.723 fish per hour per hour.

2 Problem 2

2.1 Part a

The area, A, inside the curve $r = 3\sqrt{\theta}\sin\theta^2$ is given by

$$A = \frac{1}{2}\int_0^{\sqrt{\pi}} r^2\,d\theta \tag{11}$$

$$= \frac{9}{2}\int_0^{\sqrt{\pi}} \theta\sin^2\theta^2\,d\theta \tag{12}$$

$$\sim 3.534, \tag{13}$$

where we have carried out the integration numerically. In fact, this is an elementary integral. If no calculator been allowed for this problem, we could have carried out the integration via the substitution $\varphi = \theta^2$, with $d\varphi = 2\theta\,d\theta$. The exact value of the integral so obtained is $\frac{9\pi}{8}$:

$$A = \frac{9}{2}\int_0^{\sqrt{\pi}} \theta\sin^2\theta^2\,d\theta = \frac{9}{4}\int_0^{\sqrt{\pi}} 2\theta\sin^2\theta^2\,d\theta. \tag{14}$$

Put $u = \theta^2$. Then $du = 2\theta\,d\theta$, $u = 0$ when $\theta = 0$, and $u = \pi$ when $\theta = \sqrt{\pi}$. Thus,

$$A = \frac{9}{4}\int_0^{\sqrt{\pi}} 2\theta\sin^2\theta^2\,d\theta \tag{15}$$

$$= \frac{9}{4}\int_0^{\pi} \sin^2 u\,du \tag{16}$$

$$= \frac{9}{8}\int_0^{\pi} (1-\cos 2u)\,du, \text{ by the half-angle formula for the sine,} \tag{17}$$

$$= \frac{9}{8}\left(u - \frac{1}{2}\sin 2u\right)\Big|_0^{\pi} = \frac{9\pi}{8}. \tag{18}$$

2.2 Part b

The average distance, $\overline{D}$ to the origin of a point on this curve is given by

$$\overline{D} = \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{\pi}} r\,d\theta \tag{19}$$

$$= \frac{3}{\sqrt{\pi}}\int_0^{\sqrt{\pi}} \sqrt{\theta}\sin\theta^2\,d\theta \tag{20}$$

$$\sim 1.571, \tag{21}$$

where again we have carried out the integration numerically. (This is not an elementary integral, so we didn't have a choice.)

2.3 Part c

The positive number m for which the line $y = mx$ divides the area inside this curve in halves satisfies the equation

$$\int_0^{\arctan m} r^2\, d\theta = \int_{\arctan m}^{\sqrt{\pi}} r^2\, d\theta. \tag{22}$$

It is not required to solve this equation, and in order to do so, numerical methods are required. The result is $m \sim 3.043$.

2.4 Part d

The (polar) graph of the equation $r = k\cos\theta$ is a circle of radius $k/2$ centered at the point $r = k/2$, $\theta = 0$. Consequently, the positive portion of the left side of this graph approaches the half-line $\theta = \pi/2$ as $k \to \infty$. This means that as $k \to \infty$ the intersection of the area inside the circle and the area inside the curves approaches the area inside the curve and the first quadrant, $0 \le \theta \le \pi/2$. This limiting area, A_L, is given by

$$A_L = \frac{9}{2}\int_0^{\pi/2} \theta \sin^2\theta^2\, d\theta \sim 3.324, \tag{23}$$

where, as usual when it is permitted, we have calculated the integral numerically. As we observed earlier, this integral is elementary, and its exact value is

$$A_L = \frac{9}{32}\left(\pi^2 - 2\sin\frac{\pi^2}{2}\right) \tag{24}$$

3 Problem 3

From what is given about the function f, we easily write

$$f(x) = \begin{cases} -x-1, & -2 \le x \le 0; \\ 2x-1, & 0 \le x \le 2; \\ 3-\sqrt{9-(x-5)^2}, & 2 \le x \le 5. \end{cases} \tag{25}$$

It is also given that the domain of f is the interval $[-6, 5]$ and that f is continuous on that interval.

3.1 Part a

From the properties of the definite integral, we know that

$$\int_{-6}^{-2} f(x)\,dx = \int_{-6}^{5} f(x)\,dx - \int_{-2}^{5} f(x)\,dx. \tag{26}$$

But it is given that

$$\int_{-6}^{5} f(x)\,dx = 7. \tag{27}$$

We have

$$\int_{-2}^{5} f(x)\,dx = \int_{-2}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx \tag{28}$$

$$= 0 + 2 + \left(9 - \frac{9}{4}\pi\right) = 11 - \frac{9}{4}\pi. \tag{29}$$

We can carry out the integrations of (28) by using (25), or we can carry them out by using the geometry of the graph to find the areas of several triangles, a square, and a quarter-circle.

Finally,

$$\int_{-6}^{-2} f(x)\,dx = \int_{-6}^{5} f(x)\,dx - \int_{-2}^{5} f(x)\,dxdx \tag{30}$$

$$= 7 - \left(11 - \frac{9}{4}\pi\right) = \frac{9}{4}\pi - 4. \tag{31}$$

3.2 Part b

The required integral is meaningful because the function f is given by the lower half of a circle in the region of integration, and so is continuously differentiable in that region. This guarantees that the integrand, $2f'(x) + 4$—being put together by algebra from functions continuous in that interval—is continuous there.

We have

$$\int_3^5 \left[2f'(x) + 4\right]\, dx = 2\int_3^5 f'(x)\, dx + 4\int_3^5 dx \tag{32}$$

$$= 2\left[f(5) - f(3)\right] + 4(5-3) \tag{33}$$

$$= 2\left[0 - \left(3 - \sqrt{5}\right)\right] + 8 = 2 + 2\sqrt{5}. \tag{34}$$

3.3 Part c

The function $g(x) = \int_{-2}^{x} f(t)\, dt$ gives the signed area between the curve $y = f(t)$ and the t-axis on the interval $[-2, x]$. It is visually evident that $f(x)$ is positive in $[-2, -1)$ and that the area between the curve and the horizontal axis on the interval $[-2, -1]$ is a small positive number (in fact, it is $1/2$). Similarly, $f(x)$ is negative on $(-1, 1/2)$, and the area between the curve and the horizontal axis on the interval $[-1, 1/2]$ is a negative number of small magnitude (in fact, it is $-3/4$), And, $f(x)$ being positive on $(1/2, 5)$, the area between the curve and the horizontal axis on the interval $[1/2, 5]$ is a positive number substantially larger than either of the other two magnitudes. So $g(x)$, whose derivative is—by the Fundamental Theorem of Calculus—$f(x)$, increases on the interval $[-2, -1]$ from $g(-2) = 0$ to $g(-1) = 1/2$. On the interval $[-1, 1/2]$, the value $g(x)$ decreases from $g(-1) = 1/2$ to $g(1/2) = -1/4$, and on the interval $[1/2, 5]$ the value $g(x)$ increases from $1/4$ to a relatively large positive number (in fact, to $g(5) = 11 - 9\pi/4$, as we saw in the course of our solution to Part a of this problem.) It follows from these considerations that the maximum value of g on $[-2, 5]$ is $g(5) = 11 - 9\pi/4$

3.4 Part d

We are given that f is continuous, so $\lim_{x\to 1} f(x) = f(1) = 1$. We are given that the graph of f is a straight line of slope 2 on the interval $[0, 2]$ so $f'(x) \equiv 2$ on $(0, 2)$. Hence, $\lim_{x\to 1} f'(x) = 2$. Also, $\lim_{x\to 1} 10^x = 10$ and $\lim_{x\to 1} \arctan x = \pi/4$, because both of these functions are continuous at $x = 1$. The limit of a difference is the difference of a limit, and the limit of a quotient is the quotient of the limits (provided the limit in the denominator is not zero). Thus,

$$\lim_{x\to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{10 - 3\cdot 2}{1 - \pi/4} = \frac{16}{4 - \pi}. \tag{35}$$

4 Problem 4

4.1 Part a

Now $h'(t) = -\sqrt{h}/10$, and $V = \pi r^2 h = \pi h$ (because r is the constant 1), and this means that the rate of change, with respect to time, of the volume of water in the barrel is

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} \tag{36}$$

$$= \pi \cdot \left(-\frac{\sqrt{h}}{10} \right) \text{ cubic feet per second.} \tag{37}$$

This gives $\left.\frac{dV}{dt}\right|_{t=4} = -\frac{2\pi}{5}$.

4.2 Part b

We have

$$\frac{d}{dt}\left(\frac{dh}{dt}\right) = -\frac{d}{dt}\left(\frac{\sqrt{h}}{10}\right) = -\left(\frac{1}{20\sqrt{h}}\right)\frac{dh}{dt} \tag{38}$$

$$= -\left(\frac{1}{20\cancel{\sqrt{h}}}\right) \cdot \left(-\frac{\cancel{\sqrt{h}}}{10}\right) = \frac{1}{200}. \tag{39}$$

We conclude that $h'(t)$, which is the rate at which the rate of change with respect to time of h changes, has a positive derivative at all times, and so is always increasing.

4.3 Part c

Let $h(t)$ designate the solution we seek. We are given that $h'(t) = -\sqrt{h(t)}/10$ along with the initial conditon $h(0) = 5$. Hence,

$$\frac{h'(t)}{\sqrt{h(t)}} = -\frac{1}{10}, \text{ so} \tag{40}$$

$$\int_0^t \frac{h'(\tau)}{\sqrt{h(\tau)}}\, d\tau = -\int_0^t \frac{d\tau}{10}. \tag{41}$$

We know that $h(0) = 5 > 0$, so the solution is positive when $t = 5$. Being differentiable (because it satisfies a differential equation), the solution must be continuous near $t = 5$, and so must be positive in some neighborhood of 5. Integrating (41) to some value of t where $h(t)$ remains positive, we find that

$$2\sqrt{h(t)} - 2\sqrt{h(0)} = -\frac{t}{10}, \tag{42}$$

which, on account of the initial condition, becomes

$$\sqrt{h(t)} = \sqrt{5} - \frac{t}{20} \tag{43}$$

We can rewrite this as

$$h(t) = 5 - \frac{\sqrt{5}}{10}t + \frac{t^2}{400}. \tag{44}$$

5 Problem 5

5.1 Part a

The slope of the curve $f(x) = \dfrac{1}{x^2 - 2x + k}$ at a particular value of x is given by

$$f'(x) = \frac{2 - 2x}{(x^2 - 2x + k)^2}, \tag{45}$$

so the slope of this curve at $x = 0$ is

$$f'(0) = \frac{2}{k^2}. \tag{46}$$

For positive k, this is 6 just when $k = \dfrac{1}{\sqrt{3}}$.

5.2 Part b

When $k = -8$, we have

$$\begin{aligned} \int_0^1 f(x)\,dx &= \int_0^1 \frac{dx}{x^2 - 2x - 8} && (47) \\ &= \int_0^1 \frac{dx}{(x-1)^2 - 9}, && (48) \end{aligned}$$

If we let $x-1=3\cos\theta$, then $dx=-3\sin\theta\,d\theta$; $x=1$ when $\theta=\pi/2$; and $x=0$ when $\theta=\arccos(-1/3)$. So

$$\begin{aligned}
\int_0^1 \frac{dx}{(x-1)^2-9} &= \int_{\arccos(-1/3)}^{\pi/2} \frac{-3\sin\theta\,d\theta}{9\cos^2\theta-9} && (49)\\
&= \frac{1}{3}\int_{\arccos(-1/3)}^{\pi/2} \frac{d\theta}{\sin\theta} = \frac{1}{3}\int_{\arccos(-1/3)}^{\pi/2} \csc\theta\,d\theta && (50)\\
&= \frac{1}{3}\ln|\cot x+\csc x|\Big|_{\arccos(-1/3)}^{\pi/2} && (51)\\
&= \frac{1}{3}\ln\left|\cot\frac{\pi}{2}-\csc\frac{\pi}{2}\right| - \ln\left|\cot\arccos\left(-\frac{1}{3}\right)-\csc\left(-\frac{1}{3}\right)\right| && (52)\\
&= \frac{1}{3}\ln|0-1| - \frac{1}{3}\ln\left|-\frac{1}{2\sqrt{2}}-\frac{3}{2\sqrt{2}}\right| && (53)\\
&= -\frac{1}{3}\ln\sqrt{2} = -\frac{1}{6}\ln 2. && (54)
\end{aligned}$$

Alternate Integration:

$$\begin{aligned}
\int_0^1 \frac{dx}{x^2-2x-8} &= \int_0^1 \frac{dx}{(x-4)(x+2)} && (55)\\
&= \frac{1}{6}\int_0^1 \frac{dx}{x-4} - \frac{1}{6}\int_0^1 \frac{dx}{x+2} && (56)\\
&= \frac{1}{6}\ln\left|\frac{x-4}{x+2}\right|\Bigg|_0^1 && (57)\\
&= \frac{1}{6}\ln 1 - \frac{1}{6}\ln 2 = -\frac{1}{6}\ln 2. && (58)
\end{aligned}$$

A Third Integration: Those who have studied the calculus of the hyperbolic functions and their inverses know that $\int \frac{du}{a^2-u^2} = \frac{u}{a}\tanh^{-1}\frac{u}{a}$. (Or they may derive it using a hyperbolic substitution: $x=1+3\tanh v$; $dx=\ 3\,\mathrm{sech}^2\,v\,dv$.) Using this antiderivative, we see that

$$\begin{aligned}
\int_0^1 \frac{dx}{(x-1)^2-9} &= -\frac{1}{3}\tanh^{-1}\frac{x-1}{3}\Big|_0^1 && (59)\\
&= -\frac{1}{3}\tanh^{-1}\left(\frac{1}{3}\right). && (60)
\end{aligned}$$

This is an acceptable answer on the AP Exam. To see that it is the same as the others, we resort to the defintion of the hyperbolic tangent: Suppose that $v=\tanh^{-1}w$, so that

$w = \tanh v$. Then

$$w = \frac{e^v - e^{-v}}{e^v + e^{-v}} = \frac{e^{2v} - 1}{e^{2v} + 1}, \text{ so that} \tag{61}$$

$$we^{2v} + w = e^{2v} - 1, \text{ or} \tag{62}$$

$$(1 - w)e^{2v} = 1 + w, \text{ and} \tag{63}$$

$$(e^v)^2 = e^{2v} = \frac{1 + w}{1 - w}. \tag{64}$$

Now e^{2v} must be a positive number, and this can be so only if $-1 < w < 1$, which makes the fraction on the right positive. But e^v must also be positive, so

$$e^v = \sqrt{\frac{1 + w}{1 - w}}. \tag{65}$$

We rewrite (65) as

$$v = \frac{1}{2} \ln \frac{1 + w}{1 - w}, \tag{66}$$

and we find that we have shown that

$$\tanh^{-1} w = \frac{1}{2} \ln \frac{1 + w}{1 - w}. \tag{67}$$

(We have shown as well that the domain of the inverse hyperbolic tangent function is $-1 < w < 1$.)

Now, returning to (60), we find that

$$-\frac{1}{3} \tanh^{-1} \left(\frac{1}{3}\right) = -\frac{1}{6} \ln \frac{1 + \frac{1}{3}}{1 - \frac{1}{3}} = -\frac{1}{6} \ln \frac{4}{2} \tag{68}$$

$$= -\frac{1}{6} \ln 2. \tag{69}$$

This agrees with our earlier results.

5.3 Part c

When $k = 1$, we are to evaluate, if possible, the improper integral

$$\int_0^2 \frac{dx}{x^2 - 2x + 1} = \lim_{s \to 1^-} \int_0^s \frac{dx}{(x - 1)^2} + \lim_{t \to 1^+} \int_t^2 \frac{dx}{(x - 1)^2} \tag{70}$$

$$= \lim_{s \to 1^-} \left[\frac{s}{1 - s}\right] + \lim_{t \to 1^+} \left[\frac{1}{t - 1} - 1\right] \tag{71}$$

Neither limit exists, so the improper integral diverges. (It is sufficient to observe that either one of the two limits fails to exist.)

6 Problem 6

6.1 Part a

From the graph, we see that $f(0) = 3$ and $f'(0) = -2$. The values $f''(0)$ and $f'''(0)$, along with $f^{(4)}(0)$—which we will need later—are given, too. The third-degree Taylor polynomial, $T_3(x)$, for f about $x = 0$ is

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 \tag{72}$$

$$= 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3. \tag{73}$$

6.2 Part b

The first three nonzero terms of the Maclaurin series for e^x are 1, x, and $\frac{x^2}{2}$. We compute the Maclaurin polynomial for $e^x f(x)$ by expanding the product of the expansions for the two factors:

$$(3 - 2x + \frac{3}{2}x^2 + \cdots)(1 + x + \frac{x^2}{2} + \cdots) = 3 + x + x^2 + \cdots \tag{74}$$

The desired second degree polynomial is $3 + x + x^2$.

Alternate Solution: Putting $g(x) = e^x$, we have $g^{(k)}(0) = e^0 = 1$ for all k. Also, $e^x f(x) = g(x)f(x)$, whence, putting $h(x) = e^x f(x)$,

$$h(0) = e^0 f(0) = 3; \tag{75}$$

$$h'(0) = g'(0)f(0) + g(0)f'(0) = 1 \cdot 3 + 1 \cdot -2 = 1 \tag{76}$$

$$h''(0) = g''(0)f(0) + 2g'(0)f'(0) + g(0)f''(0) = 1 \cdot 3 + 2 \cdot 1 \cdot (-2) + 1 \cdot 3 = 2. \tag{77}$$

The second-degree Maclaurin polynomial for h is therefore

$$h(0) + h'(0)x + \frac{1}{2}h''(0)x^2 = 3 + x + \frac{1}{2} \cdot 2x^2 = 3 + x + x^2 \tag{78}$$

6.3 Part c

$$\int_0^x f(t)\,dt \sim \int_0^x \left(3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3\right) dt = 3x - x^2 + \frac{1}{2}x^3 - \frac{23}{48}x^4 \tag{79}$$

Setting $x = 1$ we find that $h(1) \sim \frac{8506}{3}$.

6.4 Part d

Recapitulating Parts (a) and (c), but instead of for $T_3(x)$ for $T_4(x)$—the fourth-degree Taylor polynomial of f about $x = 0$—and using the fact that $f^{(4)}(0) = 54$ is given, we have

$$T_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} x^k \tag{80}$$

$$= 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3 + \frac{9}{4}x^4. \tag{81}$$

From this, it follows that the fifth degree Taylor polynomial for $h(x)$ about $x = 0$ is

$$\int_0^x T_4(t)\,dt = \int_0^x T_3(t)\,dt + \frac{9}{4}\int_0^x t^4\,dt \tag{82}$$

$$= 3x - x^2 + \frac{1}{2}x^3 - \frac{23}{48}x^4 + \frac{9}{20}x^5. \tag{83}$$

We are given that the series, of which this latter expression is the first five nonzero terms, meets the requirements of the alternating series test when $x = 1$, so we may conclude that the approximation of Part (c), above, differs from $h(1)$ by at most the magnitude of the corresponding fifth term, which is

$$\frac{9}{20}(1)^5 = 0.45. \tag{84}$$