AP Calculus 2021 BC FRQ Solutions

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1. **Solution:** We begin by writing $r_0 = 0$, $r_1 = 1$, $r_2 = 2$, $r_3 = 2.5$, and $r_4 = 4$.

   (a) To approximate $f'(2.25)$ from the given data, we write

   $$f'(2.25) \sim \frac{f(2.5) - f(2.0)}{2.5 - 2.0} = \frac{10 - 6}{2.5 - 2.0} = \frac{4}{0.5} = 8. \tag{1}$$

   Thus, $f(2.25) \sim 8$ mg/cm^3. This means that as we move directly outward from the center of the Petri dish, the density of bacteria is increasing at about the per-centimeter-rate of 8 milligrams per square centimeter at a point 2.25 centimeters from the center.

   (b) The required right Riemann sum to approximate $2\pi \int_0^4 r f(r)\, dr$ is

   $$2\pi \sum_{k=1}^{4} r_k f(r_k)(r_k - r_{k-1}) = 2\pi \left(1 \cdot 2 \cdot 1 + 2 \cdot 6 \cdot 1 + 2.5 \cdot 10 \cdot 0.5 + 4 \cdot 18 \cdot 1.5\right) \text{ mg.} \tag{2}$$

   This is 269π mg.

   (c) It is given that f is an increasing function. It follows that for any integer $k = 1, 2, 3, 4$ and $r_{k-1} \le r < r_k$, then $r f(r) < r f(r_k) < r_k f(r_k)$. Hence, for all such k, we must have

   $$\int_{r_{k-1}}^{r_k} r f(r)\, dr < \int_{r_{k-1}}^{r_k} r_k f(r_k)\, dr < r_k f(r_k)(r_k - r_{k-1}). \tag{3}$$

   We now see that

   $$\int_0^4 r f(r)\, dr = \sum_{k=1}^{4} \int_{r_{k-1}}^{r_k} r f(r)\, dr < \sum_{k=1}^{4} r_k f(r_k)(r_k - r_{k-1}). \tag{4}$$

   The right Riemann sum is therefore an overestimate for the corresponding integral.

(d) The average value of g on $[1, 4]$ is

$$\frac{2\pi}{4}\int_1^4 g(r)\,dr = \pi\int_1^4 \left[1 - 8\cos^3(1.57\sqrt{r})\right]dr. \tag{5}$$

Carrying out a numerical integration, we find that the required average value is about 44.186 milligrams per square centimeter.

2. **Solution:**

(a) Speed at time t, $s(t)$, of a particle whose velocity vector is $\mathbf{v}(t) = \langle x(t), y(t)\rangle$, is given by

$$s(t) = \|\mathbf{v}(t)\| = \sqrt{\mathbf{v}(t)\cdot\mathbf{v}(t)\rangle}. \tag{6}$$

Here, we have $\mathbf{v}(t) = \langle (t-1)e^{t^2}, \sin t^{1.25}\rangle$, so

$$s(t) = \sqrt{(t-1)^2 e^{2t^2} + \sin^2 t^{1.25}}. \tag{7}$$

Thus, $s(1.2) = \sqrt{(1.2-1)^2 e^{2(1.2)^2} + \sin^2 (1.2)^{1.25}} \sim 1.271.$
Acceleration at time t, $\mathbf{a}(t)$ is $\mathbf{a}(t) = \mathbf{v}(t)$, so

$$\mathbf{a}(t) = \langle e^{t^2} + 2te^{t^2}(t-1), 1.25t^{0.25}\cos t^{1.25}\rangle, \text{and} \tag{8}$$

$$\mathbf{a}(1.2) \sim \langle 6.247, 0.405\rangle. \tag{9}$$

(b) Distance traveled over the time interval $[a, b]$ is $\displaystyle\int_a^b s(\tau)\,d\tau$, so the required distance is

$$\int_0^{1.2} \sqrt{(\tau-1)^2 e^{2\tau^2} + \sin^2 \tau^{1.25}}\,d\tau \sim 1.010, \tag{10}$$

where we have carried out the integration numerically.

(c) The particle is farthest to the left for $t \in [0, \infty)$ when $x(t)$ assumes its global minimum value on that interval. At this point, either $t = 0$ or $x'(t) = 0$. But

$$x'(t) = (2t^2 - 2t + 1)e^{t^2}, \tag{11}$$

and this quantity vanishes only when the quadratic polynomial $2t^2 - 2t + 1$ vanishes. This quadratic has no real zeros, so $x'(t)$ does not vanish on $[0, \infty)$. We conclude that our particle is farthest to the left when $t = 0$.

There can be no time when the particle is farthest to the right. Such a time would be a global maximum for $x(t)$ on $[0, \infty)$, so would also have to be either when $t = 0$ or when $x'(t) = 0$. We have seen that $t = 0$ gives the global minimum on the interval, and, because $x(t)$ is non-constant, cannot give the maximum. We have also seen that $x'(t)$ does not vanish on $[0, \infty)$. Thus, there can be no global maximum.

We could also arrive at this conclusion by observing that e^{t^2} is always positive on $[0, \infty)$ and $2t^2 - 2t + 1 = t^1 + (t-1)^2$ is also positive throughout $[0, \infty)$. From this, it follows that $x'(t)$, which is the product of these two latter quantities, is positive throughout $[0, \infty)$. This guarantees that $x(t)$ increases strictly from $x(0)$ on $[0, \infty)$.

3. **Solution:**

(a) The area in the first quadrant bounded by the x-axis and the curve $y = 6x\sqrt{4 - x^2}$ is

$$3 \int_0^2 \sqrt{4 - x^2} \cdot 2x\, dx = -2(4 - x^2)^{3/2} \Big|_0^2 = -0 + 16 = 16 \text{ in}^2 \tag{12}$$

(b) If y is as above, $y' = \dfrac{c(4 - 2x^2)}{\sqrt{4 - x^2}}$ and $y' = 0$ for $0 \leq x \leq 2$, then $x = \sqrt{2}$. We are given $c > 0$, so because $y = 0$ when $x = 0$ or $x = 2$ and $y > 0$ when $0 < x < 2$, we see that y assumes its absolute minimum on $[0, 2]$ at the endpoints. Applying the Extreme Value Theorem y, which depends continuously on x throughout the closed, bounded interval $[0, 2]$, must have an absolute maximum somewhere interior to that interval. By Fermat's Theorem that maximum must occur at a value of x where $y' = 0$. There being only one such value, it must yield the maximum. Because y gives, for each x, the radius of the corresponding cross-sectional slice, we conclude that

$$1.2 = c\sqrt{2}\sqrt{4 - (\sqrt{2})^2} = 2c. \tag{13}$$

It follows that $c = 0.6$

(c) The volume of the spinning toy generated by the curve $y = cx\sqrt{4 - x^2}$ is

$$c^2\pi \int_0^2 x^2(4 - x^2)\, dx = c^2\pi \int_0^2 (4x^2 - x^4)\, dx \tag{14}$$

$$= c^2\pi \left(4\frac{x^3}{3} - \frac{x^5}{5}\right) \Big|_0^2 = c^2\pi \left(\frac{32}{3} - \frac{32}{5}\right) = \frac{64}{15} c^2\pi. \tag{15}$$

If this is to be 2π, we must have $c = \dfrac{\sqrt{30}}{8}$.

4. **Solution:**

(a) If $G(x) = \int_0^x f(t)\,dt$, then, according to the Fundamental Theorem of Calculus, $G'(x) = f(x)$. Now G is concave upward on those open intervals where $G'(x) = f(x)$ is increasing. Because we see from its graph that f is increasing on $[-4, -2]$ and on $[2, 6]$, we conclude that G is concave upward on $(-4, -2)$ and on $(2, 6)$.

(b) If $P(x) = G(x) \cdot f(x)$, then

$$P'(x) = G'(x) \cdot f(x) + G(x) \cdot f'(x), \tag{16}$$

so

$$\begin{aligned} P'(3) &= G'(3) \cdot f(3) + G(3) \cdot f'(3) &(17)\\ &= f(3) \cdot f(3) + G(3) \cdot f'(3). &(18) \end{aligned}$$

Now

$$\begin{aligned} f(3) &= -3, &(19)\\ f'(3) &= \frac{f(6) - f(2)}{6 - 2} = \frac{0 - (-4)}{6 - 2} = 1, &(20) \end{aligned}$$

and

$$\begin{aligned} G(3) &= \frac{f(0) + f(2)}{2} \cdot 2 + \frac{f(2) + f(3)}{2} \cdot 1 &(21)\\ &= \frac{4 + (-4)}{\not{2}} \cdot \not{2} + \frac{-4 + (-3)}{2} \cdot 1 = -\frac{7}{2} &(22) \end{aligned}$$

so

$$\begin{aligned} P'(3) &= (-3)^2 + 1 \cdot \left(-\frac{7}{2}\right) &(23)\\ &= \frac{11}{2}. &(24) \end{aligned}$$

(c) Now

$$G(2) = \int_0^2 f(t)\,dt = \frac{f(0) + f(2)}{\not{2}} \cdot \not{2} = 4 + (-4) = 0, \tag{25}$$

and G, by the Fundamental Theorem of Calculus, is continuous. This means that $\lim_{x \to 2} G(x) = G(2) = 0$. Also, $\lim_{x \to 2} (x^2 - 2x) = 4 - 4 = 0$, so we may apply l'Hôpital's rule to obtain

$$\lim_{x \to 2} \frac{G(x)}{x^2 - 2x} = \lim_{x \to 2} \frac{f(x)}{2x - 2}, \tag{26}$$

provided that the latter limit exists. But from the graph, we see that $\lim_{x\to 2} f(x) = -4$. Hence, the limit on the right side of (26) exists and is $-4/2 = -2$. We conclude, by l'Hôpital's rule, that

$$\lim_{x\to 2} \frac{G(x)}{x^2 - 2x} = -2. \tag{27}$$

(d) The average value, A, of the rate of change of G on the interval $[-2, 4]$ is given by

$$A = \frac{1}{4-(-2)} \int_{-2}^{4} G'(t)\,dt \tag{28}$$

$$= \frac{1}{6} \int_{-2}^{4} f(t)\,dt \tag{29}$$

$$= \frac{1}{6} \int_{-2}^{0} f(t)\,dt + \frac{1}{6} \int_{0}^{4} f(t)\,dt \tag{30}$$

$$= \frac{1}{12}\big[f(-2) + f(0)\big] \cdot 2 + \frac{1}{12}\big[f(0) + f(2)\big] \cdot 2 + \frac{1}{12}\big[f(2) + f(4)\big] \cdot 2 \tag{31}$$

$$= \frac{6+4}{6} + \frac{4-4}{6} + \frac{-4+(-2)}{6} = \frac{2}{3}. \tag{32}$$

(The trapezoidal integration is justified by the fact that f is piecewise linear and the appropriate choice of points within the interval of integration.)

(e) The function G is, by the Fundamental Theorem of Calculus, continuous on the interval $[-2, 4]$ and differentiable on the interval (-2.4); moreover, $G'(x) = f(x)$ for $-2 < x < 4$. The Mean Value Theorem therefore guarantees the existence of $\xi \in (-2, 4)$ such that

$$f(\xi) = G'(\xi) = \frac{G(4) - G(-2)}{4 - (-2)} = \frac{1}{6}\left[G(4) - G(-2)\right]. \tag{33}$$

But then by the definition of G

$$G'(\xi) = \frac{1}{6}\left[\int_0^4 f(t)\,dt - \int_0^{-2} f(t)\,dt\right] \tag{34}$$

$$= \frac{1}{6}\left[\int_{-2}^0 f(t)\,dt + \int_0^4 f(t)\,dt\right] = \frac{1}{6}\int_{-2}^4 f(t)\,dt \tag{35}$$

According to (29), this is just A, so the answer is "Yes, the Mean Value Theorem guarantees a value ξ, $-4 < \xi < 2$, for which $G'(\xi)$ is the average rate of change of G on $[-4, 2]$."

Remark: It isn't at all difficult—though it is a bit tedious—by reading the given graph, to write an explicit piecewise representation of the function f, and, thereby, of the function G.

The function f is given by

$$f(t) = \begin{cases} 3(t+4) = 3t+12; & -4 \leq f < -2 \\ 6-(t+2) = -t+4; & -2 \leq t < 0 \\ 4-4t = -4t+4; & 0 \leq t < 2 \\ -4+(t-2) = t-6; & 2 \leq t < 6. \end{cases} \tag{36}$$

Carrying out the necessary integrations, we find that

$$\tag{37}$$

$$G(x) = \begin{cases} \frac{3}{2}x^2+12x+8; & -4 \leq t < -2 \\ -\frac{1}{2}x^2+4x; & -2 \leq x < 0 \\ -2x^2+4x; & 0 \leq x < 2 \\ \frac{1}{2}x^2-6x+10; & 2 \leq x \leq 6. \end{cases} \tag{38}$$

5. **Solution:**

   (a) From $y' = xy\ln x$, $y(1) = 4$ we find that $y'(1) = 0$. The second degree Tayor polynomial, $T(x)$, for y about $x = 1$ is then

$$T(x) = y(1) + y'(1)(x-1) + \frac{1}{2}y''(1)(x-1)^2 \tag{39}$$

$$= 4 + 0\cdot(x-1) + \frac{1}{2}\cdot 2\cdot(x-1)^2 \tag{40}$$

$$= 4 + (x-1)^2. \tag{41}$$

   At $x = 2$, we then write

$$f(2) \sim T(2) = 4 + (2-1)^2 = 5. \tag{42}$$

   (b) Applying Euler's method with stepsize $h = 1/2$, beginning at $x = 1$, we write:

$$y_1 \sim y_0 + f'(x_0)h = 4 + 0\cdot\frac{1}{2} = 4; \tag{43}$$

$$y_2 \sim y_1 + f'(x_1)h = 4 + x_1y_1h\ln x_1 = 4 + \frac{3}{2}\cdot 4\cdot\frac{1}{2}\cdot\ln\frac{3}{2} \sim 5.216. \tag{44}$$

(c) We note first that, integrating by parts, we have

$$\int x\ln x\,dx = \frac{x^2}{2}\cdot\ln x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4}. \tag{45}$$

Now we multiply the equation $y' = xy\ln x$ through by $e^{\frac{x^2}{4}-\frac{x^2}{2}\ln x}$, to obtain

$$\left(e^{\frac{x^2}{4}-\frac{x^2}{2}\ln x}\right)y' - \left(e^{\frac{x^2}{4}-\frac{x^2}{2}\ln x}\cdot x\ln x\right)y = 0, \text{ or} \tag{46}$$

$$\frac{d}{dx}\left(e^{\frac{x^2}{4}-\frac{x^2}{2}\ln x}y\right) = 0. \tag{47}$$

If follows that there is a real number C such that

$$e^{\frac{x^2}{4}-\frac{x^2}{2}\ln x}y \equiv C. \tag{48}$$

But we know that $y = 4$ when $x = 1$. Hence,

$$C = e^{\frac{1^2}{4}-\frac{1^2}{2}\ln 1}\cdot 4 = 4e^{\frac{1}{4}}. \tag{49}$$

We conclude that the desired solution to the initial value problem is given (writing, as is common, $\exp(u)$ for e^u) by

$$y = 4\exp\left(\frac{x^2}{2}\ln x - \frac{x^2}{4} + \frac{1}{4}\right) \tag{50}$$

6. **Solution:**

(a) The integral test for the convergence or divergence of a series tells us that if F is a non-negative function on $[1,\infty)$ which is continuous and non-increasing on $[1,\infty)$, then $\sum_{k=1}^{\infty} F(k)$ converges if and only if the improper integral $\int_1^\infty F(t)\,dt$ converges.

Let $F(t) = \frac{1}{e^t} = e^{-t}$. Then F is positive-valued and continuous (in fact, differentiable) on $[1,\infty)$, Moreover, $F'(t) = -e^{-t}$, which is negative for all t. Hence, F is decreasing (which is surely non-increasing) on $[1,\infty)$. Because

$$\int_1^\infty e^{-t}\,dt = \lim_{T\to\infty}\int_1^T e^{-t}\,dt \tag{51}$$

$$= -\lim_{T\to\infty} e^{-t}\Big|_1^T \tag{52}$$

$$= \lim_{T\to\infty}\left(e^{-1} - e^{-T}\right) = e^{-1} \tag{53}$$

converges, we may conclude—by the integral test—that the series $\sum_{k=1}^{\infty} e^{-k}$ converges.

(b) If $a_k = \dfrac{(-1)^k}{2e^k + 3}$, and $b_k = e^{-k}$, then

$$\lim_{k\to\infty} \frac{|a_k|}{|b_k|} = \lim_{k\to\infty} \frac{e^k}{2e^k + 3} \tag{54}$$

$$= \lim_{k\to\infty} \frac{1}{2 + 3e^{-k}} = \frac{1}{2+0} = \frac{1}{2} < \infty. \tag{55}$$

It follows, by the limit comparison test, that $\sum_{k=0}^{\infty} \dfrac{1}{2e^k + 3}$ converges, so that $\sum_{k=0}^{\infty} \dfrac{(-1)^k}{2e^k + 3}$ is an absolutely convergent series.

(c) We apply the ratio test to the series for $g(x)$: $\sum_{k=0}^{\infty} a_k$, where $a_k = \dfrac{(-1)^k x^k}{2e^k + 3}$. This gives

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k l} = \lim_{k\to\infty} \left(\frac{|x|^{k+1}}{2e^{k+1} + 3} \cdot \frac{2e^k + 3}{|x|^k} \right) \tag{56}$$

$$= |x| \lim_{k\to\infty} \frac{2 + 3e^{-k}}{2e + 3e^{-k}} = e^{-1}|x|. \tag{57}$$

This is less than one when $|x| < e$. By the ratio test, the radius of convergence for the series $\sum_{k=0}^{\infty} \dfrac{(-1)^k x^k}{2e^k + 3}$ is e.

(d) The magnitude of the error in an approximation by means of an alternating series does not exceed the magnitude of the first term of the series not used in that approximation. In this case, the quantity $g(1)$ is approximated by the first two terms of the series $\sum_{k=0}^{\infty} \dfrac{(-1)^k x^k}{2e^k + 3}$, taking $x = 1$. The first term not used is the term that corresponds to $k = 2$, for which

$$\left| \frac{(-1)^k x^k}{2e^k + 3} \right| = \frac{1}{2e + 3}. \tag{58}$$

The error in the two-term approximation of $g(1)$ by means of its Maclaurin series does not exceed $\dfrac{1}{2e + 3} \sim 0.1185317$. According to the instructions in force for this examination, this result may be rounded, either to 0.118 or to 0.119. Because we want the number we seek to be an upper bound, it is preferable to round to 0.119. (We have no basis to guarantee, *a priori,* that 0.118 really is an upper bound.)