Solutions to
2022 AP Calculus AB
Free Response Questions

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1. We are given, effectively, $A(t) = 450\sqrt{\sin(0.62t)}$, where $0 \le t \le 10$, as the rate—in vehicles per hour—at which vehicles arrive at a toll plaza, with t representing the time elapsed since 5 A.M.

   (a) The total number of vehicles that arrive at the toll plaza between $t = 1$ and $t = 5$ is $\int_1^5 A(t)\,dt$. Evaluation is not required, but numerical integration gives

   $$\int_1^5 A(t)\,dt = 450\int_1^5 \sqrt{\sin(0.62t)}\,dt \sim 1502.148 \text{ vehicles.} \tag{1}$$

   (b) The average value of the rate at which vehicles arrive at the toll plaza from $t = 1$ to $t = 5$ is (to three decimal places)

   $$\frac{1}{5-1}\int_1^5 A(t)\,dt \sim 375.537 \text{ vehicles per hour,} \tag{2}$$

   where we have replaced the integral with the value we obtained numerically in the previous part of this problem.

   (c) We find that

   $$A'(t) = \frac{139.5\cos(0.62t)}{\sqrt{\sin(0.62t)}}, \tag{3}$$

   whence $A'(1) \sim 148.947 > 0$. The derivative of the function A is positive when $t = 1$, so A, the rate at which vehicles arrive at the toll plaza at 6 A.M, is increasing at that time.

(d) We are given that

$$N(t) = \int_a^t [A(x) - 400]\, dx, \tag{4}$$

where a is the time when A is 400. By the Fundamental Theorem of Calculus, this means that $N'(t) = A(t) - 400$.
Solving numerically, we find that $A(t) = 400$ when $t \sim 1.469$, so $a \sim 1.469$. Solving numerically, we find that $N't) = 0$ when $t = a$ (Duh!) and when $t \sim 3.598$.. Consequently, the maximum value of N on the interval $[a, 4]$ occurs at $t = a$, when $t \sim 3.598$, or when $t = 4$. We find that

$$N[a] = 0, \tag{5}$$
$$N[3.598] \sim 71.254, \tag{6}$$
$$N[4] \sim 62.338. \tag{7}$$

From this we conclude that, to the nearest whole vehicle, the length of the line reaches a maximum about 71 vehicles at about 8:36 A.M.

2. (a) Solving numerically, we find that B, the value near $x = 1$ where the two curves cross, is about 0.78198. So the area of the region enclosed by the two graphs is, to three decimal places,

$$\int_{-2}^{B} \left[\ln(x+3) - (x^4 + 2x^3)\right) dx = (x+3)\ln(x+3) - x - \frac{1}{2}x^4 - \frac{1}{5}x^5 \Big|_{-2}^{B} \tag{8}$$
$$\sim 3.604. \tag{9}$$

(b) The vertical distance, h, between the curves is given by $h(x) = f(x) - g(x)$ on the interval $[-2, B]$. On this interval, we have

$$h'(x) = f'(x) - g'(x) \tag{10}$$
$$= \frac{1}{x+3} - 4x^3 - 6x^2, \text{ whence} \tag{11}$$
$$h'(-0.5) = -0.6 < 0. \tag{12}$$

This derivative is continuous, and $h'(x)$ is therefore negative in some interval centered at $x = -0.5$. When its derivative is negative on an interval, the function is decreasing on that interval, so h is decreasing near $x = -0.5$.
Note: Very few elementary calculus textbooks define the terms *increasing* or *decreasing* except in reference to intervals, leaving the phrase "decreasing at a point" meaningless. The function u, given by

$$u(x) = \begin{cases} x^2 \sin \frac{1}{x} - 2x, & x \neq 0; \\ 0, & x = 0; \end{cases} \tag{13}$$

is one of a function u for which $u'(0)$ is negative, even though u is decreasing on *no* open interval centered at $x = 0$.

(c) The area of a cross section whose horizontal coordinate is $x = t$, is $A(t) = [h(t)]^2$, the function h being as above. Hence the volume of the solid in question is

$$V = \int_{-2}^{B} A(t)\, dt \tag{14}$$

$$= \int_{-2}^{B} \left[\ln(x+3) - (x^4 + 2x^3)\right]^2 dx. \tag{15}$$

One can carry out this antidifferentiation by elementary means, but the computation is too horrible to contemplate under examination conditions. We calculate numerically to find that $V \sim 5.340$, to three decimal places.

(d) The area of the cross-section at $t = x$ is $A(x)$, where A is as defined in the previous part of this problem. If the x-coordinate of the cross-section moves rightward with velocity $\frac{dx}{dt}$, then

$$\frac{d}{dt}A(x) = \frac{d}{dt}\left[f(x) - g(x)\right]^2 \tag{16}$$

$$= 2\left[\ln(x+3) - (x^4 + 2x^3)\right]\left[\frac{1}{x+3} - (4x^3 + 6x^2)\right]\frac{dx}{dt}. \tag{17}$$

Putting $x = -0.5$ and $\frac{dx}{dt} = 7$ gives $\frac{d}{dt}A(x) \sim -9.272$. At the time specified, the rate of change of the area of the cross-section with respect to time is, to three decimal places, -9.272 square units per second.

3. (a) By the Fundamental Theorem of Calculus and what is given,

$$f(x) = f(4) + \int_4^x f'(t)\, dt \tag{18}$$

$$= 3 + \int_4^x f'(t)\, dt, \tag{19}$$

In fact, f' is given by

$$f'(t) = \begin{cases} -\sqrt{4t - t^2}, & 0 \le t < 4; \\ t - 4, & 4 \le t < 6; \\ 8 - t, & 6 \le t \le 1. \end{cases} \tag{20}$$

However, we will make no use of this fact.

Because the portion of the f' curve from $t = 0$ to $t = 4$ is a semi-circle of radius 2 lying below the horizontal axis whose diameter is the horizontal axis, $\int_0^4 f'(t)\,dt = -2\pi$. The portion of the curve over the interval $[4, 5]$ is a straight line that forms, with the horizontal axis, a triangle of base 1 and altitude 1, so $\int_4^5 f'(t)\,dt = \frac{1}{2}$. From these facts, we find that

$$f(0) = 3 + \int_4^0 f'(t)\,dt = 3 - \int_0^4 f'(t)\,dt = 3 + 2\pi; \tag{21}$$

$$f(5) = 3 + \int_4^5 f'(t)\,dt = 3 + \frac{1}{2} = \frac{7}{2}. \tag{22}$$

(b) A function has an inflection point where its derivative has a local maximum or a local minimum—that is, where its derivative changes from being increasing to being decreasing or *vice versa.* We see from the given graph that $f'(x)$ has a local minimum at $x = 2$ and that $f'(t)$ has a local maximum at $x = 6$. Consequently, f has inflection points at $x = 2$ and and $x = 6$.

Note: Some elementary textbooks require that the second derivative be defined at an inflection point. If we adopt this definition, f has just one inflection point, at $x = 2$.

(c) The function g defined by $g(x) = f(x) - x$ is decreasing on the closures of those intervals where $g'(x) < 0$—that is, where $f'(x) - 1 < 0$, which is to say $f'(x) < 1$. We see from the given graph that these inequalities hold for those, and only those, values of x which are less than 5. Hence f is decreasing on the interval $[0, 5]$.

(d) The absolute minimum value of g on the interval $[0, 7]$ exists because, as the integral of a continuous derivative, g is itself a continuous, differentiable, function on that interval. We know that the absolute minimum of such a function must occur either at an endpoint of the interval or at a point where the derivative is zero. So we must evaluate g at $x = 0$, at $x = 7$, and at $x = 5$, the latter point being the point only point in the interval where $g'(x) = f'(x) - 1 = 0$. From the first part of this problem, we have

$$g(0) = f(0) - 0 = 3 + 2\pi; \tag{23}$$

$$g(5) = f(5) - 5 = \frac{7}{2} - 5 = -\frac{3}{2}. \tag{24}$$

It is easy to see from the geometry of the curve that $\int_4^6 f'(t)\,dt = 2$ and that

$\int_6^7 f'(t)\,dt = \frac{3}{2}$. Therefore,

$$g(7) = f(7) - 7 = \left[3 + \int_4^7 f'(t)\,dt\right] - 7 \tag{25}$$

$$= \left[\int_4^6 f'(t)\,dt + \int_6^7 f'(t)\,dt\right] - 4 \tag{26}$$

$$= 2 + \frac{3}{2} - 4 = -\frac{1}{2}. \tag{27}$$

We conclude that the absolute minimum value of g on $[0, 7]$ is $g(5) = -\frac{3}{2}$.

4. (a) We can approximate $r''(8.5)$, in centimeters/day^2, by

$$r''(8.5) \sim \frac{r(10) - r(7)}{10 - 7} = \frac{(-3.8) - (-4.4)}{3} = 0.2 \text{ cm/day}^2. \tag{28}$$

(b) We expect derivative of the radius of the base of a melting cone of ice to be continuous as a function of time. Under the assumption that this is so, there must be a t_0 between $t = 0$ and $t = 3$ where $r(t_0) = -6$, because $r(3) = -5.0$, and $r(0) = -6.1$—so the Intermediate Value Theorem for Continuous Functions guarantees the existence of such a t_0. **Note:** In fact, continuity of r' is not needed; derivatives always have the intermediate value property—but this is rarely shown, or even stated as a fact, in elementary calculus courses. The fact is known as "Darboux's Theorem," and it is a standard part of a good advanced calculus course.

(c) Using the values from the given table in a right Riemann sum, we have

$$\int_0^{12} r'(t)\,dt \sim r'(3)(3-0) + r'(7)(7-3) + r'(10)(10-7) + r'(10)(12-10) \tag{29}$$

$$\sim (-5.0)\cdot 3 + (-4.4)\cdot 4 + (-3.8)\cdot 3 + (-3.5)\cdot 2 = -51.0 \tag{30}$$

(d) Let $h(t)$ denote the height of the cone at time t; in addition to the information in the table, we are given $r(3) = 100$ and $h(3) = 50$. We know that the volume of the cone at time t is given by $V(t) = \frac{\pi}{3}[r(t)]^2 h(t)$, and from this we see that

$$V'(t) = \frac{\pi}{3}\left[2r(t)h(t)r'(t) + [r(t)]^2 h'(t)\right]. \tag{31}$$

Thus,

$$V'(3) = \frac{\pi}{3}\left[2\cdot 100\cdot 50\cdot(-5.0) + (100)^2\cdot(-2.0)\right] = -\frac{70000\pi}{3} \text{ cm}^3/\text{day} \tag{32}$$

5. (a) See Figure 1

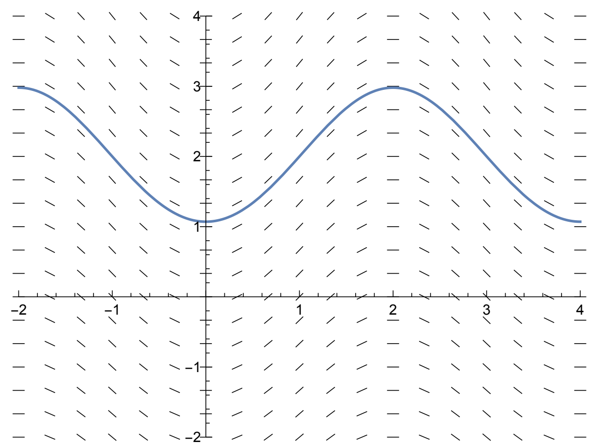

Figure 1: The Drawing for Problem 5

(b) At $x = 1$, $y = 2$, we compute from the differential equation that $y' = \frac{3}{2}$. Hence, the tangent line to the solution through $(1, 2)$ has equation

$$y = 2 + \frac{3}{2}(x - 1). \tag{33}$$

From this, we approximate $y(0.8)$ on the solution curve as

$$y(0.8) \sim 2 + \frac{3}{2}(0.8 - 1) = 2 + \frac{3}{2}(-0.2) = 1.7. \tag{34}$$

(c) If $y'' > 0$ on $[-1, 1]$, the solution curve $y = f(x)$ must be concave upward in that region, meaning that the line tangent to the curve at $x = 1$ lies below the curve there. We conclude that the approximation we have found in the previous part of this problem *underestimates* $f(0.8)$.

(d) If $y = f(x)$ is the solution to the differential equation $y' = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$

that satisfies $f(1) = 2$, then

$$\frac{dy}{dx} = \sin\left(\frac{\pi}{2}x\right)\sqrt{f(x)+7}; \tag{35}$$

$$\frac{dy}{\sqrt{y+7}} = \sin\left(\frac{\pi}{2}x\right)\,dx; \tag{36}$$

$$\int \frac{dy}{\sqrt{y+7}}\,dy = \int \sin\left(\frac{\pi}{2}x\right)\,dx; \tag{37}$$

$$\int u^{-1/2}\,du = \int \sin\left(\frac{\pi}{2}t\right)\,dt; \tag{38}$$

$$2u^{1/2} = c - \frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right); \tag{39}$$

$$y = \left[c - \frac{1}{\pi}\cos\left(\frac{\pi}{2}x\right)\right]^2 - 7. \tag{40}$$

But $y = 2$ when $x = 1$, so

$$2 = \left[c - \frac{1}{\pi}\cos\left(\frac{\pi}{2}\right)\right]^2 - 7, \tag{41}$$

whence we may take $c = 3$. Our solution is therefore

$$f(x) = \left[3 - \frac{1}{\pi}\cos\left(\frac{\pi x}{2}\right)\right]^2 - 7. \tag{42}$$

6. (a) If $x_P(t) = 6 - 4e^{-t}$, then the velocity $v_P(t)$ at time t is given by

$$v_P(t) = x'_P(t) = 4e^{-t}. \tag{43}$$

(b) Velocity at time t of particle Q is given by $v_Q(t) = \dfrac{1}{t^2} = t^{-2}$. The acceleration $a_Q(t)$ at time t is therefore given by

$$a_Q(t) = v'_Q(t) = -2t^{-3} = -\frac{2}{t^3}. \tag{44}$$

The speed $s_Q(t)$ of particle Q at time t, which is never negative satisfies

$$[s_Q(t)]^2 = v_Q(t) \cdot v_Q(t), \text{ so} \tag{45}$$

$$2s_Q(t)s'_Q(t) = 2v_Q(t) \cdot v'_Q(t), \text{ or} \tag{46}$$

$$s'_Q(t) = \frac{v_Q(t) \cdot a_Q(t)}{s_Q(t)}, \text{ as long as } s_Q(t) \neq 0. \tag{47}$$

Thus, $s'_Q(t) < 0$ when $v_Q(t) \cdot a_Q(t) < -0$, or when $(t^{-2}) \cdot (-2t^{-3}) = -2t^{-5} < 0$. It follows that the speed of particle Q is decreasing on the interval $(0, \infty)$.

(c) The position, $y_Q(t)$, of particle Q at time t, is given by

$$y_Q(t) = y_Q(1) + \int_1^t v_Q(\tau)\, d\tau \tag{48}$$

$$= 2 + \int_1^t \tau^{-2}\, d\tau \tag{49}$$

$$= 2 - \tau^{-1}\Big|_1^t \tag{50}$$

$$= 2 - \left[\frac{1}{t} - \frac{1}{1}\right] = 3 - \frac{1}{t}. \tag{51}$$

(d) As $t \to \infty$, we see that $x_P(t)$ approaches its limiting value 6 from below, while $y_Q(t)$ approaches its limiting value 3 from below. Thus, the distance from particle P to the origin never exceeds 3, while the distance from particle Q to the origin gets arbitrarily close to 6. Particle Q will eventually be farther from the origin that particle P.