Solutions to 2022 AP Calculus BC Free Response Questions

Louis A. Talman, Ph. D.
Emeritus Professor of Mathematics
Metropolitan State University of Denver

May 13, 2022

1. We are given, effectively, $A(t) = 450\sqrt{\sin(0.62t)}$, where $0 \le t \le 10$, as the rate—in vehicles per hour—at which vehicles arrive at a toll plaza, with t representing the time elapsed since 5 A.M.

   (a) The total number of vehicles that arrive at the toll plaza between $t = 1$ and $t = 5$ is $\displaystyle\int_1^5 A(t)\,dt$. Evaluation is not required, but numerical integration gives

   $$\int_1^5 A(t)\,dt = 450\int_1^5 \sqrt{\sin(0.62t)}\,dt \sim 1502.148 \text{ vehicles.} \tag{1}$$

   (b) The average value of the rate at which vehicles arrive at the toll plaza from $t = 1$ to $t = 5$ is (to three decimal places)

   $$\frac{1}{5-1}\int_1^5 A(t)\,dt \sim 375.537 \text{ vehicles per hour,} \tag{2}$$

   where we have replaced the integral with the value we obtained numerically in the previous part of this problem.

   (c) We find that

   $$A'(t) = \frac{139.5\cos(0.62t)}{\sqrt{\sin(0.62t)}}, \tag{3}$$

   whence $A'(1) \sim 148.947 > 0$. The derivative of the function A is positive when $t = 1$, so A, the rate at which vehicles arrive at the toll plaza at 6 A.M, is increasing at that time.

(d) We are given that

$$N(t) = \int_a^t [A(x) - 400]\, dx, \tag{4}$$

where a is the time when A is 400. By the Fundamental Theorem of Calculus, this means that $N'(t) = A(t) - 400$.

Solving numerically, we find that $A(t) = 400$ when $t \sim 1.469$, so $a \sim 1.469$. Solving numerically, we find that $N't) = 0$ when $t = a$ (Duh!) and when $t \sim 3.598$.. Consequently, the maximum value of N on the interval $[a, 4]$ occurs at $t = a$, when $t \sim 3.598$, or when $t = 4$. We find that

$$N[a] = 0, \tag{5}$$

$$N[3.598] \sim 71.254, \tag{6}$$

$$N[4] \sim 62.338. \tag{7}$$

From this we conclude that, to the nearest whole vehicle, the length of the line reaches a maximum about 71 vehicles at about 8:36 A.M.

2. (a) Given that $x'(t) = \sqrt{1+t^2}$ and $y'(t) = \ln(2+t^2)$, we must have for the slope of the tangent line

$$\left.\frac{dy}{dx}\right|_{(1,5)} = \frac{y'(4)}{x'(4)} \tag{8}$$

$$= \frac{\ln(2+16)}{\sqrt{1+16}} \tag{9}$$

$$= \frac{\ln 18}{\sqrt{17}}. \tag{10}$$

(b) The particle's velocity vector, $\mathbf{v}$, is given by $\mathbf{v}(t) = \langle x'(t), y'(t)\rangle$, so for speed, S, we have

$$S = \|\mathbf{v}(t)\| = \|\langle x'(t), y'(t)\rangle\| \tag{11}$$

$$= \sqrt{[y(t)]^2 + [x'(t)]^2}. \tag{12}$$

At $t = 4$ this gives

$$S(4) = \sqrt{[y'(4)]^2 + [x'(4)]^2} \tag{13}$$

$$= \sqrt{[\ln 18]^2 + \left[\sqrt{17}\right]^2} = \sqrt{17 + [\ln 18]^2}. \tag{14}$$

Acceleration, $\mathbf{a}(t)$ at time $t = 4$ is given by

$$\mathbf{a}(4) = \mathbf{v}'(4) = \langle x''(4), y''(4) \rangle. \tag{15}$$

But $x''(t) = \dfrac{t}{\sqrt{1+t^2}}$ and $y''(t) = \dfrac{2t}{2+t^2}$, so

$$x''(4) = \frac{4}{\sqrt{1+4^2}} = \frac{4}{\sqrt{17}}, \text{ and} \tag{16}$$

$$y''(4) = \frac{4}{18} = \frac{2}{9}.\text{Thus,} \tag{17}$$

$$\mathbf{a}(4) = \left\langle \frac{4}{\sqrt{17}}, \frac{2}{9} \right\rangle. \tag{18}$$

(c) The y-coordinate of position, $y(t)$, as a function of time is, by the Fundamental Theorem of Calculus, given by

$$y(t) = 5 + \int_4^t y'(\tau)\, d\tau \tag{19}$$

$$= 5 + \int_4^t \ln(2+\tau^2)\, d\tau. \tag{20}$$

Integrating numerically (This is an elementary integral, but it isn't an easy one), we find, to three decimal places,

$$y(6) = 5 + \int_4^6 \ln(2+\tau^2)\, d\tau \sim 11.570, \tag{21}$$

(d) The distance, D, the particle travels over $4 \leq t \leq 6$ is, to three decimal places,

$$D = \int_4^6 \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2}\, d\tau \tag{22}$$

$$= \int_4^6 \sqrt{1+\tau^2 + [\ln(2+\tau^2)]^2}, d\tau \sim 12.136. \tag{23}$$

3. (a) By the Fundamental Theorem of Calculus and what is given,

$$f(x) = f(4) + \int_4^x f'(t)\, dt \tag{24}$$

$$= 3 + \int_4^x f'(t)\, dt. \tag{25}$$

In fact, f' is given by

$$f'(t) = \begin{cases} -\sqrt{4t - t^2}, & 0 \le t < 4; \\ t - 4, & 4 \le t < 6; \\ 8 - t, & 6 \le t \le 1. \end{cases} \tag{26}$$

However, we will make no use of this fact.

Because the portion of the f' curve from $t = 0$ to $t = 4$ is a semi-circle of radius 2 lying below the horizontal axis whose diameter is the horizontal axis, $\int_0^4 f'(t)\,dt = -2\pi$. The portion of the curve over the interval $[4, 5]$ is a straight line that forms, with the horizontal axis, a triangle of base 1 and altitude 1, so $\int_4^5 f'(t)\,dt = \frac{1}{2}$. From these facts, we find that

$$f(0) = 3 + \int_4^0 f'(t)\,dt = 3 - \int_0^4 f'(t)\,dt = 3 + 2\pi; \tag{27}$$

$$f(5) = 3 + \int_4^5 f'(t)\,dt = 3 + \frac{1}{2} = \frac{7}{2}. \tag{28}$$

(b) A function has an inflection point where its derivative has a local maximum or a local minimum—that is, where its derivative changes from being increasing to being decreasing of *vice versa.* We see from the given graph that $f'(x)$ has a local minimum at $x = 2$ and that $f'(t)$ has a local maximum at $x = 6$. Consequently, f has inflection points at $x = 2$ and and $x = 6$.

Note: Some elementary textbooks require that the second derivative be defined at an inflection point. If we adopt this definition, f has just one inflection point, at $x = 2$.

(c) The function g defined by $g(x) = f(x) - x$ is decreasing on the closures of those intervals where $g'(x) < 0$—that is, where $f'(x) - 1 < 0$, or $f'(x) < 1$. We see from the given graph that these inequalities hold for those, and only those, values of x which are less than 5. Hence f is decreasing on the interval $[0, 5]$.

(d) The absolute minimum value of g on the interval $[0, 7]$ exists because, as the integral of a continuous derivative, g is itself a continuous, differentiable, function on that interval. We know that the absolute minimum of such a function must occur either at an endpoint of the interval or at a point where the derivative is zero. So we must evaluate g at $x = 0$, at $x = 7$, and at $x = 5$, the latter point being the point only point in the interval where $g'(x) = f'(x) - 1 = 0$. From the first part of this problem, we have

$$g(0) = f(0) - 0 = 3 + 2\pi; \tag{29}$$

$$g(5) = f(5) - 5 = \frac{7}{2} - 5 = -\frac{3}{2}. \tag{30}$$

It is easy to see from the geometry of the curve that $\int_4^6 f'(t)\,dt = 2$ and that $\int_6^7 f'(t)\,dt = \frac{3}{2}$. Therefore,

$$g(7) = f(7) - 7 = \left[3 + \int_4^7 f'(t)\,dt\right] - 7 \tag{31}$$

$$= \left[\int_4^6 f'(t)\,dt + \int_6^7 f'(t)\,dt\right] - 4 \tag{32}$$

$$= 2 + \frac{3}{2} - 4 = -\frac{1}{2}. \tag{33}$$

We conclude that the absolute minimum value of g on $[0, 7]$ is $g(5) = -\frac{3}{2}$.

4. (a) We can approximate $r''(8.5)$, in centimeters/day^2, by

$$r''(8.5) \sim \frac{r(10) - r(7)}{10 - 7} = \frac{(-3.8) - (-4.4)}{3} = 0.2 \text{ cm/day}^2. \tag{34}$$

(b) We expect derivative of the radius of the base of a melting cone of ice to be continuous as a function of time. Under the assumption that this is so, there must be a t_0 between $t = 0$ and $t = 3$ where $r(t_0) = -6$, because $r(3) = -5.0$, and $r(0) = -6.1$—so the Intermediate Value Theorem for Continuous Functions guarantees the existence of such a t_0. **Note:** In fact, continuity of r' is not needed; derivatives always have the intermediate value property—but this is rarely shown, or even stated as a fact, in elementary calculus courses. The fact is known as "Darboux's Theorem," and it is a standard part of a good advanced calculus course.

(c) Using the values from the given table in a right Riemann sum, we have

$$\int_0^{12} r'(t)\,dt \sim r'(3)(3-0) + r'(7)(7-3) + r'(10)(10-7) + r'(10)(12-10) \tag{35}$$

$$\sim (-5.0) \cdot 3 + (-4.4) \cdot 4 + (-3.8) \cdot 3 + (-3.5) \cdot 2 = -51.0 \tag{36}$$

(d) Let $h(t)$ denote the height of the cone at time t; in addition to the information in the table, we are given $r(3) = 100$ and $h(3) = 50$. We know that the volume of the cone at time t is given by $V(t) = \frac{\pi}{3}[r(t)]^2 h(t)$, and from this we see that

$$V'(t) = \frac{\pi}{3}\left[2r(t)h(t)r'(t) + [r(t)]^2 h'(t)\right]. \tag{37}$$

Thus,

$$V'(3) = \frac{\pi}{3}\left[2 \cdot 100 \cdot 50 \cdot (-5.0) + (100)^2 \cdot (-2.0)\right] = -\frac{70000\pi}{3}\ \text{cm}^3/\text{day} \tag{38}$$

5. (a) The area of region R is

$$\int_1^5 \frac{dx}{x} = \ln x \Big|_1^5 = \ln 5 - \ln 1 = \ln 5. \tag{39}$$

(b) The volume of the given solid is

$$\int_1^5 x e^{x/5}\, dx = 5xe^{x/5}\Big|_1^5 - 5\int_1^5 e^{x/5}\, dx \tag{40}$$

$$= 25e - 5e^{1/5} - 25e^{x/5}\Big|_1^5 \tag{41}$$

$$= 20e^{1/5}. \tag{42}$$

(c) The volume of the solid generated when the region W is revolved about the x-axis is

$$\pi \int_3^\infty \left(\frac{1}{x^2}\right)^2 dx = \pi \lim_{T\to\infty} \int_3^T x^{-4}\, dx \tag{43}$$

$$= -\frac{\pi}{3} \lim_{T\to\infty} x^{-3}\Big|_3^T \tag{44}$$

$$= \frac{\pi}{3} \lim_{T\to\infty} \left[3^{-3} - T^{-3}\right] = \frac{\pi}{81}. \tag{45}$$

6. (a) We have

$$\lim_{n\to\infty}\left[\frac{|x^{2n+3}|}{(2n+3)} \cdot \frac{(2n+1)}{|x^{2n+1}|}\right] = x^2 \lim_{n\to\infty} \frac{2n+1}{2n+3} \tag{46}$$

$$= x^2 \lim_{n\to\infty} \frac{\left(2 + \frac{1}{n}\right)}{\left(2 + \frac{3}{n}\right)} = x^2. \tag{47}$$

It now follows from the ratio test that the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ converges when $x^2 < 1$, that is, when $-1 < x < 1$, but diverges when $x < -1$ and when $x > 1$.

If $x = 1$, the series becomes $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, while if $x = -1$, the series becomes $-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \cdots$. Each of these two series is the negative of the other, so each of them converges if, and ony if, and other does. It therefore suffices to examine the first of the two.

To this end, consider the series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$, which is $\sum_{k=0}^{\infty}(-1)^k a_k$ with $a_k = \frac{1}{2k+1}$. In the expression for a_k, the numerator is fixed while the denominator increases strictly as $k \to \infty$. It follows that the terms a_k are strictly decreasing as $k \to \infty$. Moreover,

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{2k+1} \tag{48}$$

$$= \lim_{k\to\infty} \frac{\left(\frac{1}{k}\right)}{\left(2+\frac{1}{k}\right)} = 0. \tag{49}$$

We conclude, by the alternating series test, that our series converges.

Putting it all together, we find that the interval of convergence for the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ is $[-1, 1]$.

(b) By an argument altogether similar to what we have given above, the alternating series test is applicable to the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)}$, which gives $f(1/2)$. It follows that the difference between this sum and the value of its first term, which is $1/2$, has magnitude no greater than the magnitude of its second term. The latter magnitude is $1/24$. Thus,

$$\left| f\left(\frac{1}{2}\right) - \frac{1}{2} \right| < \frac{1}{24} < \frac{1}{10}. \tag{50}$$

(c) We may differentiate power series term-by-term, so the series that represents

$f'(x)$ is

$$\sum_{k=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots . \tag{51}$$

(d) The series for $f'(x)$, given in equation (51), is a geometric series—which converges to $\dfrac{1}{1+x^2}$ for $-1 < x < 1$. We may thus use this fact it to evaluate $f'(1/6)$:

$$f'\left(\frac{1}{6}\right) = \frac{1}{1+\frac{1}{36}} = \frac{36}{37}. \tag{52}$$