# Solutions to 2023 AP Calculus AB Free Response Questions

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1. (a) The integral $\int_{60}^{135} f(t)\,dt$ gives, in gallons, the amount of gas pumped into the gas tank during the time interval $60 \le t \le 135$. This amount is given by

$$\int_{60}^{135} f(t)\,dt \sim f(90)\cdot 30 + f(120)\cdot 30 + f(135)\cdot 15 \tag{1}$$

$$= 0.15\cdot 30 + 0.10\cdot 30 + 0.05\cdot 15 = 8.25 \text{ gallons} \tag{2}$$

where we have used "$\sim$" to mean "is approximately equal to."

(b) The function $f$ is given differentiable, presumably on at least the interior of its domain, and therefore certainly on the interval $(60, 120)$ because $(60, 120) \subseteq (0, 150)$. It also follows from the differentiability of $f$ on $(0, 150)$ that $f$ is continuous (and differentiable) on $[60, 120]$. We may therefore apply the Mean Value Theorem to $f$ on $[60, 120]$ to conclude that there is a number, $c$, in the interval $(60, 120)$ such that

$$f'(c)(120 - 60) = f(120) - f(60) = 0. \tag{3}$$

We conclude that there must be a number with the required properties.

(c) If the rate of flow of gasoline be modeled by

$$g(t) = \frac{t}{500}\cos\left[\left(\frac{t}{120}\right)^2\right],$$

for $0 \le t \le 150$, then $\bar{g}$, the average rate of flow for that time interval is given by

$$\begin{aligned}
\bar{g} &= \frac{1}{150-0}\int_0^{150} g(t)\,dt && (4)\\
&= \frac{1}{150\cdot 500}\int_0^{150} t\cos\left(\frac{t}{120}\right)^2\,dt && (5)\\
&= \frac{120\cdot 120}{2\cdot 150\cdot 500}\int_0^{150} \frac{2}{120}\cdot\frac{t}{120}\cdot\cos\left(\frac{t}{120}\right)^2\,dt && (6)\\
&= \frac{12}{125}\sin\left(\frac{t}{120}\right)^2\Bigg|_0^{150} && (7)\\
&= \frac{12}{125}\sin\frac{25}{16} \sim 0.095997. && (8)
\end{aligned}$$

(d) With $g$ as given, we have

$$\begin{aligned}
g'(t) &= \frac{1}{500}\cdot\cos\left(\frac{t}{120}\right)^2 - \frac{t}{500}\cdot\frac{2t}{120}\cdot\frac{1}{120}\cdot\sin\left(\frac{t}{120}\right)^2 && (9)\\
&= \frac{1}{500}\cdot\cos\left(\frac{t}{120}\right)^2 - \frac{t^2}{3600000}\sin\left(\frac{t}{120}\right)^2 && (10)
\end{aligned}$$

Thus,

$$g'(140) = \frac{1}{500}\cos\frac{49}{36} - \frac{49}{9000}\sin\frac{49}{36}. \qquad (11)$$

2. (a) Stephen's velocity for $0 \le t \le 90$, we are told, is given, in m/sec, by

$$v(t) = 2.38e^{-0.02t}\sin\left(\frac{\pi}{56}t\right), \qquad (12)$$

so Stephen changes directions at just those times $t$ between $t = 0$ and $t = 90$ where $v(t)$ changes sign. All factors of $v(t)$ are positive except for $\sin\left(\frac{\pi}{56}t\right)$, so the sign of this latter factor determines the sign of $v$. But $\sin u$ changes sign only at $u = \pi$ if $0 \le u \le \frac{90\pi}{56} < 2\pi$. Thus, $t = 56$ seconds gives the only time at which Stephen changes direction of travel.

(b) Stephen's acceleration at time $\big(t, a(t)\big)$, is given by

$$\begin{aligned}
a(t) &= v'(t) && (13)\\
&= \frac{d}{dt}\left[2.38e^{-0.02t}\sin\frac{\pi}{56}t\right] && (14)\\
&= e^{-0.02t}\left(0.133518\cos\frac{\pi}{56}t - 0.0476\sin\frac{\pi}{56}t\right). && (15)
\end{aligned}$$

Thus, Stephen's acceleration at time $t = 60$ is

$$a(60) \sim -0.0360162 \text{ m/sec}^2. \tag{16}$$

The acceleration is negative. But

$$v(60) \sim -0.259512 \tag{17}$$

is also negative, and speed $s(t)$, which is never negative, satisfies $s(t)^2 = v(t)^2$, which means that

$$\frac{d}{dt}\left[s(t)^2\right] = 2s(t)s'(t) = 2v(t)v'(t), \tag{18}$$

so the sign of $s'(t)$ is the same as that of the product $v(t)v'(t)$. We have seen that $v(60)$ and $v'(60)$ are both negative, and we conclude that $s'(60)$ must be positive—which, in turn, means that speed is increasing when $t = 60$.

(c) The distance $S$ between Stephen's position at time $t = 20$ secconds and time $t = 80$ seconds is given, in meters, by

$$\begin{aligned} S &= \left| \int_{20}^{80} v(t)\, dt \right| && (19)\\ &= \left| 2.38 \int_{20}^{80} e^{-0.02t} \sin \frac{\pi}{56} t\, dt \right| && (20)\\ &\sim 23.383997 \text{ meters.} && (21) \end{aligned}$$

(d) The distance $D$ that Stephen travels during the time interval $0 \leq t \leq 90$ is given by

$$\begin{aligned} d &= \int_{0}^{90} |v(t)|\, dt && (22)\\ &= 2.38 \int_{0}^{90} \left| e^{-0.02t} \sin \frac{\pi}{56} t \right| dt && (23)\\ &= 2.38 \int_{0}^{56} e^{-0.02t} \sin \frac{\pi}{56} t\, dt - 2.38 \int_{56}^{90} e^{-0.02t} \sin \frac{\pi}{56} t\, dt && (24)\\ &\sim 62.1642 \text{ meters.} && (25) \end{aligned}$$

3. (a) See Figure 3a.

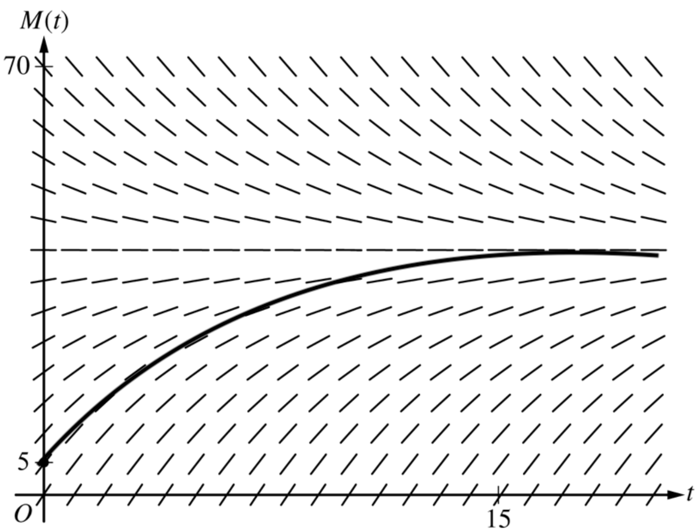


Figure 1: Problem 3a

(b) The line tangent, at $(0, 5)$, to the graph of Problem 3a is

$$M = 5 + \frac{1}{4}(40 - 5)t = 5 + \frac{35}{4}t \tag{26}$$

This gives the approximate value $M = 5 + \frac{35}{2} = \frac{45}{2}$ C° for the temperature of the milk at $t = 2$.

(c) Because

$$\frac{dM}{dt} = \frac{1}{4}(40 - M), \tag{27}$$

we have

$$\frac{d^2M}{dt^2} = \frac{d}{dt}\left[\frac{1}{4}(40 - M)\right] \tag{28}$$

$$= -\frac{1}{4}\frac{dM}{dt} \tag{29}$$

$$= -\frac{1}{16}(40 - M). \tag{30}$$

When $0 \le M < 40$, it is clear that $M'' < 0$; in particular, $M'' < 0$ when $t = 0$ and $M = 5$. By continuity, $M$ is near 5 when $t$ is near 2, so we can expect $M''$

to be negative for values of $t$ near zero. This means that the graph of $M$ as a function of $t$ is concave downward near the point $(0, 5)$ so that its tangent lines lie above the curve in that region. Consequently, the approximation of Problem 3b is an underestimate for the actual value of $M(2)$.

(d) We are given the initial value problem

$$\frac{dM}{dt} = \frac{1}{4}(40 - M); \tag{31}$$

$$M(0) = 5. \tag{32}$$

Let us suppose that the function $\varphi$ gives a solution to this problem, so that

$$\varphi'(t) = \frac{1}{4}[40 - \varphi(t)], \tag{33}$$

and

$$\varphi(0) = 5. \tag{34}$$

Then we may write

$$\frac{4\varphi'(t)}{40 - \varphi(t)} = 1. \tag{35}$$

From this, it follows that

$$4\int_0^t \frac{\varphi'(\tau)}{40 - \varphi(\tau)}\, d\tau = \int_0^t d\tau, \tag{36}$$

In order to carry out the integration on the left side of (36), we make the substituion $M = M(t) = \varphi(t)$; $dM = \varphi'(t)\, dt$. We carry out the integration onf the right, and (36) becomes

$$\int_{\varphi(0)}^{\varphi(t)} \frac{dM}{40 - M} = \frac{t}{4}. \tag{37}$$

As long as $t > 0$ is not too big, we know, by the continuity of $\varphi$ and the fact that $\varphi(0) = 5$, that $\varphi(t) < 40$. Thus, for positive values $t$ that are not too large, we have

$$-\ln[40 - \varphi(t)] + \ln 35 = \frac{t}{4}, \tag{38}$$

which, upon back-substituting and eliminating the logarithm, becomes

$$\frac{35}{40 - M} = e^{t/4}. \tag{39}$$

Equation (39) is equivalent to

$$35e^{-t/4} = 40 - M, \text{ or} \tag{40}$$

$$M(t) = 40 - 35e^{-t/4}. \tag{41}$$

This is the solution we sought.

4. (a) The function $f$, as given, does not have a relative minimum at $x = 6$. This is so because $f'(x) > 0$ on $(5, 6)$ and on $(6, 7)$—making $f$ a strictly increasing function on the interval $[5, 7]$.

(b) A function $f$ is concave downward on any open interval where $f'$ is a decreasing function. The function for which the graph of $f'$ is given can be seen to be a decreasing function on the interval $(-2, 0)$ and on the interval $(4, 6)$. Consequently, the function $f$ is concave downward on the interval $(-2, 0)$, and concave downward on the interval $(4, 6)$.

(c) By the Fundamental Theorem of Calculus, we can write

$$f(x) = f(2) + \int_2^x f'(t)\,dt. \tag{42}$$

On the interval $[0, 4]$, we see from the graph of $f'$ that $f'(x) = x - 2$. Thus, when $0 \le x \le 4$, we have (because $f(2) = 1$ is given)

$$f(x) = 1 + \int_2^x (t-2)\,dt \tag{43}$$

$$= 1 + \left[\left(\frac{t^2}{2} - 2t\right)\Big|_2^x\right] \tag{44}$$

$$= \frac{1}{2}x^2 - 2x + 3. \tag{45}$$

Thus,

$$\lim_{x\to 2} \frac{6f(x) - 3x}{x^2 - 5x + 6} = \lim_{x\to 2} \frac{3x^2 - 15x + 18}{x^2 - 5x + 6} \tag{46}$$

$$= \lim_{x\to 2} \frac{3\cancel{(x^2 - 5x + 6)}}{\cancel{x^2 - 5x + 6}} = \lim_{x\to 2} 3 = 3. \tag{47}$$

**Alternate Solution:** It is also possible to use the fact (given) that $f(2) = 1$, and the fact (which we can read from the graph) that $f'(x) = 0 = \lim_{x\to 2} f'(x)$ for all $x$ near $x = 2$, to employ l'Hôpital's Rule to solve this problem.

Because $f'$ is given, we know that $f$ is differentiable, and therefore continuous on the interval $[-2, 8]$; moreover, it is clear from the graph of $f'$ that $f'$ is continuous on that interval, and at $x = 2$, whence $\lim_{x\to 2} f'(x) = 0$. So both the numerator and the denominator of $\dfrac{6f(x) - 3x}{x^2 - 5x + 6}$ are continuous and continuously differentiable at $x = 2$. We also have

$$\lim_{x\to 2} [6f(x) - 3x] = 6 \cdot 1 - 3 \cdot 2 = 0, \tag{48}$$

and

$$\lim_{x\to 2} \left[x^2 - 5x + 6\right] = 2^2 - 5 \cdot 2 + 6 = 0. \tag{49}$$

It is therefore legitimate to see if l'Hôpital's rule can be used to evaluate the limit we seek:
We have,

$$\lim_{x\to 2} \frac{6f(x) - 3x}{x^2 - 5x + 6} = \lim_{x\to 2} \frac{6f'(x) - 3}{2x - 5} = \frac{6 \cdot 0 - 3}{2 \cdot 2 - 5} = 3. \tag{50}$$

The limit exists, so l'Hôpital's rule justifies the first of the equalities in (50).

(d) If a differentiable function, $g$, has an absolute minimum at $x = a$ in an interval $[\alpha, \beta]$, the value of its derivative, $g'(a)$ must vanish or $a$ must be either $\alpha$ or $\beta$. We see from the given graph that the derivative $f'$ vanishes only at $x = -1$, $x = 2$ and $x = 6$. We have already [see Problem (4a)] ruled out $x = 6$ as a possibility for the function $f$ of this problem: we saw there that $f$ doesn't have even a relative minimum there—and we know that an absolute minimum must be a relative minimum.

There can't be a relative minimum (or, consequently, an absolute minimum) for $f$ at $x = -1$ because $f'(x) > 0$ when $-2 < x < -1$—meaning that $f$ is increasing immediately to the left of $x = 1$).

We also know that $f$ must have an absolute minimum in the interval $[-2, 8]$, because $f$ is differentiable, and therefore continuous, throughout that interval, so that the Extreme Value Theorem guarantees an absolute minimum somewhere therein. We have ruled out $x = 6$ and $x = -1$. So the absolute minimum lies at one of the points $\{-2, 2, 8\}$.

We know [see Problem (4c)] that

$$f(x) = f(2) + \int_2^x f'(t)\, dt. \tag{51}$$

We see (by considering what has been given and using the areas between the $f'$ curve and the $x$-axis) to evaluate the integral, that

$$f(-2) = 1 + (2 + 1 - 1) = 2; \tag{52}$$

$$f(2) = 1; \tag{53}$$

$$f(8) = 1 + [2 + (8 - 2\pi)] = 11 - 2\pi \sim 4.72. \tag{54}$$

We see that $f$ assumes its absolute minimum value on $[-2, 8]$ at $x = 2$, where $f(2) = 1$.

5. (a) If $h(x) = f[g(x)]$, then. by the Chain Rule, $h'(7) = f'[g(7)] \cdot g'(7)$. From the given table, we see that $g'(7) = 8$, $g(7) = 0$, $f'(0) = 3/2$. Thus,

$$h'(7) = f'[g(7)] \cdot g'(7) \tag{55}$$
$$= f'[0] \cdot 8 \tag{56}$$
$$= \frac{3}{2} \cdot 8 = 12. \tag{57}$$

(b) If the derivative of the function $k$ is given by $k'(x) = [f(x)]^2 \cdot g(x)$, then, by the Power Rule, the Product Rule and the Chain Rule,

$$k''(x) = 2f(x)g(x)f'(x) + [f(x)]^2 g'(x). \tag{58}$$

Reading again from the table as necessary, we have

$$k''(4) = f(4)g(4)f'(4) + [f(4)]^2 g'(4) \tag{59}$$
$$= 4 \cdot (-3) \cdot 3 + 4^2 \cdot 2 \tag{60}$$
$$= -36 + 32 = -4. \tag{61}$$

We find that $k''(4) = -4 < 0$, so graph of the function $k$ is concave downward at $x = 4$.

(c) We are given

$$m(x) = 5x^3 + \int_0^x f'(t)\, dt. \tag{62}$$

This means, by the Fundamental Theorem of Calculus, that

$$m(x) = 5x^3 + [f(x) - f(0)], \tag{63}$$

so that

$$m(2) = 5 \cdot 2^3 + f(2) - f(0) = 40 + 7 - 10 = 37. \tag{64}$$

Also by the Fundamental Theorem of Calculus,

$$m'(x) = 15x^2 + \frac{d}{dx} \int_0^x f'(t)\, dt \tag{65}$$
$$= 15x^2 + f'(x). \tag{66}$$

Thus,

$$m'(2) = 15 \cdot 2^2 + f'(2) = 60 + (-8) = 52. \tag{67}$$

(d) The notions "increasing (decreasing) at the point" are problematic, because very few textbooks or instructors define the notions. And we must take care in making such a definition, because it is possible for a derivative to be (say) positive at a point while its primitive is not increasing in any open interval centered at that point.

The function $m$, as given, is twice differentiable, which means that $m'$ must be continuous. Thus, because $m'(2) = 52 > 0$, $m'(x)$ must be positive for all values of $x$ that lie in some open interval centered at $x = 2$, guaranteeing that $m$ is an increasing function on that interval and "at the point" $(2, 37)$.

6. (a) If $6xy = 2 + y^3$, then, differentiating implicitly yields

$$6y + 6xy' = 3y^2y', \text{ or} \tag{68}$$

$$2y = (y^2 - 2x)y' \tag{69}$$

$$y' = \frac{2y}{y^2 - 2x}, \tag{70}$$

as required.

(b) Horizontal tangent lines are to be found at points $(x, y)$ where $y' = 0$, or, in this case, where

$$\frac{2y}{y^2 - 2x} = 0. \tag{71}$$

At such a point, we would have to have $y = 0$, but if $y = 0$, the equation $6xy = 2 + y^3$ becomes the equaion $0 = 2$, which has no solutions in $x$. (Or in anything else.) Thus, no horizontal lines are tangent to this curve.

(c) If $6xy = 2 + y^3$, then

$$x = \frac{2 + y^3}{6y}, \text{ so that} \tag{72}$$

$$\frac{dx}{dy} = \frac{3y^2 \cdot 6y - 6(2 + y^3)}{36y^2} \tag{73}$$

$$= \frac{18y^3 - 12 - 6y^3}{36y^2} \tag{74}$$

$$= \frac{y^3 - 1}{3y^2} \tag{75}$$

Thus, $\frac{dx}{dy} = 0$, giving a vertical tangent line, precisely when $y = 1$, and

$$x = \frac{2 + 1^3}{6 \cdot 1} = \frac{1}{2}. \tag{76}$$

The only vertical tangent line is the vertical line through the point $\left(\frac{1}{2}, 1\right)$.

(d) If $6xy = 2 + y^3$, and $x$, $y$ both depend differentiably on a third variable $t$, then implicit differentiation with respect to $t$ gives

$$6\left(\frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt}\right) = 3y^2 \frac{dy}{dt}. \tag{77}$$

When $x = \frac{1}{2}$, $y = -2$ and $\frac{dx}{dt} = \frac{2}{3}$, this yields

$$6\left[\frac{2}{3} \cdot (-2) + \frac{1}{2}\frac{dy}{dt}\right] = 3 \cdot (-2)^2 \frac{dy}{dt}, \text{ or} \tag{78}$$

$$-8 + 3\frac{dy}{dt} = 12\frac{dy}{dt}. \tag{79}$$

It follows that, under the given conditions, $\frac{dy}{dt} = -\frac{8}{9}$.