Solutions to
2023 AP Calculus BC
Free Response Questions

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1. (a) The integral $\int_{60}^{135} f(t)\,dt$ gives, in gallons, the amount of gas pumped into the gas tank during the time interval $60 \leq t \leq 135$. This amount is given by

$$\int_{60}^{135} f(t)\,dt \sim f(90)\cdot 30 + f(120)\cdot 30 + f(135)\cdot 15 \tag{1}$$

$$= 0.15\cdot 30 + 0.10\cdot 30 + 0.05\cdot 15 = 8.25 \text{ gallons} \tag{2}$$

where we have used "$\sim$" to mean "is approximately equal to."

(b) The function f is given differentiable, presumably on at least the interior of its domain, and therefore certainly on the interval $(60, 120)$ because $(60, 120) \subseteq (0, 150)$. It also follows from the differentiability of f on $(0, 150)$ that f is continuous (and differentiable) on $[60, 120]$. We may therefore apply the Mean Value Theorem to f on $[60, 120]$ to conclude that there is a number, c, in the interval $(60, 120)$ such that

$$f'(c)(120 - 60) = f(120) - f(60) = 0. \tag{3}$$

We conclude that there must be a number with the required properties.

(c) If the rate of flow of gasoline be modeled by

$$g(t) = \frac{t}{500}\cos\left[\left(\frac{t}{120}\right)^2\right],$$

for $0 \leq t \leq 150$, then $\bar{g}$, the average rate of flow for that time interval is given by

$$\bar{g} = \frac{1}{150-0}\int_0^{150} g(t)\,dt \tag{4}$$
$$= \frac{1}{150\cdot 500}\int_0^{150} t\cos\left(\frac{t}{120}\right)^2\,dt \tag{5}$$
$$= \frac{120\cdot 120}{2\cdot 150\cdot 500}\int_0^{150} \frac{2}{120}\cdot\frac{t}{120}\cdot\cos\left(\frac{t}{120}\right)^2\,dt \tag{6}$$
$$= \frac{12}{125}\sin\left(\frac{t}{120}\right)^2\Big|_0^{150} \tag{7}$$
$$= \frac{12}{125}\sin\frac{25}{16} \sim 0.095997. \tag{8}$$

(d) With g as given, we have

$$g'(t) = \frac{1}{500}\cdot\cos\left(\frac{t}{120}\right)^2 - \frac{t}{500}\cdot\frac{2t}{120}\cdot\frac{1}{120}\cdot\sin\left(\frac{t}{120}\right)^2 \tag{9}$$
$$= \frac{1}{500}\cdot\cos\left(\frac{t}{120}\right)^2 - \frac{t^2}{3600000}\sin\left(\frac{t}{120}\right)^2 \tag{10}$$

Thus,

$$g'(140) = \frac{1}{500}\cos\frac{49}{36} - \frac{49}{9000}\sin\frac{49}{36}. \tag{11}$$

2. (a) If the position vector, $\mathbf{r}$ of the moving particle is given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, with $x'(t) = e^{\cos t}$, $y(t) = 2\sin t$, then the velocity vector is $\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t)\rangle = \langle e^{\cos t}, 2\cos t\rangle$, and the acceleration vector is $\mathbf{a}(t) = \mathbf{v}'(t) = \langle x''(t), y''(t)\rangle$. We have $\mathbf{r}(t) = \langle x(t), 2\sin t\rangle$, so the desired acceleration vector is

$$\mathbf{a}(t) = \langle x''(t), y''(t)\rangle \tag{12}$$
$$= \langle -e^{\cos t}\sin t, -2\sin t\rangle. \tag{13}$$

This gives

$$\mathbf{a}(1) = \langle -e^{\cos 1}\sin 1, -2\sin 1\rangle \tag{14}$$
$$= \langle -1.44441, -1.68294\rangle. \tag{15}$$

(b) Speed, $s(t)$, at time t is given by

$$s(t) = \sqrt{\mathbf{v}(t)\cdot\mathbf{v}(t)} \tag{16}$$
$$= \sqrt{[x'(t)]^2 + [y'(t)]^2} \tag{17}$$
$$= \sqrt{e^{2\cos t} + 4\cos^2 t}. \tag{18}$$

Thus, we seek the smallest value of t in $[0, \pi]$ for which

$$\sqrt{e^{2\cos t} + 4\cos^2 t} = 1.5. \tag{19}$$

We solve numerically, and we find that this equation has two solutions in $[0, \pi]$: $t \sim 1.25447$, and $t \sim 2.35808$. The smaller of these is $t \sim 1.25447$.

(c) The slope of the line tangent to the path of the particle at $t = 1$ is given by

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{y'(1)}{x'(1)} \tag{20}$$

$$= \frac{2\cos 1}{e^{\cos 1}} \sim 0.62953. \tag{21}$$

For $x(1)$, we write (using the Fundamental Theorem of Calculus)

$$x(1) = x(0) + \int_0^1 x'(\tau)\, d\tau \tag{22}$$

$$= 1 + \int_0^1 e^{\cos\tau}\, d\tau \tag{23}$$

$$\sim 3.34157, \tag{24}$$

where we have carried out the integration numerically.

The x coordinate of the moving particle at time $t = 1$ is approximately 3.34157.

(d) The total distance traveled by the particle over the time interval $0 \le t \le \pi$ is

$$\int_0^\pi |\mathbf{v}(\tau)|\, d\tau = \int_0^\pi \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2}\, d\tau \tag{25}$$

$$= \int_0^\pi \sqrt{e^{2\cos\tau} + 4\cos^2\tau}\, d\tau \sim 6.03461. \tag{26}$$

Once again, we have integrated numerically.

3. (a) See Figure 3a. (The slight dip at the right end of the curve doesn't belong there; $M = 40$ is a horizontal asymptote to the curve. But I was too lazy to figure out how to coax cooperation out of the software I used to draw the curve on a copy of the slope-field from the exam.)

(b) The line tangent, at $(0, 5)$, to the graph of Problem 3a is

$$M = 5 + \frac{1}{4}(40 - 5)t = 5 + \frac{35}{4}t \tag{27}$$

For M when $t = 2$, this gives the approximate value $5 + \frac{35}{2} = \frac{45}{2}$ C°.

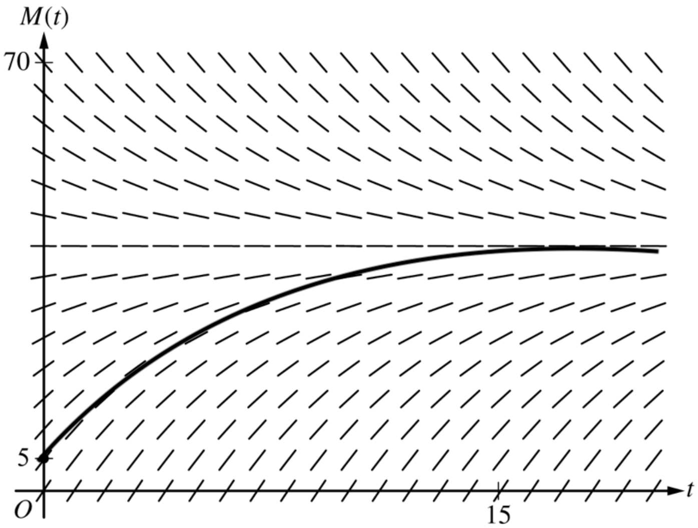

Figure 1: 3a

(c) Because

$$\frac{dM}{dt} = \frac{1}{4}(40 - M), \tag{28}$$

we have

$$\frac{d^2M}{dt^2} = \frac{d}{dt}\left[\frac{1}{4}(40 - M)\right] \tag{29}$$

$$= -\frac{1}{4}\frac{dM}{dt} \tag{30}$$

$$= -\frac{1}{16}(40 - M). \tag{31}$$

When $0 \leq M < 40$, it is clear that $M'' < 0$; in particular, $M'' < 0$ when $t = 0$ and $M = 5$. This means that the graph of M is concave downward near the point $(0, 5)$ so that its tangent lines lie above the curve in that region. Consequently, the approximation of Problem 3b is an underestimate for the actual value of $M(2)$.

(d) We are given the initial value problem

$$\frac{dM}{dt} = \frac{1}{4}(40 - M); \tag{32}$$

$$M(0) = 5. \tag{33}$$

Let us suppose that the function φ gives a solution to this problem, so that

$$\varphi'(t) = \frac{1}{4}[40 - \varphi(t)], \tag{34}$$

and

$$\varphi(0) = 5. \tag{35}$$

By continuity, $\varphi(t) > 0$ when t is close to 0. So, at least when t is near 0, we may write

$$\frac{4\varphi'(t)}{40 - \varphi(t)} = 1. \tag{36}$$

From this, it follows that

$$4\int_0^t \frac{\varphi'(\tau)}{40 - \varphi(\tau)}\, d\tau = \int_0^t d\tau, \tag{37}$$

at least when t is close to zero. In order to carry out the integration on the left side of (37), we make the substituion $M = M(t) = \varphi(t)$; $dM = \varphi'(t)\, dt$. We carry out the integration on the right, and (37) becomes

$$\int_{\varphi(0)}^{\varphi(t)} \frac{dM}{40 - M} = \frac{t}{4}. \tag{38}$$

As long as $t > 0$ is not too big, we know, by the continuity of φ and the fact that $\varphi(0) = 5$, that $\varphi(t) < 40$. Thus, at least for positive values t that are not too large, we have

$$-\ln[40 - \varphi(t)] + \ln 35 = \frac{t}{4}, \tag{39}$$

which, upon back-substituting and eliminating the logarithm, becomes

$$\frac{35}{40 - M} = e^{t/4}. \tag{40}$$

Equation (40) is equivalent to

$$35e^{-t/4} = 40 - M, \text{ or} \tag{41}$$

$$M(t) = 40 - 35e^{-t/4}. \tag{42}$$

This is the solution we sought.

4. (a) The function f, as given, does not have a relative minimum at $x = 6$. This is so because $f'(x) > 0$ on $(5, 6)$ and on $(6, 7)$—making f a strictly increasing function on the interval $[5, 7]$.

(b) A function f is concave downward on any open interval where f' is a decreasing function. The function for which the graph of f' is given can be seen to be a decreasing function on the interval $(-2, 0)$ and on the interval $(4, 6)$. Consequently, the function f is concave downward on the interval $(-2, 0)$, and concave downward on the interval $(4, 6)$.

(c) By the Fundamental Theorem of Calculus, we can write

$$f(x) = f(2) + \int_2^x f'(t)\, dt. \tag{43}$$

On the interval $[0, 4]$, we see from the graph of f' that $f'(x) = x - 2$. Thus, when $0 \le x \le 4$, we have (because $f(2) = 1$ is given)

$$f(x) = 1 + \int_2^x (t - 2)\, dt \tag{44}$$

$$= 1 + \left(\left[\frac{t^2}{2} - 2t \right] \Big|_2^x \right) \tag{45}$$

$$= \frac{1}{2}x^2 - 2x + 3. \tag{46}$$

Thus,

$$\lim_{x\to 2} \frac{6f(x) - 3x}{x^2 - 5x + 6} = \lim_{x\to 2} \frac{3x^2 - 15x + 18}{x^2 - 5x + 6} \tag{47}$$

$$= \lim_{x\to 2} \frac{3\cancel{(x^2 - 5x + 6)}}{\cancel{x^2 - 5x + 6}} = \lim_{x\to 2} 3 = 3. \tag{48}$$

Alternate Solution: It is also possible to use the fact (given) that $f(2) = 1$, and the fact (which we can read from the graph) that $f'(x) = 0 = \lim_{x\to 2} f'(x)$, to employ l'Hôpital's Rule to solve this problem.

Because f' is given, we know that f is differentiable, and therefore continuous on the interval $[-2, 8]$; moreover, it is clear from the graph of f' that f' is continuous on that interval, and at $x = 2$, whence $\lim_{x\to 2} f'(x) = 0$. So both the numerator and the denominator of $\dfrac{6f(x) - 3x}{x^2 - 5x + 6}$ are continuous and continuously differentiable at $x = 2$. We also have

$$\lim_{x\to 2} [6f(x) - 3x] = 6 \cdot 1 - 3 \cdot 2 = 0, \tag{49}$$

and

$$\lim_{x\to 2}\left[x^2-5x+6\right]=2^2-5\cdot 2+6=0. \tag{50}$$

It is therefore legitimate to see if we can use l'Hôpital's rule to evaluate the limit we seek:
We have,

$$\lim_{x\to 2}\frac{6f(x)-3x}{x^2-5x+6}=\lim_{x\to 2}\frac{6f'(x)-3}{2x-5}=\frac{6\cdot 0-3}{2\cdot 2-5}=3. \tag{51}$$

The limit exists, so l'Hôpital's rule justifies the first of the equalities in (51).

(d) If a differentiable function, g, has an absolute minimum at $x=a$ in an interval $[\alpha,\beta]$, then the value of its derivative, $g'(a)$ must vanish or a must be one of α and β. We see from the given graph that $f'(x)$ vanishes only at $x=-1$, $x=2$, and $x=6$. We have already [see Problem (4a)] ruled out $x=6$ as a possibility for the function f of this problem: we saw there that f doesn't have even a relative minimum there—and we know that an absolute minimum must be a relative minimum.

There can't be a relative minimum (or, consequently, an absolute minimum) for f at $x=-1$ because $f'(x)>0$ when $-2<x<-1$—meaning that f is increasing immediately to the left of $x=1$).

We also know that f must have an absolute minimum in the interval $[-2,8]$, because f is differentiable, and therefore continuous, throughout that interval, so that the Extreme Value Theorem guarantees an absolute minimum somewhere therein. We have ruled out $x=6$ and $x=-1$. So the absolute minimum lies at one of the points $\{-2,2,8\}$.

We know [see Problem (4c)] that

$$f(x)=f(2)+\int_2^x f'(t)\,dt. \tag{52}$$

We see (by considering what has been given and using the areas—which we can decompose into triangles and squares from which circumscribed quarter-disks have been removed—between the curve $y=f'(x)$ and the x-axis) to evaluate the integral, that

$$f(-2)=1+(2+1-1)=3; \tag{53}$$

$$f(2)=1; \tag{54}$$

$$f(8)=1+[2+(8-2\pi)]=11-2\pi\sim 4.72. \tag{55}$$

We see that f assumes its absolute minimum value on $[-2,8]$ at $x=2$, where $f(2)=1$.

5. (a) The area of the region that lies below the curve $y = f(x)$ but above the curve $y = g(x)$ as shown is

$$\int_0^3 [f(t) - g(t)]\ dt = \int_0^3 f(t)\,dt - \int_0^3 g(t)\,dt \tag{56}$$

$$= 10 - \int_0^3 \frac{12\,dt}{3+t} \tag{57}$$

$$= 10 - \left(12 \ln|3+t| \Big|_0^3 \right) \tag{58}$$

$$= 10 - 12\,[\ln 6 - \ln 3] \qquad = 10 - 12\ln 2 \sim 1.68223. \tag{59}$$

(b) We calculate

$$\int_0^\infty [g(x)]^2\,dx = \lim_{T\to\infty} \int_0^T \left(\frac{12}{3+x}\right)^2 dx \tag{60}$$

$$= -\lim_{T\to\infty} \left(\frac{144}{3+x}\right)\Big|_0^T \tag{61}$$

$$= -\lim_{T\to\infty} \left[\left(\frac{144}{3+T}\right) - \left(\frac{144}{3+0}\right) \right] \tag{62}$$

$$= 48. \tag{63}$$

The improper integral converges to 48.

(c) We put

$$u = x; \qquad dv = f'(x)dx \tag{64}$$

and we may take

$$du = dx; \qquad v = f(x). \tag{65}$$

Integrating by parts, we find that

$$\int_0^3 x f'(x)\,dx = \int_0^3 u\,dv \tag{66}$$

$$= uv\Big|_0^3 - \int_0^3 v\,du \tag{67}$$

$$= xf(x)\Big|_0^3 - \int_0^3 f(x)\,dx \tag{68}$$

$$= 3 \cdot f(3) - 0 \cdot f(0) - 10 \tag{69}$$

$$= 3 \cdot 2 - 10 = -4. \tag{70}$$

6. We are given that f has derivatives of all orders throughout $\mathbb{R}$ and that:

$$f(0) = 2; \tag{71}$$
$$f'(0) = 3; \tag{72}$$
$$f''(x) = -f(x^2) \tag{73}$$
$$f'''(x) = -2x \cdot f'(x^2). \tag{74}$$

(a)

$$f^{(4)}(x) = \frac{d}{dx} f'''(x) \tag{75}$$
$$= \frac{d}{dx}\left[-2x \cdot f'(x^2)\right] \tag{76}$$
$$= -2f'(x^2) - 4x^2 f''(x^2). \tag{77}$$

Thus,

$$f(0) = 2; \tag{78}$$
$$f'(0) = 3; \tag{79}$$
$$f''(0) = -f(0^2) = -f(0) = -2; \tag{80}$$
$$f'''(0) = -2 \cdot 0 \cdot f'(0) = 0; \tag{81}$$
$$f^{(4)}(0) = -2f'(0) - 4 \cdot 0^2 f''(0^2) = -6 - 0 = -6. \tag{82}$$

Put $a_n = \frac{1}{n!} f^{(n)}(0)$ for $n = 0, \ldots, 4$. Then

$$a_0 = \frac{f(0)}{0!} = 2; \tag{83}$$
$$a_1 = \frac{f'(0)}{1!} = 3: \tag{84}$$
$$a_2 = \frac{f''(0)}{2!} = -1; \tag{85}$$
$$a_3 = \frac{f^{(3)}(0)}{3!} = 0; \text{ and} \tag{86}$$
$$a_4 = \frac{f^{(4)}(0)}{4!} = -\frac{1}{4}. \tag{87}$$

So $T_4(x)$, the desired fourth degree Taylor polynomial, is given by

$$T_4(x) = \sum_{n=0}^{4} a_n x^n \tag{88}$$
$$= 2 + 3x - x^2 - \frac{1}{4}x^4. \tag{89}$$

(b) If $|f^{(5)}(x)| \leq 15$ for $0 \leq x \leq 0.5$, the Lagrange error bound gives

$$|f(x) - T_4(x)| \leq \frac{15}{5!} \cdot |x^n| \tag{90}$$

for $0 \leq x \leq 0.5$. Thus,

$$|f(0.1) - T_4(0.1)| \leq \frac{1}{8}\,|0.1|^5 \tag{91}$$

$$|f(0.1) - T(0.1)| \leq 1.25 \times 10^{-6} < 10^{-5}, \text{ as required.} \tag{92}$$

(c) If $g(0) = 4$ and $g'(x) = e^x f(x)$, then

$$g(0) = 4; \tag{93}$$

$$g'(0) = e^0 f(0) = 1 \cdot 2 = 2; \tag{94}$$

$$g''(0) = e^0 f(0) + e^0 f'(0) = 2 + 1 \cdot 3 = 5. \tag{95}$$

Thus, the second degree Taylor polynomial, $P_2(x)$, for $g(x)$, expanded at $x = 0$ is

$$P_2(x) = g(0) + g'(0)x + \frac{g''(0)}{2}x^2; \tag{96}$$

$$= 4 + 2x + \frac{5}{2}x^2. \tag{97}$$