Solutions to
2025 AP Calculus AB
Free Response Questions

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We use the symbol "$\sim$" to mean "is approximately equal to" throughout this document.

1. We are given $C(t) = 7.6 \arctan 0.2t$.

   A. The average number of acres affected by the invasive species from $t = 0$ to $t = 4$ is

$$\frac{1}{4-0}\int_0^4 C(s)\,ds = \frac{7.6}{4}\int_0^4 \arctan 0.2s\,ds \tag{1}$$

$$= 1.9\left[s\arctan 0.2s - 2.5\ln(s^2+25)\right]\Big|_0^4 \sim 2.778 \tag{2}$$

   B. The instantaneous rate of change of C is

$$C'(t) = \frac{38}{25+t^2}, \tag{3}$$

   while $\overline{C}$, the average rate of change of C over the time interval $0 \le t \le 4$, is

$$\overline{C} = \frac{C(4)-C(0)}{4-0} = \frac{7.6\arctan 0.8 - 7.6\arctan 0}{4-0}. \tag{4}$$

   So we must solve the equation $\dfrac{38}{25+t^2} = \dfrac{C(4)-C(0)}{4-0}$ for t. We have

$$\frac{38}{25+t^2} = \frac{C(4)-C(0)}{4-0}; \tag{5}$$

$$C(4)(25+t^2) = 152; \tag{6}$$

$$t^2 = \frac{152}{C(4)} - 25 = \frac{152}{7.6\arctan 0.8} - 25 \sim 4.64100553; \tag{7}$$

$$t \sim \sqrt{4.64100553} \sim 2.154 \text{ weeks.} \tag{8}$$

C. The end behavior of the rate of change in the number of acres affected by the species is given by

$$\lim_{t\to\infty} C'(t) = \lim_{t\to\infty} \frac{38}{25+t^2} \tag{9}$$

$$= \lim_{t\to\infty} \frac{38t^{-2}}{25t^{-2}+1} = 0. \tag{10}$$

D. We seek the maximum value of

$$A(t) = C(t) - 0.1 \int_4^t \ln(s)\, ds \tag{11}$$

in the interval $4 \leq t \leq 36$. To this end, we note that

$$A'(t) = \frac{38}{25+t^2} - 0.1 \ln t. \tag{12}$$

Machine calculation shows that the only critical point for A (i.e., solution of $A'(t) = 0$ in the interval $4 \leq t \leq 36$ is at $t \sim 11.44169985$. We find that

$$A(4) \sim 5.12803 \tag{13}$$

$$A(11.4416998) \sim 7.31698 \tag{14}$$

$$A(36) \sim 1.74306 \tag{15}$$

Thus, we conclude that, for $4 \leq t \leq 36$, $A(t)$ reaches its maximum value of about 7.317 acres at the time $t \sim 11.442$ weeks.

2. A. The area A of R is given by

$$A = \int_0^3 [g(x) - f(x)]\; dx \tag{16}$$
$$= \int_0^3 \big[(x + \sin \pi x) - (x^2 - 2x)\big]\; dx \tag{17}$$
$$= \int_0^3 \big[3x - x^2 + \sin \pi x\big]\; dx \tag{18}$$
$$= \left[\frac{3}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{\pi}\cos \pi x\right]\Bigg|_0^3 \tag{19}$$
$$= \left(\frac{9}{2} + \frac{1}{\pi}\right) - \left(-\frac{1}{\pi}\right) = \frac{9}{2} + \frac{2}{\pi}. \tag{20}$$

B. The volume V described is given by

$$V = \int_0^3 x\,[g(x) - f(x)]\; dx \tag{21}$$
$$= \int_0^3 \big[3x^2 - x^3 + x \sin \pi x\big]\; dx \tag{22}$$
$$= \left[x^3 - \frac{1}{4}x^4 - \frac{1}{\pi}x\cos \pi x + \frac{1}{\pi^2}\sin \pi x\right]\Bigg|_0^3 \tag{23}$$
$$= \frac{27}{4} + \frac{3}{\pi}. \tag{24}$$

C. The volume V described is given by

$$V = \pi \int_0^3 \left([g(x) + 2]^2 - [f(x) + 2]^2\right)\; dx \tag{25}$$
$$= \pi \int_0^3 \big(12x - 7x^2 + 4x^3 - x^4 + 4\sin \pi x + 2x \sin \pi x + \sin^2 \pi x\big)\; dx \tag{26}$$

Note: It can easily—but somewhat tediously—be shown that $V = \dfrac{249}{10}\pi + 14 \sim 92.226$.

D. The two tangent lines are parallel to each other when $f'(x) - g'(x)$, so—because $f'(x) = 2x - 1$—we must solve the equation

$$2x - 1 = 1 + \pi \cos \pi x, \tag{27}$$

given that $0 \leq x \leq 1$. Numeric solution of this equation leads to $x \sim 0.676$ as the value of x in $(0, 1)$ for which the two tangent lines are parallel.

3. A. Approximating $R'(1)$ using the average rate of change of R over the interval $0 \le t \le 2$ gives

$$R'(1) \sim \frac{R(2) - R(0)}{2 - 0} \tag{28}$$

$$= \frac{100 - 90}{2} = 5 \text{ words per minute per minute.} \tag{29}$$

B. The function R is given differentiable, and differentiable functions are continuous, so R is continuous. We see from the table that $R(8) = 150$ and $R(10) = 162$. Also, $150 \le 155 \le 162$. By the Intermediate Value Theorem for Continuous Functions, there must be a number c such that $0 < 8 < c < 10$ and $R(c) = 155$.

C. We can approximate $\int_0^{10} R(t)\,dt$ by

$$\int_0^{10} R(t)\,dt \sim \frac{1}{2}[R(0) + R(2)](2 - 0) + \frac{1}{2}[R(2) + R(8)](8 - 2) + \frac{1}{2}[R(8) + R(10)](10 - 8) \tag{30}$$

$$\sim \frac{1}{2}[90 + 100] \cdot 2 + \frac{1}{2}[100 + 150] \cdot 6 + \frac{1}{2}[150 + 162] \cdot 2 \tag{31}$$

$$\sim 190 + 3 \cdot 250 + 312 = 1252 \tag{32}$$

D. Based on the model given, by the end of ten minutes, the teacher has read

$$\int_0^{10} W(t)\,dt = \int_0^{10} \left(100 + 8t - \frac{3}{10}t^2\right)\,dt \tag{33}$$

$$= \left[100t + 4t^2 - \frac{1}{10}t^3\right]\Bigg|_0^{10} \tag{34}$$

$$= 1300 \text{ words.} \tag{35}$$

4. A. If $g(x) = \displaystyle\int_6^x g(t)\,dt$, then, by the Fundamental Theorem of Calculus, $g'(x) = f(x)$. Combining this fact with what we read from the picture given, we see that $g'(8) = f(8) = 1$.

   B. The graph given for the function f is also, as we have seen above, the graph of g'. Inflection points for g are to be found at points where g' changes either from increasing to decreasing or from decreasing to increasing[1]. But from the picture and what is given, we see that g' is decreasing on each of the intervals $[-6,-3]$ and $[3,6]$, while it is increasing on each of the intervals $]-3,3]$ and $[6,12]$. From these observations, we conclude that g has points of inflection at $x=-3$, at $x=3$ and at $x=6$.

   C. From the definition and the graph given in the statement of the problem, we have

$$g(12) = \int_6^{12} f(t)\,dt = \frac{1}{2}(12-6)\cdot 3 = 9, \tag{36}$$

   which is the area of a triangle of base 6 and height 3. On the other hand

$$g(0) = \int_6^{0} f(t)\,dt = -\frac{1}{2}\pi\cdot 3^2 = -\frac{9}{2}\pi, \tag{37}$$

   which is the negative of the area of a semicircle of radius 3.

   D. g attains its absolute minimum on the interval $[-6,12]$ at either a critical point or an endpoint. The critical points are those points, x, of $(-6,12)$ where $g'(x) = f(x) = 0$, or $x=0$ and $x=6$. Summing signed areas of appropriate semicircles and triangles, we see that

$$g(-6) = 0; \tag{38}$$
$$g(0) = -\frac{9}{2}\pi \tag{39}$$
$$g(6) = 0 \tag{40}$$
$$g(12) = 9. \tag{41}$$

   Consequently, the absolute minimum value of $g(x)$ for $x \in [-6,12]$ is found at $x=0$ and is $g(0) = -\frac{9}{2}\pi$.

[1]Some authors impose a requirement that the second derivative exist at an inflection point. We are ignoring such a requirement. This could pose a problem for the readers.

5. We are given

$$x_H(t) = e^{t^2-4t}; \tag{42}$$
$$v_J(t) = 2t(t^2-1)^3. \tag{43}$$

A. The velocity $v_H(t)$ of particle H at time t is

$$v_H(t) = x_H'(t) \tag{44}$$
$$= (2t-4)e^{t^2-4t}, \tag{45}$$

from which we find that

$$x_H'(1) = (2\cdot 1-4)e^{1^2-4\cdot 1} = -2e^{-3} \tag{46}$$

B. The particles H and J are moving in opposite directions when their velocities have opposing signs, or when $v_H(t)\cdot v_J(t) < 0$. To solve this inequality, we write

$$v_H(t)\cdot v_J(t) < 0;\ \text{or} \tag{47}$$
$$(2t-4)e^{t^2-4t}\cdot 2t(t^2-1)^3 < 0. \tag{48}$$

The exponential factor can't be negative or zero, and can be ignored. Now $2t-4<0$ when $t<2$; $2t<0$ when $t<0$; and $(t^2-1)^3<0$ when $-1<t<1$. The inequality (48) holds precisely when an odd number of the factors on its left side are negative. This is so when $-1<t<0$ or $1<t<2$. We are interested only in those t that lie in the interval $(0,5)$, so our solution is the open interval $(1,2)$.

C. If $s_J(t)$ denotes the speed of particle$J$ at time t, then $s_J(t)\geq 0$ satisfies

$$\big[s_J(t)\big]^2 = \big[v_J(t)\big]^2; \tag{49}$$

whence

$$2s_J(t)s_J'(t) = 2v_J(t)v_J'(t), \tag{50}$$

so that $s_J'(t)$ has the same sign as the product $v_J(t)v_J'(t)$. But

$$v_J(t)v_J'(t) = \big(2t(t^2-1)^3\big)\cdot\big[2(t^2-1)^3+12t^2(t^2-1)^2\big], \tag{51}$$

so that $v_J(2)\cdot v_J'(2) = 52488 > 0$. Consequently $s'(2)>0$. Because $s'(t)$ is continuous at the point $t=2$, we know that $s'(t)$ is positive on some open interval centered at $t=2$. The function s must be increasing on that interval[2] .

D. By the Fundamental Theorem of Calculus,

$$x_J(2)-x_J(0) = \int_0^2 v_J(t)\,dt. \tag{52}$$

[2]The phrase "increasing at $t=2$" is generally not defined by most authors of calculus textbooks.

From what has been given, we see that

$$x_J(2) = 7 + \int_0^2 \left[2t(t^2-1)^3\right] dt \tag{53}$$

$$= 7 + \left[\frac{1}{4}(t^2-1)^4\right]\Big|_0^2 = 27. \tag{54}$$

6. A. If the curve G be given by $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$, then with a few exceptions, as long as we consider (x, y) to be a point on G,

$$\frac{d}{dx}\left(y^3 - y^2 - y + \frac{1}{4}x^2\right) = \frac{d}{dx}0; \tag{55}$$

so that

$$3y^2y' - 2yy' - y' + \frac{1}{2}x = 0; \tag{56}$$

and

$$\left(3y^2 - 2y - 1\right) y' = -\frac{1}{2}x. \tag{57}$$

From this it follows[3] that

$$y' = \frac{-x}{2(3y^2 - 2y - 1)}, \tag{58}$$

as required.

B. When $x = 2$ and $y = -1$, the equation $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$ becomes

$$(-1)^3 - (-1)^2 - (-1) + \frac{1}{4}(2)^2 = 0, \tag{59}$$

or

$$-1 - 1 + 1 + \frac{1}{4} \cdot 4 = 0, \tag{60}$$

which is a true statement. So the point $(2, -1)$ does lie on the curve G. The line tangent to the curve G at the point $(2, -1)$ is given by the equation

$$y = -1 + m(x - 2), \tag{61}$$

where $m = y'\Big|_{(2,-1)}$. But

$$y'\Big|_{(2,-1)} = \frac{-x}{2(3y^2 - 2y - 1)}\Bigg|_{(2,-1)} \tag{62}$$

$$= \frac{-\cancel{2}}{\cancel{2}[3(-1)^2 - 2(-1) - 1]} = -\frac{1}{4}. \tag{63}$$

[3] As long as $y \neq 1$ and $y \neq -\frac{1}{3}$, where the denominator of the quotient in (58) vanishes. These are the exceptions mentioned earlier. There is, it turns out, no point on G for which $y = -\frac{1}{3}$. There are two points on G where $y = 1$, *both of which are points where* $\frac{dy}{dx}$ *can't be found by implicit differentiation.* One of these turns out to be the point needed in the next part of the question.

The tangent line is therefore given by the equation

$$y = -1 - \frac{1}{4}(x-2). \tag{64}$$

Because the tangent line lies close to the curve when x is close to 2, we can obtain an approximation, y_0 of the value of y for a point $P\colon (1.6, y_0)$ near $(2, -1)$. We obtain

$$\begin{aligned} y_0 &= -1 - \frac{1}{4}\left(\frac{8}{5} - 2\right) & (65) \\ &= -1 + \frac{1}{10} = -\frac{9}{10} = -0.9. & (66) \end{aligned}$$

Thus, the y-coordinate of the point P is given approximately[4] by $y_0 \sim -0.9$.

C. In order to find vertical tangents to the curve G, given by $F(x, y) = 0$ where

$$F(x, y) = y^3 - y^2 - y + \frac{1}{4}x^2, \tag{67}$$

we must find points on G where $\frac{dx}{dy} = 0$. We again employ implicit differentiation, but with respect to y this time[5]:

$$\begin{aligned} \frac{d}{dy}\left(y^3 - y^2 - y + \frac{1}{4}x^2\right) &= \frac{d}{dy}0; & (68) \\ 3y^2 - 2y - 1 + \frac{1}{2}x\frac{dx}{dy} &= 0; & (69) \end{aligned}$$

or

$$\frac{dx}{dy} = \frac{2\left(1 + 2y - 3y^2\right)}{x}. \tag{70}$$

Solving for zeros of this derivative, we find that $y = 1$ or $y = -\frac{1}{3}$. We are interested only in positive values of y and positive values of x, so we discard the second solution. We have only, now, to find all positive values of x for which $F(x, 1) = 0$, or

$$\begin{aligned} 1^3 - 1^2 - 1 + \frac{1}{4}x^2 &= 0; & (71) \\ x^2 &= 4. & (72) \\ & & (73) \end{aligned}$$

The only positive value for x that satisfies this equation is $x = 2$. The point S we seek is $S : (2, 1)$, and the y-coordinate of this point is $y = 1$.

[4]More advanced methods give $y_0 \sim -0.90031$.

[5]See footnote 3 for the reason we take this approach.

D. We differentiate the equation $2xy + \ln y = 8$ implicitly with respect to t, treating both x and y as functions of t:

$$\frac{d}{dt}(2xy + \ln y) = \frac{d}{dt}8; \tag{74}$$

$$2\dot{x}y + 2x\dot{y} + \frac{\dot{y}}{y} = 0. \tag{75}$$

Now we set $x = 4$, $y = 1$, and $\dot{x} = 3$ to obtain

$$2 \cdot 3 \cdot 1 + 2 \cdot 4 \cdot \dot{y} + \frac{\dot{y}}{1} = 0; \tag{76}$$

$$6 + 9\dot{y} = 0; \tag{77}$$

$$\frac{dy}{dt} = \dot{y} = -\frac{2}{3}. \tag{78}$$