Solutions to
2025 AP Calculus BC
Free Response Questions

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We use the symbol "$\sim$" to mean "is approximately equal to" throughout this document.

1. We are given $C(t) = 7.6 \arctan 0.2t$.

   A. The average number of acres affected by the invasive species from $t = 0$ to $t = 4$ is

$$\frac{1}{4-0}\int_0^4 C(s)\,ds = \frac{7.6}{4}\int_0^4 \arctan 0.2s\,ds \tag{1}$$

$$= 1.9\left[s \arctan 0.2s - 2.5\ln(s^2+25)\right]\Big|_0^4 \sim 2.778 \tag{2}$$

   B. The instantaneous rate of change of C is

$$C'(t) = \frac{38}{25+t^2}, \tag{3}$$

   while $\overline{C}$, the average rate of change of C over the time interval $0 \le t \le 4$, is

$$\overline{C} = \frac{C(4)-C(0)}{4-0} = \frac{7.6\arctan 0.8 - 7.6\arctan 0}{4-0}. \tag{4}$$

   So we must solve the equation $\dfrac{38}{25+t^2} = \dfrac{C(4)-C(0)}{4-0}$ for t. We have

$$\frac{38}{25+t^2} = \frac{C(4)-C(0)}{4-0}; \tag{5}$$

$$C(4)(25+t^2) = 152; \tag{6}$$

$$t^2 = \frac{152}{C(4)} - 25 = \frac{152}{7.6\arctan 0.8} - 25 \sim 4.64100553; \tag{7}$$

$$t \sim \sqrt{4.64100553} \sim 2.154 \text{ weeks.} \tag{8}$$

C. The end behavior of the rate of change in the number of acres affected by the species is given by

$$\lim_{t\to\infty} C'(t) = \lim_{t\to\infty} \frac{38}{25+t^2} \tag{9}$$

$$= \lim_{t\to\infty} \frac{38t^{-2}}{25t^{-2}+1} = 0. \tag{10}$$

D. We seek the maximum value of

$$A(t) = C(t) - 0.1 \int_4^t \ln(s)\, ds \tag{11}$$

in the interval $4 \le t \le 36$. To this end, we note that

$$A'(t) = \frac{38}{25+t^2} - 0.1 \ln t. \tag{12}$$

Machine calculation shows that the only critical point for A (i.e., solution of $A'(t) = 0$ in the interval $4 \le t \le 36$ is at $t \sim 11.44169985$. We find that

$$A(4) \sim 5.12803 \tag{13}$$

$$A(11.4416998) \sim 7.31698 \tag{14}$$

$$A(36) \sim 1.74306 \tag{15}$$

Thus, we conclude that, for $4 \le t \le 36$, $A(t)$ reaches its maximum value of about 7.317 acres at the time $t \sim 11.442$ weeks.

2. We are given $r = 2\sin^2\theta$ for $0 \le \theta \le \pi$.

   A. The rate of change of r with respect to θ is

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{d}{d\theta}\left(2\sin^2\theta\right) && (16)\\ &= 4\sin\theta\cos\theta. && (17) \end{aligned}$$

   Consequently, the rate of change of r with respect to θ when $\theta = 1.3$ is

$$4 \cdot (\sin 1.3) \cdot (\cos 1.3) \sim 1.031. \qquad (18)$$

   B. We begin by finding the intersection points of the two curves; for this, we solve the equation

$$\begin{aligned} 2\sin^2\theta &= \frac{1}{2}; && (19)\\ \sin^2\theta &= \frac{1}{4}; && (20)\\ \sin\theta &= \pm\frac{1}{2}; && (21)\\ & && (22) \end{aligned}$$

   The only solutions, θ, for which $0 \le \theta \le \pi$, are $\theta = \dfrac{\pi}{6}$ and $\theta = \dfrac{5\pi}{6}$. Consequently, the desired area between the two curves is

$$\begin{aligned} \frac{1}{2}\int_{\pi/6}^{5\pi/6}\left[\left(2\sin^2\theta\right)^2 - \frac{1}{4}\right]d\theta &= \frac{1}{2}\int_{\pi/6}^{5\pi/6}\left[(1-\cos 2\theta)^2 - \frac{1}{4}\right]d\theta && (23)\\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6}\left[\frac{3}{4} - 2\cos 2\theta + \cos^2 2\theta\right]d\theta && (24)\\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6}\left[\frac{3}{4} - 2\cos 2\theta + \left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right)\right]d\theta && (25)\\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6}\left[\frac{7}{4} - 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right]d\theta && (26)\\ &= \frac{1}{2}\left(\frac{7\sqrt{3}}{8} + \frac{5\pi}{6}\right) \sim 2.067 && (27) \end{aligned}$$

   C. If $0 \le \theta \le \dfrac{\pi}{2}$, the distance from the y-axis to the point (r, θ), which is the maximum value of $x(\theta) = r(\theta)\cos\theta$, must be either at one of the endpoints or at a critical point for $x(\theta)$. But the endpoints of this interval correspond to points on the y axis, where $x(\theta) = 0$, so we can discard these points. The critical points are at zeros of the function $x'(\theta) = 4\sin\theta\cos^2\theta - 2\sin^3\theta$. These zeros are to be found where $\sin\theta = 0$ and where

$$4\cos^2\theta - 2\sin^2\theta = 0; \qquad (28)$$

or, equivalently, where

$$4 - 6\sin^2\theta = 0. \tag{29}$$

This can be rewritten as

$$\sin^2\theta = \frac{2}{3}, \tag{30}$$

whence

$$\sin\theta = \pm\sqrt{\frac{2}{3}}. \tag{31}$$

We are interested only in first quadrant values for θ, so we can discard the negative solution; and we find that our only critical point is at $\theta = \theta_0 = \arcsin\left(\sqrt{\frac{2}{3}}\right)$. We conclude that

$$\theta_0 = \arcsin\left(\sqrt{\frac{2}{3}}\right) \sim 0.955 \tag{32}$$

is the value we want. The maximum value for $x(\theta)$ when $0 \leq \theta \leq \frac{\pi}{2}$ is thus

$$x(\theta_0) = \frac{4\sqrt{3}}{9} \sim 0.770. \tag{33}$$

D The distance from the origin to the point with polar coordinates $(r(\theta), \theta)$ is precisely $r(\theta)$. Treating θ as a function of time t, we have

$$\begin{aligned}
\frac{d}{dt} r[\theta(t)] &= \frac{d}{dt}\left[2\sin^2\theta(t)\right] \qquad (34)\\
&= 4\,\dot{\theta}(t)\sin\theta(t)\cos\theta(t). \qquad (35)
\end{aligned}$$

We have been given a certain time $t = t_1$ when $\theta(t_1) = 1.3$ and $\dot{\theta}(t_1) = 15$. Thus, at the time $t = t_1$, the rate, $\dot{r}$, at which the particle's distance from the origin changes with respect to time is given by

$$\begin{aligned}
\dot{r} &= 4\,\dot{\theta}(t_1)\sin\theta(t_1)\cos\theta(t_1) \qquad (36)\\
&= 60 \cdot \sin 1.3 \cdot \cos 1.3 \sim 15.465. \qquad (37)
\end{aligned}$$

3. A. Approximating $R'(1)$ using the average rate of change of R over the interval $0 \leq t \leq 2$ gives

$$R'(1) \sim \frac{R(2) - R(0)}{2 - 0} \tag{38}$$

$$= \frac{100 - 90}{2} = 5 \text{ words per minute per minute.} \tag{39}$$

B. The function R is given differentiable, and differentiable functions are continuous, so R is continuous. We see from the table that $R(8) = 150$ and $R(10) = 162$. Also, $150 \leq 155 \leq 162$. By the Intermediate Value Theorem for Continuous Functions, there must be a number c such that $0 < 8 < c < 10$ and $R(c) = 155$.

C. We can approximate $\int_0^{10} R(t)\, dt$ by

$$\int_0^{10} R(t)\, dt \sim \frac{1}{2}[R(0) + R(2)](2 - 0) + \frac{1}{2}[R(2) + R(8)](8 - 2) + \frac{1}{2}[R(8) + R(10)](10 - 8) \tag{40}$$

$$\sim \frac{1}{2}[90 + 100] \cdot 2 + \frac{1}{2}[100 + 150] \cdot 6 + \frac{1}{2}[150 + 162] \cdot 2 \tag{41}$$

$$\sim 190 + 3 \cdot 250 + 312 = 1252 \tag{42}$$

D. Based on the model given, by the end of ten minutes, the teacher has read

$$\int_0^{10} W(t)\, dt = \int_0^{10} \left(100 + 8t - \frac{3}{10}t^2\right)\, dt \tag{43}$$

$$= \left[100t + 4t^2 - \frac{1}{10}t^3\right]\Bigg|_0^{10} \tag{44}$$

$$= 1300 \text{ words.} \tag{45}$$

4. A. If $g(x) = \int_6^x g(t)\,dt$, then, by the Fundamental Theorem of Calculus, $g'(x) = f(x)$. Combining this fact with what we read from the picture given, we see that $g'(8) = f(8) = 1$.

B. The graph given for the function f is also, as we have seen above, the graph of g'. Inflection points for g are to be found at points where g' changes either from increasing to decreasing or from decreasing to increasing[1]. But from the picture and what is given, we see that g' is decreasing on each of the intervals $[-6,-3]$ and $[3,6]$, while it is increasing on each of the intervals $]-3,3]$ and $[6,12]$. From these observations, we conclude that g has points of inflection at $x=-3$, at $x=3$ and at $x=6$.

C. From the definition and the graph given in the statement of the problem, we have

$$g(12) = \int_6^{12} f(t)\,dt = \frac{1}{2}(12-6)\cdot 3 = 9, \tag{46}$$

which is the area of a triangle of base 6 and height 3. On the other hand

$$g(0) = \int_6^{0} f(t)\,dt = -\frac{1}{2}\pi \cdot 3^2 = -\frac{9}{2}\pi, \tag{47}$$

which is the negative of the area of a semicircle of radius 3.

D. g attains its absolute minimum on the interval $[-6,12]$ at either a critical point or an endpoint. The critical points are those points, x, of $(-6,12)$ where $g'(x) = f(x) = 0$, or $x=0$ and $x=6$. Summing signed areas of appropriate semicircles and triangles, we see that

$$g(-6) = 0; \tag{48}$$

$$g(0) = -\frac{9}{2}\pi \tag{49}$$

$$g(6) = 0 \tag{50}$$

$$g(12) = 9. \tag{51}$$

Consequently, the absolute minimum value of $g(x)$ for $x \in [-6,12]$ is found at $x=0$ and is $g(0) = -\frac{9}{2}\pi$.

[1] Some authors impose a requirement that the second derivative exist at an inflection point. We are ignoring such a requirement. This could pose a problem for the readers.

5. We are given that the function f is the solution the initial value problem

$$y' = (3-x)y^2; \tag{52}$$
$$y(1) = -1. \tag{53}$$

A. The function f is a solution of the differential equation. Thus, $f'(x) = (3-x)[f(x)]^2$, and we must have

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left\{(3-x)[f(x)]^2\right\} && (54)\\ &= 2f(x)f'(x)(3-x) - [f(x)]^2 && (55)\\ &= 2(3-x)^2[f(x)]^3 - [f(x)]^2. && (56)\end{aligned}$$

We have

$$f(1) = -1; \tag{57}$$
$$f'(1) = (3-1)(-1)^2 = 2; \tag{58}$$

whence we conclude that

$$f''(1) = 2(3-1)^2[f(1)]^3 - [f(1)]^2 = -9. \tag{59}$$

B. $T_2(x)$, the second degree Taylor polynomial for f about $x = 1$, is given by

$$\begin{aligned} T_2(x) &= f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 && (60)\\ &= -1 + 2(x-1) - \frac{9}{2}(x-1)^2. && (61)\end{aligned}$$

C. The Lagrange Remainder for $T_2(x)$ is

$$R_2(x) = \frac{1}{6}f'''(\xi)(x-1)^3, \tag{62}$$

the number ξ lying somewhere between x and 1. We have been given that $\left|f'''(x)\right| \leq 60$ for all x lying in $[1, 1.1]$, so we may conclude that

$$\begin{aligned} \left|f(1.1) - T_2(1.1)\right| &= \left|R_2(1.1)\right| && (63)\\ &\leq \frac{1}{6}\cdot 60 \cdot \left|1.1 - 1\right|^3 = 10 \cdot \frac{1}{1000} = \frac{1}{100}. && (64)\end{aligned}$$

D. Euler's method for the differential equation $y' = F(x, y)$ with intial condition $y_0 = -1$, starting at $x = 1$ with steps of size h, takes $x_0 = 1$ and iterates the recursion

$$x_{k+1} = x_k + h; \tag{65}$$
$$y_{k+1} = y_k + F(x_k, y_k)h; \tag{66}$$

for $k = 0, 1, 2, \ldots$. We are to approximate the solution of $y = (3 - x)y^2$; $y(1) = -1$ at $x = 1.4$ with two steps of equal size, so we will take $h = 0.2$. We obtain

$$x_1 = 1 + 0.2 = 1.2; \tag{67}$$
$$y_1 = -1 + (3 - 1)(-1)^2(0.2) = -0.6; \tag{68}$$
$$x_2 = 1.2 + 0.2 = 1.4; \tag{69}$$
$$y_2 = -0.6 + (3 - 1.2)(-0.6)^2(0.2) = -0.4704. \tag{70}$$

6. We consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-4)^{n+1}}{(n+1)3^n}$.

   A. We have

$$\lim_{n\to\infty} \left| \frac{(x-4)^{n+2}}{(n+2)3^{n+1}} \cdot \frac{(n+1)3^n}{(x-4)^{n+1}} \right| = \frac{1}{3}|x-4| \lim_{n\to\infty} \frac{(n+1)}{(n+2)} \tag{71}$$
$$= \frac{1}{3}|x-4| \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} \tag{72}$$
$$= \frac{1}{3}|x-4|. \tag{73}$$

   This is less than one when $|x - 4| < 3$ or, equivalently, when $1 < x < 7$. By the Ratio Test, this is the interior of the interval of convergence. When $x = 7$ the series becomes

$$\sum_{n=1}^{\infty} \frac{(7-4)^{n+1}}{(n+1)3^n} = 3\sum_{n=1}^{\infty} \frac{1}{n+1}, \tag{74}$$

   which is a divergent harmonic series. When $x = 1$ the series becomes

$$\sum_{n=1}^{\infty} \frac{(1-4)^{n+1}}{(n+1)3^n} = 3\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}, \tag{75}$$

   which is a convergent alternating harmonic series. We conclude that the interval of convergence for this series is $1 < x \leq 7$.

   B. If $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(x-4)^{n+1}}{(n+1)3^n}$ throughout the series' interval of convergence, we can obtain the series for f' by term-by-term differentiation. Thus,

$$f(x) = \sum_{n=1}^{\infty} \frac{d}{dx}\left[\frac{(x-4)^{n+1}}{(n+1)3^n}\right] \tag{76}$$
$$= \sum_{n=1}^{\infty} \frac{(x-4)^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{x-4}{3}\right)^n \tag{77}$$

   throughout the interior of the interval of convergence. The first three non-zero terms of this series are $\frac{x-4}{3}$, $\frac{(x-4)^2}{3^2}$, and $\frac{(x-4)^3}{3^3}$, and the general tern is $\frac{(x-4)^n}{3^n}$.

C. We know that when the geometric series with common ratio r converges, we can write

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}. \tag{78}$$

Consequently, when x lies inside the interval $(1, 7)$, we have

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} \frac{(x-4)^n}{3^n} && (79) \\ &= \frac{x-4}{3} \cdot \frac{1}{\left(1 - \frac{x-4}{3}\right)} && (80) \\ &= \frac{x-4}{3-(x-4)} = \frac{x-4}{7-x}. && (81) \end{aligned}$$

D. As we have seen above, the interior of the interval of convergence for f, and therefore for f', is $(1, 7)$. Because the number 8 lies outside the closure of this interval, the Taylor series for f' diverges at $x = 8$.

Alternately, we can see that the series diverges when $x = 8$ because it becomes $\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$, which is a divergent geometric series.